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Electromagnetic Slow Wave Systems

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John Wiley & Sons, Inc., New York · London · Sydney

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Library of Congress Catalog Card Number: 64-13212

Printed in the United States of America

Eng'g
621.3815
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This Book Is Dedicated to

my parents,

Ayn Rand,

and the glory of GOD

Preface

With the advent of World War II there was an explosion of activity in the microwave electronics field. The British very suddenly found themselves in need of a means for generating high-power signals at wavelengths of a few centimeters. For their airborne radars they turned to a high-frequency electronic tube called the magnetron. It had been invented by A. W. Hull in 1921 and subsequently modified into a circular form which converts d.c. electronic energy into radio frequency electromagnetic energy. Around 1940 magnetrons were operated at frequencies of 3000 Mc with pulsed powers of tens of kilowatts for a microsecond or so. The magnetron was brought to the United States during the early phases of the war and developed under military supervision so as to yield peak powers of megawatts in the kilomegacycle range.

In 1939 the first operational high-frequency tube, known as the klystron, was developed by W. W. Hansen and the Varian brothers working under the auspices of Stanford University. During the war the relatively low-power reflex klystron oscillator was relegated to the humble role of local oscillator in superheterodyne-type radar receivers. Today, however, the importance of klystrons and magnetrons is reversed: The Hansen Laboratories at Stanford University and others are building megawatt klystron tubes for radar oscillators, microwave amplifiers, and linear accelerators.

These linear accelerators utilize a propagating structure for electromagnetic waves known as a *slow wave structure*, so called because it supports a wave traveling at slower than light velocity. The structure is designed so that the wave velocity is approximately the velocity of electrons (or occasionally, protons) in order that the electric field of the wave will travel with the electrons and accelerate them. Accelerators may have an interaction length ranging from a few hundred feet to several miles. Microwave amplifiers and oscillators operating in the kilomegacycle range also employ slow wave structures for much the same reason. A slight readjustment of the electron beam velocity from the value required

in the accelerator will cause time-average radio frequency power to flow from the electron beam into the slow wave circuit and thence out into the useful load.

The slow wave structures for many accelerators, amplifiers, and oscillators are built by cascading resonant cavities, each providing a very strong electric field in a narrow frequency range for good power transfer between the particles and the circuit. Such cavity-chain slow wave structures usually provide tremendous power gains over short interaction lengths but are necessarily limited to a frequency bandwidth anywhere from a fraction of a percent up to 10% or so. What is needed for commercial communication is a wider-bandwidth amplifier tube capable of amplifying information in frequency bands hundreds of megacycles wide without distortion or interference. This brings us to the subject of the traveling wave tube.

In the early 1940's, R. Kompfner, reflecting upon the relative inefficiency and low bandwidth of the klystron, reasoned that if an electron beam were to interact continuously with a wave on a helix it would interact more efficiently. Furthermore, the helix would not be strongly resonant at any one frequency and therefore would have a high bandwidth. His first traveling wave tube, tested in 1943, had a power gain of 6 db and a noise figure of 13 db at a wavelength of 9.1 cm.

Subsequently, Bell Telephone Laboratory workers, among whom J. R. Pierce is perhaps most prominent, developed the traveling wave tube theory and built many tubes with gains of 30 db and more and bandwidths of more than 25%, in a frequency range extending well over 100 kMc. A 4-kMc tube with a 20% bandwidth, for example, can easily handle multichannel FM and television communication, since a simple black and white television channel requires a bandwidth of only 4 Mc or so. The ordinary helix type of traveling wave tube amplifier was the forerunner of a host of similar tubes: backward wave oscillators, resistive wall amplifiers, double-stream amplifiers, and crossed-field or *M*-type carcinotrons. Many of these tubes employ a slow wave structure in the form of a helix of one or more windings; others employ a ridged waveguide to slow the electromagnetic wave to the beam velocity; still others employ a chain of resonant cavities, mutually coupled through small holes.

This book treats for the most part the slow wave systems that employ the chain of resonant cavities as a slow wave circuit, as opposed to the wider-bandwidth helices and ridged waveguides. The primary reason for this emphasis is the fact that a consistent analysis of propagation in the chain of cavities which are either empty or else interacting with an electron beam or other active medium is not as well understood in detail as is the propagation in a helix or loaded-waveguide type of structure operated far

from any one of its resonant frequencies. The customary treatment of propagation in a cavity chain coupled to an electron beam, say, is based upon an equivalent circuit, the parameters of which are often not clearly related to the physical parameters of the structure. In this book I analyze the propagation in such structures directly from Maxwell's rather than Kirchhoff's equations, making reasonable assumptions in order to simplify the mathematics, but always in terms of parameters directly related to the structure.

Veteran workers in the microwave field, as well as beginning students of electromagnetic theory, are strongly urged to read the Introduction for a review of the basic concepts of slow wave structures and my definitions of terms such as "wave," "mode," and "harmonic," which are often bandied about rather carelessly. The Introduction also explains my uncommon notation which has the advantage of preciseness at the cost of being somewhat cumbersome.

Chapter 1 reviews the periodicity properties of various slow wave structures and formulates the general problem. It emphasizes the general description of the propagation characteristics of the chain of resonant cavities, although I do summarize the Marcuvitz-Schwinger technique as a useful tool for calculating equivalent circuit elements for obstacles in waveguides, with which the propagation can be studied. Chapter 2 reviews somewhat briefly the basic aspects of helices, tape ladder lines, and the ridged waveguide. Information about other structures such as meander lines, different types of ridged waveguides, and dielectric-loaded structures will be found in the literature. Chapters 3 and 4 deal with the properties of a cavity chain in which the cavities are lightly coupled to each other. About the only restrictions I impose on the chain in these chapters are that (1) it has a certain longitudinal symmetry, and (2) its passband of interest is narrow compared to the width of the adjacent stopband(s). I remove this second restriction in Chapter 5. Chapter 6 introduces a powerful variational analysis for general helical slow wave structures; practically all the results in this and the remainder of the book have not yet appeared either in journal or in book form. The variational analysis is convenient and straightforward for helical structures and almost indispensable for an understanding of the general resonant-slot or resonant-loop coupled cavity chain treated in Chapter 7.

Chapters 8-10 discuss both general and specific aspects of propagation in helix traveling wave tubes and in cavity-chain traveling wave tubes in which the cavities interact with a longitudinally confined beam at a frequency in one of the passbands. Chapter 9 treats the cavity-chain tubes operated in one of the narrow passbands in much the same way as J. R. Pierce treats the helix tubes in his Chapters 5-11 of *Traveling Wave Tubes*.

Chapter 10—although devoted to small-signal effects—should help workers design stagger-tuned and synchronously tuned klystron amplifiers with both reasonable gain *and* reasonable bandwidth.

During the last few years parametric amplifiers have come into vogue as low noise devices, particularly suited to radio astronomy work. Chapter 11, I believe, clarifies the nature of the waves in a slow wave structure consisting of a chain of resonant cavities in each of which a ferroelectric sample is operated parametrically. An interesting gain-passband width relation exists under certain conditions for the three-frequency amplifier. H. Suhl has analyzed the interactions in a cavity operated with a sample of ferromagnetic rather than ferroelectric material, and I use his results to write the equations for the waves in a ferromagnetic parametric amplifier, both for electromagnetic and semistatic operation. Traveling wave masers have proven to be devices of comparable or lower noise characteristics so I develop the quantum electrodynamic equations for a three-frequency continuously operated maser in the last section of Chapter 11.

The problem of slow wave structure synthesis proved to be so formidable that the best I can do in Chapter 12 is present bounds for dispersion curves and perturbation formulas for small changes in geometry, as well as an outline of a computer program for analyzing various helices. Since the problem is one of optimization of the impedance and bandwidth at a given center frequency and phase velocity I believe we ought to compute the characteristics of all helices of interest and catalogue them.

The numerical computations in this book for various beam tubes and the parametric amplifier, although theoretical, should prove valuable for two reasons. First, the equations for the computations are all really based on variational expressions which have unquestionably yielded accurate descriptions of the propagation characteristics of empty helices and cavity chains. Second, the presence of the electron beam or active parametric or maser material merely augments Maxwell's equations without changing the nature of the analysis. The crucial question which always remains is: How many modes of the slow wave structure must we employ in order to obtain a satisfactory description of the waves in the composite system? Although the answer depends upon our definition of "satisfactory," let me point out that the variational method usually gives a better estimate of the propagation constants or phase shifts of the waves at a given frequency than it does of the fields which go with the waves.

Those readers who are just embarking upon a research program of unknown outcome may be interested to learn that this entire book is an outgrowth of essentially one simple idea. The idea concerns an accurate way to represent electromagnetic fields on an open surface between two regions. The concept enabled Professor H. A. Haus, T. J. Goblick, and

I, in 1957, at the Research Laboratory of Electronics, M.I.T., to obtain the single-passband characteristics of the cavity-chain type of slow wave structure, as discussed in Chapter 4. We then developed a variational formulation of the analysis which established it on firmer ground. Later, in the course of my Ph.D. work with Professor M. Chodorow and Dr. G. Kino at Stanford University I formulated the variational analysis in a manner more satisfying to me and also verified the fact that this analysis is a logical extension of the Bethe theory of small-hole coupling between cavities.

I made a careful check on the detailed derivations of certain equations and the computer programs to verify the fact that the results agreed with physical knowledge and were compatible among themselves. The pronoun "we" used throughout the book is editorial.

Some workers in this field may view with some uneasiness the methods of analysis employed. I can only say that I have never found an instance in which any one of these methods, when properly interpreted, failed to give a good description of periodic slow wave system phenomena.

My thanks go to the Board of Regents of the University of California at Berkeley for providing me with free time on those blessed IBM 704 and 7090 computers in Campbell Hall. I acknowledge with gratitude the splendid research atmosphere in the Electrical Engineering Department, Professor R. M. Saunders, Chairman. I have also benefited from conversations with Professors W. J. Welch, T. E. Everhart, A. J. Lichtenberg, and C. K. Birdsall of this department.

Good luck in the reading, *garde à vous*, and a pox on headache.

R. M. Bevensee

Berkeley, California
January 1964

This research was supported in part by the Air Force Office of Scientific Research of the Office of Aerospace Research; the Department of the Army, Army Research Office; and the Department of the Navy, Office of Naval Research, under Grant No. AF-AFOSR-139-63.

Contents

Introduction 1

Chapter I. General Properties of Slow Wave Structures 8

- 1-1 Synopsis 8
- 1-2 Periodicity Theorem 9
- 1-3 The Central Problem of Slow Wave Systems. Dispersion Curve Solutions 14
- 1-4 Significance of the Group Velocity 17
- 1-5 Early Transmission Filters 19
- 1-6 The Periodically Lumped-Loaded Transmission Line 20
- 1-7 The Periodic Cavity Chain 24

Chapter II. Conventional Analyses of Helices and Loaded Waveguides 39

- 2-1 Synopsis 39
- 2-2 The Sheath Helix 40
- 2-3 The Tape Helix 44
- 2-4 Multifilar Helices 49
- 2-5 Tape Ladder Lines 51
- 2-6 The Ridged Waveguide 57

Chapter III. Radiation and Cavity Coupling through Small Holes and Apertures 63

- 3-1 Synopsis 63
- 3-2 Objectives 64
- 3-3 Review of the Bethe Theory 65
- 3-4 The Centerhole-Coupled Cavity Chain 72
- 3-5 Nearly Uniform $\hat{H}_0$ and Inflecting $\hat{E}_0$ on the Hole 74
- 3-6 Nearly Uniform $\hat{E}_0$ and Inflecting $\hat{H}_0$ on the Hole 76
- 3-7 Elliptical Hole in a Plane Sheet 77
- 3-8 Fields and Coupling on Waveguide Apertures 80

3-9	Equations for Two Coupled Cavities	83
3-10	Expressions for the Coupling Integral	$-\int_{\text{hole}} \vec{e}_1^{(1)} \times \vec{H}_1^{(1)} \cdot \vec{i}_z dS$
		87
Chapter IV. The Single-Passband Behavior of an Infinite Cavity Chain		92
4-1	Synopsis	92
4-2	Dispersion Relation from the Short-Circuit Mode Expansion	94
4-3	A Theorem about the Passband Shape	101
4-4	The Power Flow-Stored Energy-Group Velocity Relation	107
4-5	The Maximization of Electronic Gain Per Cavity and the Fractional Passband Width	108
4-6	Dispersion Relation from the Open-Circuit Mode Expansion	110
4-7	Equivalent Circuits for the n th Passband	112
Chapter V. General Theorems about Symmetric Cavity Chains		120
5-1	Synopsis	120
5-2	Symmetry of the Structure	122
5-3	The General Field Expansions	125
5-4	Effect of the Irrotational Modes upon the Dispersion Characteristics	134
5-5	Correction to the First Dispersion Curve Due to the Modes of the Second Passband	140
5-6	The Frequency-Squared Sum Rule	148
Chapter VI. Variational Analyses of "Cold" Slow Wave Structures		151
6-1	Synopsis	151
6-2	Examples of the Variational Technique	153
6-3	A Variational Analysis for Wire and Tape Helices	166
6-4	The Variational Analysis of Ring Helices	181
6-5	Contrawound Helices	188
6-6	The Variational Analysis of the Ridged Waveguide	190
6-7	Variational Expressions for $j\omega$ and k^2 of a Periodic Cavity Chain	195
6-8	Critique of the Variational Analyses	201
Chapter VII. Resonant-Slot and Resonant-Loop Coupled Structures		205
7-1	Synopsis	205
7-2	A Resonant-Slot Coupled Chain	207
7-3	Calculation of the Interaction Parameter $E_z(0)^2/W_s$	214
7-4	A Slot-Coupled Chain with Successive Slots Rotated	215

- 7-5 Variational Description of the Dispersion in Resonant-Slot and Resonant-Loop Coupled Structures 220
- 7-6 Loop-Coupled Cavity Chains; the Centipede 227
- 7-7 The Cloverleaf Chain 238

Chapter VIII. Analyses of Electron Beam Interaction with Various Slow Wave Structures 244

- 8-1 Synopsis 244
- 8-2 Variational Treatment of the Interaction between One Helix Mode and the Beam 245
- 8-3 Extended Variational Treatment of the Helix-Beam Interaction 252
- 8-4 Interaction between the Beam and Certain Cavity Chains 257
- 8-5 Interaction of a Resonant-Slot Coupled Structure with the Beam 263
- 8-6 An Equivalence Principle for all Types of Cavity Chains 265

Chapter IX. The Cavity-Chain Beam Tube 270

- 9-1 Synopsis 270
- 9-2 Description of the Structure and Definitions of Terms 274
- 9-3 Nature of the Waves for Uniform Cavity Electric Field 280
- 9-4 The Growing Wave for Operation near the $7\pi/4$ Synchronous Point 291
- 9-5 Expressions for Electromagnetic Power Flow across a Coupling Surface 292
- 9-6 Nature of the Waves for Uniform Cavity Electric Field, in a Voltage Range about the Synchronous Voltage at π -Cutoff 295
- 9-7 Behavior of the Growing Wave with Sinusoidal e_z Cavity Field; Operation near the $3\pi/4$ and π Synchronous Points 297
- 9-8 Excitation of the Growing Wave in the First Cavity of the Fig. 9-1 Structure 302
- 9-9 Effect of a Reactive Discontinuity (Detuned Cavity) 307
- 9-10 Output Cavity Match to the Growing Wave; Input Cavity Match to the Backward Wave 311
- 9-11 Effect on the Growing Wave of the Impedance Walls 318

Chapter X. Design Considerations for Cavity-Chain and Multicavity Klystron Beam Amplifiers 323

- 10-1 Synopsis 323
- 10-2 Outline of the Cavity-Chain Analysis 325

- 10-3 Decoupled and Coupled Synchronously Tuned Tube Models with High Gain and Narrow Bandwidth 334
- 10-4 Effect of Stagger-Tuning on the Performances of the High-Gain, Narrow-Bandwidth Tube Models 346
- 10-5 Various Tube Models with an Optimized Output Section 347
- 10-6 Response Characteristics of Constituent Cavities Appropriate for Multicavity Klystrons 354

Chapter XI. Parametric and Maser Slow Wave Systems 359

- 11-1 Synopsis 359
- 11-2 Parametric Power Conversion on Two Coupled Transmission Lines 362
- 11-3 The Three-Frequency Ferroelectric Parametric Amplifier 368
- 11-4 The Three-Frequency Ferromagnetic Cavity-Chain Amplifier 387
- 11-5 The Cavity Chain Maser 394

Chapter XII. Aspects of Slow Wave Structure Synthesis 406

- 12-1 Synopsis 406
- 12-2 Machine Computation of the Characteristics of a Given Helix 407
- 12-3 Ideal Interaction Parameter and Impedance Z_I for a Helix or Cavity Chain 413
- 12-4 Upper and Lower Bounds to $\omega(\beta_0)$ 416
- 12-5 A Perturbation Formula for $\omega(\beta)$ of a General Helix 422
- 12-6 Resonant-Slot and Resonant-Loop Dispersion Curve Shaping in a Cavity Chain 425

Appendix A. Proofs Based upon the Variational Expressions 429

- A-1. The Power Flow–Stored Energy–Group Velocity Relation. General Expression for Power Flow in a Cavity Chain 429
- A.2 Derivation of the Pierce Equation for a Weak Beam Interacting with One Mode of a Slow Wave Structure 432

Appendix B. Measurement of Tube and Cavity Parameters 435

- B-1. Perturbation Formula for Electric Field 435
- B-2. Definition of the Cavity Parameter R_{sh}/Q 436
- B-3. Helix Impedance Based upon Perturbation of Electric Field 437
- B-4. Helix Gain Parameter C_P from the Kompfner Dip Condition 439
- B-5. C_P for the Cavity Chain 440

Appendix C. Dispersion and $E_z(0)^2/W_s$ in the Cloverleaf Slow Wave Structure 442

C-1. Dispersion Relation 442

C-2. Interaction Parameter, $\frac{1}{L} \int_0^L \frac{|\hat{E}_z(0)|^2 dz}{W_s}$ 444

Appendix D. Development of the Small-Signal Beam Equations 446

Appendix E. Solution of the Beam and Cavity Equations 449

E-1. Uniform Field Pattern $\bar{e}_z$ 449

E-2. Sinusoidal Field Pattern $\bar{e}_z = e_m \sin(\pi z/L)$ 451

Index 455

Introduction¹

In this Introduction the basic concepts of slow wave systems will be introduced, terms will be defined, and the notation we use will be explained. The defined terms are listed in a group at the end of the introduction.

Everyone knows that d.c. fields do not travel through space carrying radio and television programs and other information. A radar beam will not distinguish a target unless the wavelength of the beam is comparable to or smaller than the target dimensions, so that kilomegacycle beams with wavelengths of a few centimeters are preferred. Another advantage of high frequencies accrues from the fact that a higher center frequency of a communication system also implies a higher bandwidth, as a general rule. This is because equipment can be built at successively higher frequencies so as to maintain a certain percentage bandwidth, say 1%. One percent of 100 kc is scarcely enough bandwidth for a single AM radio station but 1% of 1000 Mc is 10 Mc, about the bandwidth required by two black-and-white television channels.

For all these reasons it is advantageous to generate very high-frequency radio waves in the kilomegacycle range. The fact that we cannot easily build ordinary triode oscillators in this range forced the invention of the two-cavity klystron amplifier, followed by the reflex klystron oscillator. In these tubes, d.c. energy of electrons is converted directly to r.f. (radio-frequency) power by the interaction of the r.f. fields with electrons within one or two cavities which are resonant at the operating frequency. The extraction of d.c. electronic power can be improved by allowing a beam

¹ The reader unfamiliar with the basic properties of slow wave structures is urged to read J. C. Slater, *Microwave Electronics*, D. Van Nostrand and Co., Princeton, N.J., 1950, Chapter 8, in conjunction with this Introduction.

of electrons to travel in synchronism with an r.f. wave through many cavities over a long interaction distance so as to obtain cumulative interaction. Actually there are two types of cumulative interaction, one resulting in the conversion of d.c. electronic power into r.f. circuit power which flows into the external load, and one resulting in a flow of circuit power into electron energy as in the linear accelerator. The point is simply this: For more efficient amplifiers as well as accelerators the electrons should interact with the circuit wave over a long distance.

The trouble is that electrons travel with velocities less than light velocity, whereas r.f. fields in waveguides travel with velocities equal to or greater than that of light. What is required, then, is a *slowing structure* to reduce the field velocity to the electronic velocity. Such a structure can assume one of two basic forms, as illustrated by Figs. 1-1 and 1-2. In Fig. 1-1 the circuit field is induced to travel at the velocity of light along a wire wound in the shape of a spiral or helix, so that the velocity along the axial or z direction is considerably less than light velocity. The tighter the helix is wound, the smaller the pitch p and the more slowly the wave appears to travel in the z direction. If an electron beam is introduced, traveling along the axis with the same axial velocity as the wave, cumulative interaction will occur and we have a form of traveling wave tube. In Fig. 1-2 we have a waveguide loaded periodically by a series of obstacles, fins in this instance, which slow the wave much as they would slow a stream of water coursing through the pipe. The smaller the centerholes the greater the reduction in circuit field velocity through the pipe. This structure belongs to the class of cavity-chain slow wave structures, each of which consists of a chain of lightly coupled resonant cavities. Electrons, of course, can be introduced to flow along the axis with a velocity which allows them either to absorb power from or to deliver power to the circuit wave.

If we agree that these slow wave structures are good candidates for use in microwave amplifiers and linear accelerators, the next question is: How shall we describe them electromagnetically at a given frequency? Specifically, we want to know if they will support a strong component of axial electric field for exciting electrons and at what velocity this field tends to travel through each structure. We realize first of all that the electric and magnetic fields at each point satisfy the *Maxwell equations*,

$$\nabla \times \hat{E} = -\frac{\partial \hat{B}}{\partial t} = -j\omega\mu \cdot \hat{H} \quad (i)$$

$$\nabla \times \hat{H} = \frac{\partial \hat{D}}{\partial t} = j\omega\epsilon \cdot \hat{E} \quad \sigma = 0 \quad (ii)$$

in the absence of electrons. Sinusoidal time dependence $e^{j\omega t}$ is always assumed unless otherwise stated. We shall always neglect ohmic loss, unless otherwise stated, as a realistic approximation for most slow wave structures of interest. The caret is added as a reminder that the quantity is an r.f. vector which varies spacially in a yet-unknown manner.

In order to study the spacial variation of the fields, $\hat{E}_z$ in particular, in the Fig. 1-1 helix, say, we let the structure be infinitely long. This stipulation is not just an impractical one inserted to simplify matters, as indeed it does; it turns out that if we know the fields which can propagate through the infinitely long structure we can terminate it at both ends and specify the fields in between so as to satisfy the boundary conditions. The advantage of analyzing an infinitely long structure rests in the symmetry it possesses. Imagine yourself at any point (r, θ, z) in cylindrical coordinates of the Fig 1-1 helix and then move along z and also in the angular direction at some fixed radius so as to follow the wire. You will find that the geometry does not change, and this has an important bearing on the functional form of the fields which satisfy Eqs. i and ii. The deduction of this functional form is explained in Chapter 1, but at the moment it suits our purposes merely to give the expression for the $\hat{E}_z$ field. It is

$$\hat{E}_z(r, \theta, z) = \sum_{n=-\infty}^{+\infty} A_n F_n(r) e^{j(n\theta - \beta_n z) + j\omega t}, \quad \beta_n = \beta_0 + \frac{2n\pi}{p} \quad (\text{iii})$$

We say that this $\hat{E}_z$ belongs to a *wave* at frequency ω which satisfies Maxwell's equations and all the boundary conditions. This is the definition of one of several terms which we shall define very carefully for use throughout the book. Expansion iii is reminiscent of a Fourier expansion for a periodic function; it says that $\hat{E}_z$ everywhere is composed of an infinite number of components, which we shall define as *modes*. A *mode* will denote a particular solution to Maxwell's equations or the wave equation (for $\hat{E}$ or $\hat{H}$ alone, as obtained from i and ii) at a certain frequency, subject to very particular boundary conditions over the entire boundary surface, excepting those boundaries at $z = \pm\infty$. Although it is not apparent now, each mode of iii can be chosen to satisfy Maxwell's equations. Occasionally the boundary conditions are "open" in the sense that the relation between tangential $\hat{E}$ and tangential $\hat{H}$ is left unspecified. Each mode of iii satisfies "open" boundary conditions because the relations between the components of $\hat{E}$ and $\hat{H}$ in the mode are not known until all the relative amplitudes A_m of the various modes are determined so as to satisfy the boundary conditions at the helix radius. These boundary conditions are continuity of fields across the open surface

at $r = a$ and $\hat{E}_{\tan} = 0$ on the wire surface. In iii F_n is a function of radius which we need not specify as yet. The exponent is interesting, for it says that the n th mode has a *propagation constant* β_n which differs from a reference propagation constant β_0 by the term $2n\pi/p$. The constant β_0 is called the *fundamental propagation constant*, not because the fundamental mode is always more predominant than any other mode but because it serves as a reference.

With β_0 as the reference propagation constant, the entire $\hat{E}_z$ field, as well as all other field components, phase shift by $e^{-j\beta_0 p}$ in moving from z to $z + p$. To prove this we need only change z to $z + p$ in iii at fixed r and θ , whereupon the n th mode exponent changes to

$$\beta_n z + \left(\beta_0 + \frac{2n\pi}{p} \right) p = \beta_n z + \beta_0 p,$$

the phase shift $2n\pi$ being irrelevant. Therefore we see that

$$\hat{E}_z(r, \theta, z + p) = \hat{E}_z(r, \theta, z) e^{-j\beta_0 p} \quad (\text{iv})$$

The term $\beta_0 p$ is called the *periodic phase shift* of the structure, and iv illustrates the periodicity theorem which can be proved for any slow wave structure in which the geometry repeats at intervals of p . The determination of β_0 at a given frequency ω is the central problem of a slow wave structure, and we shall devote many pages to its solution for various structures. A partial solution for the helix of Fig. 1-1 can be depicted in the form of a curve, Fig. i, known appropriately as an ω - β_0 or *dispersion curve*. Every slow wave structure has such a curve or series of curves with certain properties to be described next.

Suppose we choose point A on the curve of Fig. i and draw a dashed

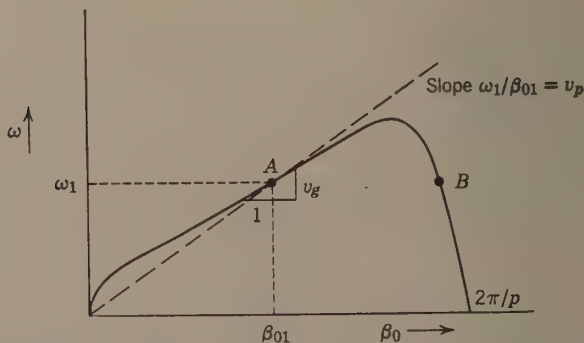


Fig. i. ω - β_0 or dispersion curve for a helix. v_p is the phase velocity and v_g the group velocity at point A . p is the pitch.

line from the origin through this point. We now want to show that the slope of this line, ω_1/β_{01} , is the velocity or, more precisely, the *phase velocity* of the $n = 0$ mode. This is the velocity with which an observer must travel to keep in step with this mode. To see why, we refer to iii and let the observer advance a distance Δz in time Δt . If the observer is truly in step with the $n = 0$ mode, the exponent will not change:

$$-\beta_{01}z + \omega_1 t = -\beta_{01}(z + \Delta z) + \omega_1(t + \Delta t).$$

But this means $0 = -\beta_{01}\Delta z + \omega_1\Delta t$ or $\Delta z/\Delta t = \omega_1/\beta_{01}$. Since the observer is moving with the velocity $\Delta z/\Delta t$, the $n = 0$ mode must be moving with the phase velocity ω_1/β_{01} . If an electron beam had the velocity of the dashed line it would interact strongly with the $n = 0$ mode of the circuit wave at frequency ω_1 . This same argument is valid for any mode n : The phase velocity of mode n is ω_1/β_{n1} ; the other β_n 's are not apparent in Fig. i but they are obtained from β_{01} by the relation of iii. It is perhaps a curious fact that all the modes in the wave travel with different velocities, but they do travel in such a way as to make the entire field phase shift by $e^{-j\beta_0 p}$ from any point z to the point $z + p$ according to iv.

There is another velocity of considerable importance, called the *group velocity*. It is defined as the slope $\partial\omega/\partial\beta_n$, at the point on the ω - β_n curve corresponding to frequency ω . Physically the group velocity is the energy velocity in a lossless structure, and its sign determines the direction of r.f. power flow in the wave. The group velocity happens to be the same for all the modes because $\partial\omega/\partial\beta_n = (\partial\beta_n/\partial\omega)^{-1} = (\partial\beta_0/\partial\omega)^{-1} = \partial\omega/\partial\beta_0$. Thus the group velocity is a parameter of an entire wave. In Fig. i the group velocity at point *A* is positive, which means the r.f. power in the wave flows in the $+z$ direction. The group velocity at point *B* for an independent wave at the same frequency ω_1 is negative, which means the r.f. power in this wave flows in the $-z$ direction. In Chapter 1 we will develop the precise relation between power flow, stored energy, and group velocity of a wave.

We can repeat practically all the preceding remarks with minor modifications for the slow wave structure of Fig. 1-2. This structure, unlike the helix, has rotational symmetry, and the period is denoted by L instead of p . The $\hat{E}_z$ field in a given wave of this structure can be generally written in the form

$$\hat{E}_z(r, \theta, z) = \sum V_m(z) \bar{e}_{mz}(r, \theta), \quad V_m(z + L) = V_m(z) e^{-j\beta_0 L} \quad (\text{v})$$

a sum over modes of the circular waveguide. The m th such mode is defined in Chapter 1 so that its $\bar{e}_m$ and $\bar{h}_m$ vector mode patterns satisfy a certain equation over the cross section of the waveguide. The bar will

always denote a vector *mode pattern* which forms a component of a wave only when it is multiplied by a complex amplitude, in this case $V_m(z)$. Since the geometry remains invariant on moving from any point z to the corresponding point $z + L$ a period away, we can expand $V_m(z)$ in a Fourier series as

$$V_m(z) = \sum_{n=-\infty}^{\infty} V_{mn} \epsilon^{-j\beta_n z + j\omega t}, \quad \beta_n = \beta_0 + \frac{2n\pi}{L} \quad (\text{vi})$$

This expansion will be called an expansion in *harmonics*, and introduces the third term we want to define carefully for the entire book. A *harmonic* is merely a component of a wave containing the explicit factor $\epsilon^{-j2n\pi z/L}$, where p may substitute for L . A harmonic may be a component of a mode, as in vi, or it may be a component of a wave which is expanded directly in harmonics because there is no unique expansion for a given wave. A harmonic rarely satisfies all the boundary conditions, and it may or may not satisfy Maxwell's equations.

According to these definitions each mode of iii might be called a harmonic instead; we have simply preferred to say that $\hat{E}_z$ is expanded in modes rather than harmonics. There is another way of representing the $\hat{E}$ field in a cavity chain like that of Fig. 1-2. It reveals the usefulness of modes as distinct from harmonics. We can define a complete set of resonant modes within a single cavity of the structure by closing the holes. Then the $\hat{E}$ field at every point in this cavity can be expanded with these modes as

$$\hat{E}(r, \theta, z) = \sum_n V_n \bar{E}_n \quad (\text{vii})$$

V_n being the complex amplitude of this mode and $\bar{E}_n$ its vector electric field pattern. In the adjacent cavity $\hat{E}$ would be

$$\hat{E}(r, \theta, z + L) = \epsilon^{-j\beta_0 L} \sum_n V_n \bar{E}_n, \quad \bar{E}_n(z) = \bar{E}_n(z + L) \quad (\text{viii})$$

for the same reason that iv was written. We discuss these expansions further in Chapter 1; the sums could be resolved further into an expansion in harmonics but it is not necessary.

Summary of Terms

1. β_0 is the fundamental propagation constant of a wave.
2. β_n is the propagation constant of the n th mode or harmonic in the wave, related to β_0 as $\beta_n = \beta_0 + 2n\pi/p$. p is the periodic length, sometimes denoted by L .

3. $\beta_0 p$ (or $\beta_0 L$) is the periodic phase shift of the entire electric field and the entire magnetic field in a wave, from any point z to the point $z + p$ (or $z + L$).
4. A dispersion curve is a plot of frequency ω vs. β_n for the n th mode or harmonic in a wave.
5. The phase velocity, ω/β_n , is the velocity with which a constant phase front of the n th mode or harmonic in a wave travels along the structure.
6. The group velocity, $\partial\omega/\partial\beta_n$, is the velocity of energy flow in the wave and the same for all components of the wave. Its sign determines the direction of r.f. power flow.

General Properties of Slow Wave Structures

I-1. Synopsis

This chapter develops in more detail statements made in the Introduction about the properties of slow wave structures. The periodicity theorem, a general theorem for all periodic structures, is illustrated for the helix by invoking the symmetry properties to develop a mode expansion for $\hat{E}_z$ belonging to a given wave. This expansion introduces the fundamental propagation constant β_0 , as well as the related propagation constant β_n for the n th mode in the wave. The periodic phase shift is shown to be $\beta_0 p$. Analogous statements are made about the chain of coupled cavities.

The periodicity theorem helps us to formulate the central problem of a slow wave system: the determination of the mode components of electric and magnetic field in a particular wave which obeys Maxwell's equations subject to periodic boundary conditions. The solutions for lossless passive structures are usually expressed graphically in the form of ω - β curves, where a given curve has a number of related branches, the n th one of which represents the behavior of ω vs. β_n for the n th mode. The shapes of the various branches indicate the phase velocities ω/β_n , of the various modes and the group velocity of all of them at any frequency. The various ω - β curves for a structure, each of which represents a *passband* within which waves propagate without attenuation in lossless structures, are separated by frequency intervals called *stopbands*. In a stopband the fields of a wave decay exponentially away from a source of excitation. We shall discuss measurement techniques for obtaining the curves.

It is shown that the group velocity is not only the velocity of energy

transport through a lossless periodic structure but also the velocity of a low-frequency envelope of the carrier wave.

The propagation characteristics of several early lumped transmission filters are reviewed, as deduced from their resonant behaviors. Periodically lumped-loaded transmission lines, which prove so helpful in understanding periodic cavity chains, are then studied. It is shown how the qualitative shapes of the ω - β curves may be deduced from the properties of the resonant modes of a unit cell. Two basic methods of expanding the fields are illustrated: one utilizing traveling harmonics; the other, standing wave modes. The former are appropriate if the line is lightly loaded; the latter if it is heavily loaded.

The qualitative aspects of propagation along a periodically loaded transmission line also serve to describe a periodic cavity chain. The passband behaviors of a common centerhole-coupled chain are deduced from the properties of its resonant modes. It is pointed out that the fields can be expanded in either of two basic ways: in traveling wave form by means of the waveguide modes or in standing wave form by means of the resonant cavity modes. The Marcuvitz-Schwinger technique employs the waveguide modes and is very useful for studying the propagation when the periodic waveguide obstacles are small (light loading). In this technique we obtain equivalent circuit elements to represent the obstacles in the transmission line for the propagating guide mode. The periodic phase shift $\beta_0 L$ can then be calculated at any frequency from the equivalent circuit.

The reader is presumed to be familiar with the Introduction while reading this chapter.

1-2. The Periodicity Theorem¹

This theorem states we can express the fields of a wave in an infinitely long slow wave structure in a special form appropriate to the structural symmetry. The form indicates that the electric and magnetic field components of the wave phase shift by the factor $e^{-j\beta_0 p}$ from any point z to the point $z + p$, p being the periodic length. The Fourier-type expansions for the fields, as determined by symmetry, help us to identify the predominant field components and to obtain the fundamental propagation constant at the frequency of interest. Consider first the typical helix slow wave structure of Fig. 1-1. This tape helix has a radius a , pitch angle ψ defined as $\cot^{-1}(2\pi a/p)$, and periodic length p along the axial

¹ This theorem is a generalization of the Floquet theorem for differential equations with periodic coefficients. See Whittaker and Watson [1, Section 19.4].

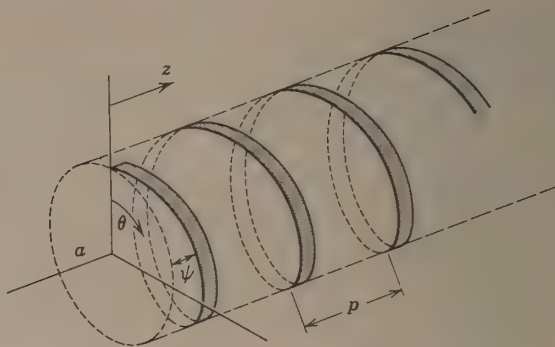


Fig. 1-1. A helix slow wave structure. p is the pitch, ψ the pitch angle, and a the radius.

or z direction. It turns out (Sections 2-2 and 2-3) that this helix will support one wave in which the predominant mode components travel with a phase velocity nearly that of light along the helix winding. Since ψ is normally a small angle in the range 5° – 20° , the axial phase velocity of the modes is slow enough to allow interaction with an electron beam. We observe immediately that the helix geometry remains invariant if we move from any point at $[r, \theta, z]$ to a new point $[r, \theta + (2\pi/p)z_1, z + z_1]$, z_1 being an arbitrary translation. Then any component $\hat{F}$ of electric or magnetic field at the second point must differ by only a complex factor from that component at the first point. Let us call this factor $a(z_1)$ and write

$$\hat{F}\left[r, \theta + \frac{2\pi}{p}z_1, z + z_1\right] = a(z_1)\hat{F}[r, \theta, z] \quad (1-1)$$

If we now move from the second point to a third point by translation z_2 and angle $(2\pi/p)z_2$ we can relate the new field component to that at the second point as

$$\hat{F}\left[r, \theta + \frac{2\pi}{p}(z_1 + z_2), z + z_1 + z_2\right] = a(z_2)a(z_1)\hat{F}[r, \theta, z] \quad (1-2)$$

However, the effect of both rotation-translations is equivalent to a single one through the axial distance $z_1 + z_2$ and angle $(2\pi/p)(z_1 + z_2)$, so we can equally well write

$$\hat{F}\left[r, \theta + \frac{2\pi}{p}(z_1 + z_2), z + z_1 + z_2\right] = a(z_2 + z_1)\hat{F}[r, \theta, z] \quad (1-2')$$

The only function for which $a(z_2)a(z_1) = a(z_2 + z_1)$ for all values of z_2

and z_1 is the exponential one. So we may write Eq. 1-1 as

$$\hat{F}\left[r, \theta + \frac{2\pi}{p} z_1, z + z_1\right] = \epsilon^{-j\beta_0 z_1} \hat{F}[r, \theta, z] \quad (1-3)$$

for all r, θ, z and any z_1 .

Equation 1-3 is a statement of the periodicity theorem for the helix, where β_0 is called the fundamental propagation constant of the structure. It is easy to see from Eq. 1-3 that a translation by a period of the structure, $z_1 = p$, does not change the θ dependence and merely multiplies the field component by $\epsilon^{-j\beta_0 p}$. For this reason $\beta_0 p$ is called the periodic phase shift. The minus sign preceding it means that if β_0 is a positive real number the wave moves in the $+z$ direction with time, since the time dependence is $\epsilon^{j\omega t}$. If β_0 had an imaginary component, as it does at some frequencies, the field would decay exponentially away from a source of excitation; in this case the wave cannot carry any real power in a lossless structure because there can be no divergence of real power within any periodic cell. Therefore β_0 is pure real if the wave carries real power in either the $+z$ or $-z$ direction. If, however, an electron beam or other active material were present, the symmetry considerations would still yield Eq. 1-3, but then β_0 could be complex, representing growth or decay of a wave amplitude in the z direction. In a lossless structure, which we always assume unless otherwise stated, any growth or decay of the circuit electromagnetic power in a wave must be accompanied by equal decay or growth of power flowing from the material.

For a lossless tape helix structure without energy sources let us write explicitly the form of the $\hat{E}_z$ -field, say, so as to introduce a mode expansion for $\hat{E}_z$. The $\hat{E}_z$ field generates, through Maxwell's equations, the TM (or transverse magnetic) portion of the helix field. We must expand $\hat{E}_z$ in a set of functions which can represent that field completely along the r and θ coordinates. It is natural to build up the θ representation with a (complete) set of $\epsilon^{jn\theta}$ functions. However, $\hat{E}_z[r, \theta, z]$ must remain constant as we move along the helix, except for the factor $\epsilon^{-j\beta_0 z}$, so let us write it as

$$\hat{E}_z[r, \theta, z] = \epsilon^{-j\beta_0 z} \sum_{m,n} A_{mn} R_m(r) \epsilon^{jn(\theta - 2\pi z/p)} \quad (1-4)$$

Here A_{mn} is a complex amplitude and $R_m(r)$ is the unspecified m th radial function. To find R_m we substitute Eq. 1-4 into the wave equation for $\hat{E}_z$:

$$(\nabla^2 + k^2)E_z = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \hat{E}_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} E_z + \frac{\partial^2}{\partial z^2} \hat{E}_z + k^2 \hat{E}_z = 0 \quad (1-5)$$

whereupon we find that m must equal n and that $R_n(r)$ obeys the

differential equation of the modified Bessel function.² We take $R_n(r) = A_n I_n(\sqrt{\beta_n^2 - k^2}r)$ inside the helix radius, $r \leq a$, and

$$R_n(r) = B_n K_n(\sqrt{\beta_n^2 - k^2}r)$$

outside the helix over the range $a \leq r \leq \infty$. If the helix were surrounded by an outer coaxial tube we would take a linear combination of I_n and K_n components so as to make $\hat{E}_{zn} = 0$ on the tube surface. The propagation constant for the n th mode, β_n , is defined as $\beta_0 + 2n\pi/p$ for obvious reasons. Now a slow wave contains all slow modes by implication, which means the phase velocity ω/β_0 of the $n = 0$ mode as well as $\omega/\beta_n = \omega(\beta_0 + 2n\pi/p)^{-1}$ of every n th mode is less than light velocity c . Light velocity $c = \omega/k$, k being the free-space propagation constant, all of which means that a slow mode will have a propagation constant $\beta_n > k$ and the arguments of the preceding Bessel functions will all be real.

Waves with fast mode components may also propagate in a closed slow wave structure which is surrounded by an outer tube; these modes have axial propagation constants less than k and radial propagation constants $(\beta_n^2 - k^2)^{1/2}$ which are imaginary. Fields of such modes are expressed by expansions of the form of Eq. 1-4 but with $R_n(r)$ given by the ordinary Bessel function $J_n(\sqrt{k^2 - \beta_n^2}r)$ inside the helix and by a linear combination of the J_n and N_n (Neumann) functions outside the helix.

If the helix is not surrounded by a metal cylinder we may write $\hat{E}_z$ in the form

$$\hat{E}_z[r, \theta, z] = \sum_{n=-\infty}^{\infty} \left\{ \frac{A_n I_n(\sqrt{\beta_n^2 - k^2}r)}{B_n K_n(\sqrt{\beta_n^2 - k^2}r)} \right\} e^{jn\theta - j\beta_n z}, \quad \beta_n = \beta_0 + \frac{2n\pi}{p} \quad (1-6)$$

according to previous remarks. The form of $\hat{H}_z$ which generates the TE or transverse electric portion of the helix field would be similar. These expressions represent a mode expansion of $\hat{E}_z$ or $\hat{H}_z$, where A_n is the amplitude of the n th TM mode inside the helix and B_n is the amplitude of the n th mode (really another mode) outside the helix. Each of these modes satisfies Maxwell's equations, Eqs. i and ii of the Introduction, identically, with unspecified relations between $\hat{E}_{\tan}$ and $\hat{H}_{\tan}$ on the boundary surface $r = a$. The relative amplitudes of all these modes in the wave are determined by the boundary conditions of continuous fields across the open surface at $r = a$ and zero tangential electric and normal magnetic field on the metal surface. The term β_0 in Eq. 1-6 is called the "fundamental" propagation constant, not because the $n = 0$ mode is necessarily

² See S. Ramo and J. R. Whinnery [2, the end of Section 3.24]. Their Sections 3.25-3.27 contain formulas for large arguments, derivatives, and recurrences. See also J. R. Pierce [3, Appendix 1] for relations among the I_n and K_n functions.

predominant in the wave but only because this propagation constant serves as a reference for all of them, through the relation $\beta_n = \beta_0 + 2n\pi/p$. Later we discuss waves in which the fundamental mode component is rather small by comparison with other components.

The propagation characteristics of any slow wave structure such as this helix are represented by a number of ω - β curves. Each one of these curves consists of a series of branches with the n th branch representing the frequency dependence of β_n belonging to the n th mode in the wave. Because all the β_n are related to β_0 according to Eq. 1-6 we can say the n th branch is merely the zero or ω - β_0 branch shifted along the β axis by $2n\pi/p$.

Now let us consider the symmetry of the infinitely long cavity-chain slow wave structure of Fig. 1-2. This structure is obviously symmetric with respect to translation by periodic length L and, unlike the helix, trivially symmetric with respect to rotation by any angle θ . Therefore we can represent the vector electric or magnetic field or one of its components as

$$\hat{F}(x, y, z + L) = e^{-j\beta_0 L} \hat{F}(x, y, z), \quad \text{any } x, y, z \quad (1-7)$$

analogous to Eq. 1-3 for the helix. Here $\beta_0 L$ is the periodic phase shift

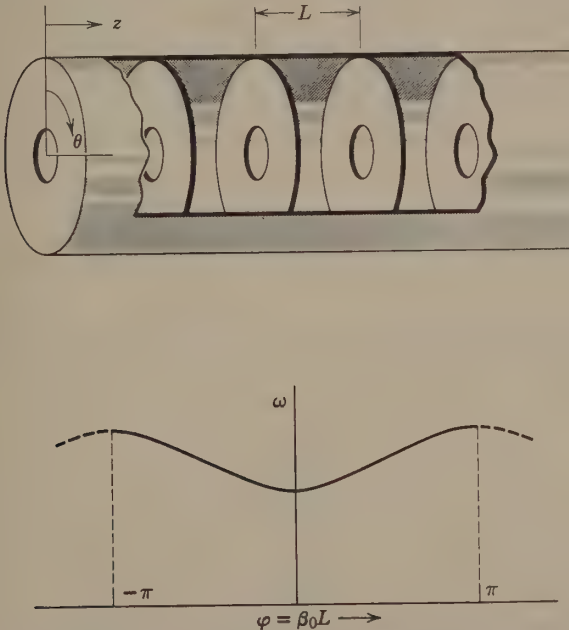


Fig. 1-2. A cavity-chain slow wave structure (from Pierce [3]) and its first dispersion curve. L is the periodic spacing of the disks.

per cavity. As in the helix, the transverse electric field, for example, may be expanded as

$$\hat{E}_t(x, y, z) = e^{-j\beta_0 z} \sum_{m,n} V_{mn} e^{-j2n\pi z/L} \bar{e}_m(x, y) \quad (1-8)$$

where the subscript m denotes the m th TM or TE waveguide mode and n denotes its axial n th harmonic. As explained in the Introduction, a harmonic is identified by the factor $e^{-j2n\pi z/L}$. These waveguide modes are defined in the course of discussing the Marcuvitz-Schwinger technique in Section 1-7.

Unlike the helix, the symmetry of this structure suggests that β_0 may be restricted to the range $-\pi/L \leq \beta_0 \leq \pi/L$, which is called the "first Brillouin zone" after that pioneer in this area of research.³ For if β_0 lay outside this range we could write it as $\beta_0' + 2k\pi/p$, where β_0' lies in the range, then absorb the $e^{-j2k\pi/p}$ factor into the summation, and renumber the harmonics. Therefore $\beta_0 L$ is redundant outside the range $-\pi \leq \beta_0 L \leq \pi$. Clearly the ω - β curve for this structure is constructed by shifting the ω - β_0 branch defined in the range $-\pi/L \leq \beta_0 \leq \pi/L$ along the β axis by $2n\pi/L$ for the n th branch.

It is true that the periodicity theorem, as embodied in Eq. 1-4 for the helix and Eq. 1-8 for the cavity chain, helps us to obtain the waves only for the infinitely long structure. If, however, we know the value of β_0 for each wave, and the proper linear combination of modes (or harmonics) composing each wave, we can specify completely the excitation in a finite structure. We merely choose the amplitudes of the various *waves* so as to match boundary conditions at both ends of the structure.

1-3. The Central Problem of Slow Wave Systems. Dispersion Curve Solutions

The problem we discuss throughout this book is that of determining at a particular frequency or frequencies solutions to Maxwell's equations

$$\nabla \times \hat{E} = -j\omega\mu \cdot \hat{H} \quad (1-9a)$$

$$\nabla \times \hat{H} = j\omega\epsilon \cdot \hat{E} + \hat{J} \quad (1-9b)$$

and the equations for the electron beam or other active material if such is present, subject to periodic boundary conditions in a structure like that of Fig. 1-1 or Fig. 1-2. We can state this problem from either of two

³ A classic work on slow wave structure propagation is the book by L. Brillouin [4]. It deals mostly with propagation in lumped-circuit electric filters and crystal lattices rather than microwave structures like distributed helices and cavity chains so we shall not refer to it very often.

viewpoints. Either we want solutions to the equations at a particular frequency, each solution being an independent wave with its own fundamental propagation constant, or we can seek the allowed values of frequency which belong to a given value of propagation constant.

Of course, the determinantal equation for ω in terms of β_0 is, in principle, the same whether ω is specified and β_0 is regarded as unknown or vice versa. For example, we can write the expressions for the helix fields as in Eq. 1-6 and ask for two values of β_0 at a particular frequency—one for a forward and one for a backward wave—as Sensiper [5] did, or we can work with a variational expression for frequency ω (Chapter 6) which adjusts the value of ω for an implicitly known value of β_0 . In the cavity chain of abrupt periodic discontinuities or in the general helix we shall find that it is often definitely advantageous to formulate the problem by taking the propagation constant β_0 as known and ω as to be determined. On the other hand, the properties of the various independent waves in a slow wave structure coupled to an electron beam may be conveniently developed from a variational expression for complex propagation constant Γ at a given ω (Chapter 8).

In solving Maxwell's Eqs. 1-9 subject to periodic boundary conditions we represent the fields by a linear superposition of modes, of either the propagating or the resonant variety. But it is interesting to observe in retrospect that the principle of superposition was seriously questioned at one time. Euler in 1748 analyzed the one-dimensional string vibrating in a plane and discovered the fact that the general form for displacement d is the traveling wave function $d(x \pm vt)$, v being the velocity. However, he could not believe that an arbitrary function of x could be represented by sines and cosines (Fourier did not prove his theorem until 1807). Therefore he rejected the principle of superposition! Nowadays we accept this concept readily enough and worry about whether the functions employed in the field expansions can represent satisfactorily the physical continuous field distribution in the sense of small mean-square error throughout a periodic cell. In periodic volumes of simple shape we can prove that all the waveguide mode-harmonics or a set of resonant cavity modes are complete for an arbitrary continuous field, except perhaps for certain field components on the bounding surface of the volume.

The solutions to Maxwell's equations in an infinitely long and empty structure (or one containing lossless permeability and permittivity) are displayed in the form of a series of ω - β or dispersion curves, several of which are shown in the latter portions of Figs. 1-3 through 1-6. The frequency range between the extremes of an ω - β curve is called a passband because β_0 is real and the fields carry real power through the structure. A frequency range between two passbands is a stopband,

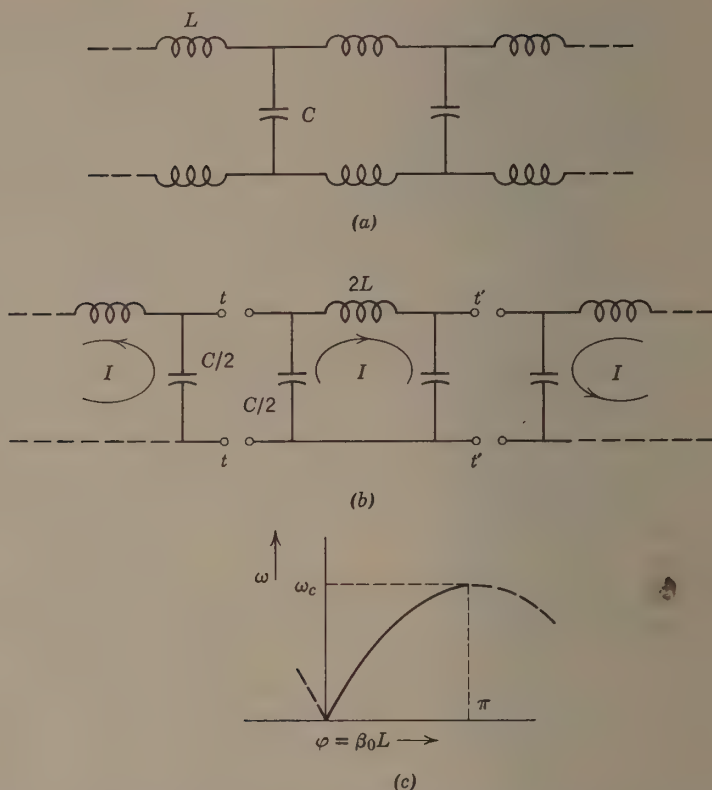


Fig. 1-3. A low-pass electric filter. (a) Physical filter, (b) currents and isolation of a periodic cell at π phase shift per period, (c) ω - ϕ curve.

within which no real power can flow because the fields decay exponentially away from a source of excitation. The phase velocity of the n th mode or harmonic is ω/β_n as we have already observed. The group velocity is $\partial\omega/\partial\beta_n$; it is the velocity of energy propagation through the structure. In the Introduction we pointed out the fact that the group velocity is the same for all components of a wave. It is discussed further in the next section.

An ω - β curve is measured experimentally by short-circuiting both ends of a finite length of the structure with low-loss metal planes so placed that they do not disturb the field configuration in the infinitely long structure. For example, the symmetry planes for the first passband of the Fig. 1-6 structure would bisect the two end cavities because the electric field there is axially directed both at the zero and at the π phase shift cutoff frequencies (Fig. 1-6b,c) and at all intermediate frequencies.

Observe, however, that the symmetry planes for the second passband bounded by the resonances of the modes of Fig. 1-6*d,e* would be a pair of *magnetic* shorting planes in the same positions where the electric field is transverse, and hence physically unrealizable. The ordinary tape helix of Fig. 1-1 is distinct in that no transverse electric or magnetic shorting plane can be inserted anywhere without disturbing either the electric or magnetic field; strictly speaking, the helix cannot be resonated as a cavity in order to measure the dispersion characteristics of the infinitely long structure.

It follows that if a structure has mirror symmetry about a transverse plane which bisects any cavity or periodic cell a finite length can be resonated between either electric or (hypothetical) magnetic shorting planes at a number of discrete frequencies in the passband under study. These frequencies correspond to periodic phase shifts of $0, \pi/N, 2\pi/N, \dots, N\pi/N$ if the section is N periods long. These resonances fall at equal intervals of $\beta_0 L = n\pi/N$ on the dispersion or $\omega - \beta_0$ curve for the passband.

1-4. Significance of the Group Velocity

We first want to prove that the group velocity $\partial\omega/\partial\beta$ of a wave is the velocity of energy transport through the structure, provided it is lossless, devoid of energy sources, and operated in a passband where β_0 is real. The proof begins with a statement of the energy theorem,⁴

$$\nabla \cdot \left(\hat{E}^* \times \frac{\partial \hat{H}}{\partial \omega} + \frac{\partial \hat{E}}{\partial \omega} \times \hat{H}^* \right) = -j(\epsilon \hat{E}^* \cdot \hat{E} + \mu \hat{H}^* \cdot \hat{H}) \quad (1-10)$$

where the asterisk denotes a complex conjugate. We now generalize Eq. 1-4 for the helix structure and Eq. 1-8 for the cavity chain by writing $\hat{E}$ and $\hat{H}$ in the form

$$\hat{E}(x, y, z) = e^{-j\beta_0 z} \hat{E}_p(x, y, z) \quad (1-11a)$$

$$\hat{H}(x, y, z) = e^{-j\beta_0 z} \hat{H}_p(x, y, z) \quad (1-11b)$$

in which $\hat{E}_p$ and $\hat{H}_p$ are periodic with axial period p or L . With these,

⁴ To derive this theorem we use Maxwell's equations 1-9 as follows. The first one is differentiated with respect to frequency and dot-multiplied by $\hat{H}^*$ (* denotes complex conjugate). It is also conjugated and dot-multiplied by $\partial \hat{H} / \partial \omega$. The second equation, with $\hat{J} = 0$, is first conjugated and dot-multiplied by $-\partial \hat{E} / \partial \omega$, and also differentiated with respect to ω and dot-multiplied by $-\hat{E}^*$. Upon adding these four equations and canceling terms we obtain the energy theorem.

the left side of Eq. 1-10 becomes

$$\nabla \cdot \left[\hat{E}_p^* \times \left(\frac{\partial \hat{H}_p}{\partial \omega} - jz \frac{\partial \beta_0}{\partial \omega} \hat{H}_p \right) + \left(\frac{\partial \hat{E}_p}{\partial \omega} - jz \frac{\partial \beta_0}{\partial \omega} \hat{E}_p \right) \times \hat{H}_p^* \right] \quad (1-12)$$

Integration of this over a periodic volume of length p , or L , yields

$$L \frac{\partial \beta_0}{\partial \omega} \int_A (\hat{E}_p^* \times \hat{H}_p + \hat{E}_p \times \hat{H}_p^*) \cdot \hat{i}_z dS = \int_V (\epsilon |E_p|^2 + \mu |\hat{H}_p|^2) dV \quad (1-13)$$

because $\hat{E}_p$ is perpendicular to the metal surfaces in the cross section. Here A denotes the cross-sectional area. But the total time-average power flow P_{av} in the axial or z direction is just

$$\frac{1}{4} \int_A (\hat{E}_p^* \times \hat{H}_p + \hat{E}_p \times \hat{H}_p^*) \cdot \hat{i}_z dS,$$

while the total time-average stored energy, W_s , in the volume is one-quarter of the right side. Therefore we obtain

$$P_{av} = \frac{W_s}{p} \frac{\partial \omega}{\partial \beta_0} \quad (1-14)$$

This expression says that the power flow can be regarded as the stored energy traveling with the group velocity and verifies the interpretation of the group velocity as an energy velocity.

Equation 1-14 means that, at a resonant frequency of a cavity-chain slow wave structure where a resonant mode is excited within each periodic cell such that no power flows, the slope of the ω - β curve must be zero. We use this fact in later discussion.

The group velocity has a further significance not apparent from the foregoing discussion, namely, as the velocity with which a low-frequency envelope of the carrier wave propagates. That is, if a modulating wave-form represented by

$$A[t, z = 0] = \int_{-\omega_1}^{+\omega_1} A(\omega) \epsilon^{j\omega t} d\omega, \quad \omega_1 \ll \omega_0, \text{ carrier frequency} \quad (1-15)$$

is impressed at $z = 0$ upon the carrier wave which propagates as $\epsilon^{j\omega t - j\beta_0 z}$ the envelope of the carrier at any point z will vary in time as

$$\begin{aligned} A[t, z] &= \int_{-\omega_1}^{\omega_1} A(\omega) \exp \left\{ j\omega \left[t - z \left(\frac{\partial \beta_0}{\partial \omega} \right)_{\omega_0} \right] \right\} d\omega \\ &= A \left[t - z \left(\frac{\partial \beta_0}{\partial \omega} \right)_{\omega_0}, z = 0 \right] \end{aligned} \quad (1-16)$$

Hence the envelope propagates with velocity $(\partial\omega/\partial\beta_0)_{\omega_0}$, the group velocity.⁵ We might say the group velocity is the velocity with which information in the form of amplitude modulation travels through the structure.

1-5. Early Transmission Filters

We first study several filters each with only one nonzero resonant frequency in order to become familiar with the very useful method of specifying the shapes of the dispersion curves from knowledge of the resonant frequencies. It is convenient to abbreviate the periodic phase shift $\beta_0 L$ by φ , and talk about an ω - φ curve instead of an ω - β_0 curve. In 1900, M. Pupin referred to Lagrange's analysis of the propagation on a discontinuous string and built, by analogy, the low-pass electric filter shown in Fig. 1-3. We shall analyze this line as if it had no distributed transmission line inductance and capacitance per unit length.

Quite obviously the line will propagate at low frequencies because ωL approaches zero at d.c. frequency. As frequency increases the wave is slowed until the structure resonates with a π phase shift of current between successive cavities, Fig. 1-3*b*. The phase shift must be π at resonance or no net current would flow through C and we would not have a resonance. Clearly, no current then flows across terminal pairs t and t' , so we can open them and compute the cutoff frequency as $\omega_c = (2LC/4)^{-1/2}$. The slope of the first passband must be as shown in Fig. 1-3*c*. At no frequency above ω_c will the structure again resonate, so a stopband extends from ω_c to ∞ .

In 1906, Campbell built the first high-pass electric filter shown in Fig. 1-4. Again we neglect the distributed inductance and capacitance. This structure clearly propagates at high frequencies, down to the only resonant frequency ω_c of Fig. 1-4*b*. This frequency must be a π phase shift resonance or no net voltage would appear across each inductance L' . Therefore we can open the terminal pairs t and t' and compute the cutoff frequency as $\omega_c = (4L'C')^{-1/2}$. The ω - φ curve can only have the slope shown in Fig. 1-4*c* since no other resonances exist and the series reactance is ∞ at $\omega = 0$, 0 at $\omega = \infty$.

This procedure of first determining the resonant frequencies and then sketching the ω - φ curves between them is illustrated further in the next two sections.

⁵ J. R. Pierce [3, p. 81].

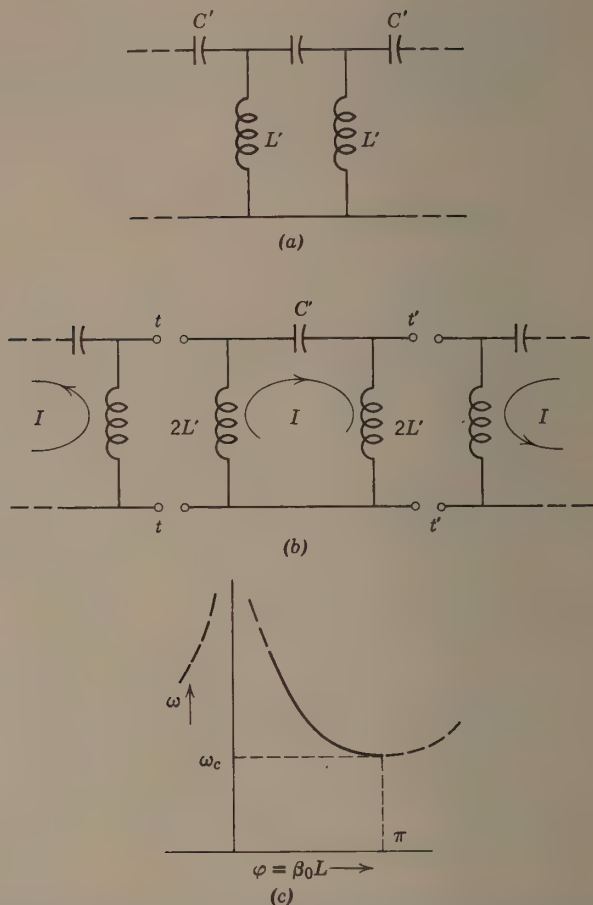


Fig. 1-4. A high-pass electric filter. (a) Physical structure, (b) currents and isolation of a periodic cell at π phase shift per period, (c) ω - ϕ curve.

1-6. The Periodically Lumped-Loaded Transmission Line

A study of a periodically loaded TEM (transverse electric and magnetic) structure is informative because it illustrates all the important properties of the cavity chain (Fig. 1-6). First we illustrate the determination of the first few resonant frequencies in order to predict the qualitative shapes of the ω - ϕ curves. Then we discuss the two basic ways of representing the fields within a given unit cell.

Consider the structure shown in Fig. 1-5a, consisting of a distributed transmission line loaded periodically by capacitors $2C_L$. Each capacitor

has been bisected so as to associate C_L with one cell and C_L with the adjacent cell. Voltage $V(z)$ and current $I(z)$ satisfy the equations

$$\frac{dV}{dz} = -\omega L(jI) \quad (1-17a)$$

$$\frac{d(jI)}{dz} = \omega C(z)V, \quad \int_{\text{acr } C_L} C(z) dz = C_L \quad (1-17b)$$

At low frequencies this line behaves qualitatively like the low-pass electric filter of Fig. 1-3. The field configurations at $\omega = 0$ are illustrated in Fig. 1-5b; the amplitudes V and jI are plotted relative to the upper horizontal line as the zero. No voltage exists and the current is continuous across the cell boundaries at $z = 0, L$ so we have short circuited those boundaries. This zero phase shift cell mode should be regarded as a degenerate resonant mode. As frequency increases the structure reaches the next resonance at ω_1 , for which the true fields are sketched in Fig. 1-5c. Note the rapid—actually discontinuous—change of current across C_L . This is clearly a π phase shift resonance and is so labeled on the ω - φ curve; we could open the terminals at $z = 0, L$ and define the resonance within one cell alone. At the next resonant frequency ω_2 of the line the fields vary as shown in Fig. 1-5d. This is the half-wave short-circuit resonance with a π phase shift between cells. Hence we have placed the appropriate dot on Fig. 1-5f. The third finite resonance at ω_3 is sketched in Fig. 1-5e; by opening the terminals at $z = 0, L$ we see that it is a zero phase shift, open-circuit cell resonance.

Two questions now arise: Does every ω - φ curve of this transmission line have zero slope at a frequency corresponding either to zero or to π phase shift, and is every zero or π phase shift frequency also a resonant frequency of a cell? The answer is yes to both questions. In the first place, the structure is symmetrical in both $\pm z$ directions so that both $\pm\varphi(\omega)$ belong to a frequency ω . And the two field harmonics with the propagation constants $(\mp\varphi \pm 2\pi)/L$ belong to the same frequency by the properties of Eq. 1-8. It follows that each ω - φ curve is symmetric around both $\varphi = 0$ and π and must therefore have zero slope there (except at $\omega = 0$). In the second place, a zero slope implies no power flow along the line, as demonstrated in the preceding section. No power flow means a resonance condition in each cell.

We see, then, that to determine the ω - φ curves qualitatively in the Fig. 1-5a structure we have only to obtain all the open- and short-circuit resonances of a unit cell, plot them as zero or π phase shift resonances on an ω - φ diagram and connect the points so as to form alternate passbands and stopbands. Nevertheless knowing how to sketch the curves

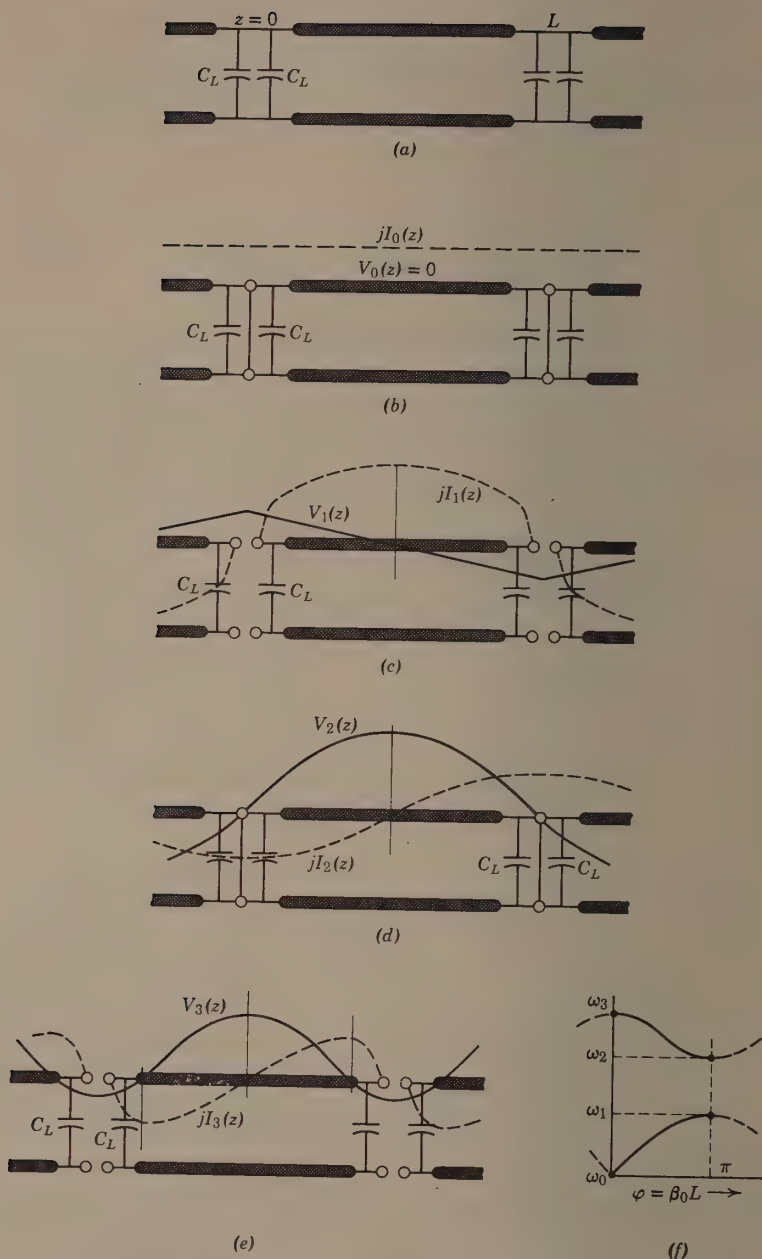


Fig. 1-5. A periodically lumped-loaded TEM line. (a) Physical line, (b) current and voltage at the d.c. "resonant" frequency ω_0 at which the cell boundaries are effectively shorted at $z = 0, L$; (c) excitations at the first nonzero resonant frequency ω_1 , at which the cell boundaries are effectively opened at $z = 0, L$; (d) excitations at resonant frequency ω_2 with the boundaries effectively shorted; (e) excitations at resonant frequency ω_3 with the boundaries effectively open-circuited; (f) dispersion curves between these resonant frequencies.

does not tell us the best way in which to represent the voltage and current and proceed to calculate the curves in detail. This problem is, of course, elementary for the periodic lumped-loaded transmission line; we have only to start from an initially unknown voltage and current at any point in the line, work back across $2C_L$ and the unloaded line to the voltage and current a period away, and then relate the two voltages and currents by the factor $\epsilon^{j\varphi}$ (see Slater [6]). However, let us consider two indirect but basic methods of representing voltage $V(z)$, say, which will suggest ways of expanding the fields in a cavity chain.

One extreme case is that of very light capacitive loading in Fig. 1-5: $C_L \ll C_0L$, where C_0 is the distributed capacitance per unit length. The function $V(z)$ will then have a strong fundamental component of traveling wave and relatively weak components of spacial harmonics at frequencies within the first passband. A useful representation of $V(z)$ is, in this case,

$$V(z) = V_0 \epsilon^{-j\beta_0 z} + \sum_{n \neq 0} V_n \epsilon^{-j(\beta_0 + 2n\pi/L)z} \quad (1-18)$$

in which amplitude $V_0 \gg V_n$ and β_0 is very nearly the propagation constant $\omega/c = k$ of the unloaded line. We are assuming here that the frequency is not too near the π cutoff resonance ω_1 of Fig. 1-5f; otherwise both V_0 and V_{-1} components would predominate ($\beta_0 \simeq \pi/L$) in the standing wave or near-standing wave pattern resembling Fig. 1-5c. This expansion is rapidly convergent for the voltage in a lightly loaded line and is complete as a Fourier series.

The other extreme case of loading is specified by very large shunt capacitors: $C_L \gg C_0L$. The boundaries of the unit cell of Fig. 1-5a are now more completely short circuited so that the passbands are relatively narrow. A better expansion for $V(z)$ within one unit cell is now

$$V(z) = \sum_{n=1}^{\infty} V_n \sin \frac{(n-1)\pi z}{L} \quad (1-19)$$

in which only one V_n is predominant at a frequency in the n th passband, for $L \simeq (n-1)\lambda/2$ where λ is the unloaded line wavelength. For example, Fig. 1-5d,e shows $V(z)$ as approximately a half-sinusoid in the second passband ($n=2$). To obtain the voltage $V(z+L)$ in the next cell we need only multiply Eq. 1-19 by $\epsilon^{-j\varphi}$, where φ is to be determined. Equation 1-19 is also complete for the voltage *within* a unit cell but fails utterly to represent any voltage *on* the boundaries at $z=0, L$. To represent boundary voltage we may consider a second expansion

$$V(z) = \sum_{n=1} V_n' \cos \frac{(n-1)\pi z}{L} \quad (1-20)$$

which is complete for $V(z)$ within the cell *and* on the boundaries.

The point of this discussion is simply that we are free to expand the voltage $V(z)$ within a unit cell of the line *either* in terms of the short-circuit modes of Fig. 1-5*b,d*, for example, *or* the open circuit modes shown in Fig. 1-5*c,e*. Although the latter do not vary strictly as cosine functions it can be proved by a variational argument⁶ that the set of open-circuit resonant cell modes, as well as the set of short-circuit modes, is complete within the cell. The former set is complete for the voltage on those boundaries and the latter is complete for the derivative of voltage, that is, current on the boundaries.

1-7. The Periodic Cavity Chain

A. General Properties

A side view of the very common centerhole-coupled chain of Fig. 1-2 appears in Fig. 1-6*a*. This cylindrical structure is the usual one employed for linear accelerator work. We shall first determine the shapes of the first two passbands from the resonant frequencies of the cavity modes, just as for the periodically lumped-loaded line. Then we discuss the two basic ways of expanding the fields within a cavity or periodic cell.

We know from the remarks of the preceding section that each ω - φ curve, $\varphi = \beta_0 L$, has a zero slope at the $\varphi = 0$ and π phase shift frequencies where the structure resonates. So we can determine the cutoff frequencies as resonant frequencies of a single cavity. The resonance of lowest frequency is that of the "straight-through" mode of Fig. 1-6*b*, resonating between holes which may be electrically shorted without disturbing the fields. In defining this and other resonances it proves convenient to represent the electric and magnetic fields of the mode in the forms $V_1 \bar{E}_1$ and $I_1 \bar{H}_1$, respectively.⁷ Here V_1 is the scalar *amplitude* of the electric field and $\bar{E}_1$ is its vector field *pattern* as denoted by the bar; I_1 is the scalar amplitude, and $\bar{H}_1$ is the vector pattern of the magnetic field. With this convention, Maxwell's equations for the first resonant mode may be split into a vector and a scalar portion,

$$\nabla \times \bar{E}_1 = P_1 \bar{H}_1, \quad P_1 = \omega_1 \sqrt{\mu \epsilon} \quad (1-21a)$$

$$\nabla \times \bar{H}_1 = P_1 \bar{E}_1, \quad \bar{E}_1 \times \bar{n} = 0 \quad \text{on walls and coupling holes}$$

$$P_1 V_1 = -j\omega_1 \mu I_1, \quad P_1 I_1 = j\omega_1 \epsilon V_1 \quad (1-21b)$$

⁶ See P. M. Morse and H. Feshbach [7, p. 738].

⁷ See E. V. Condon [8].

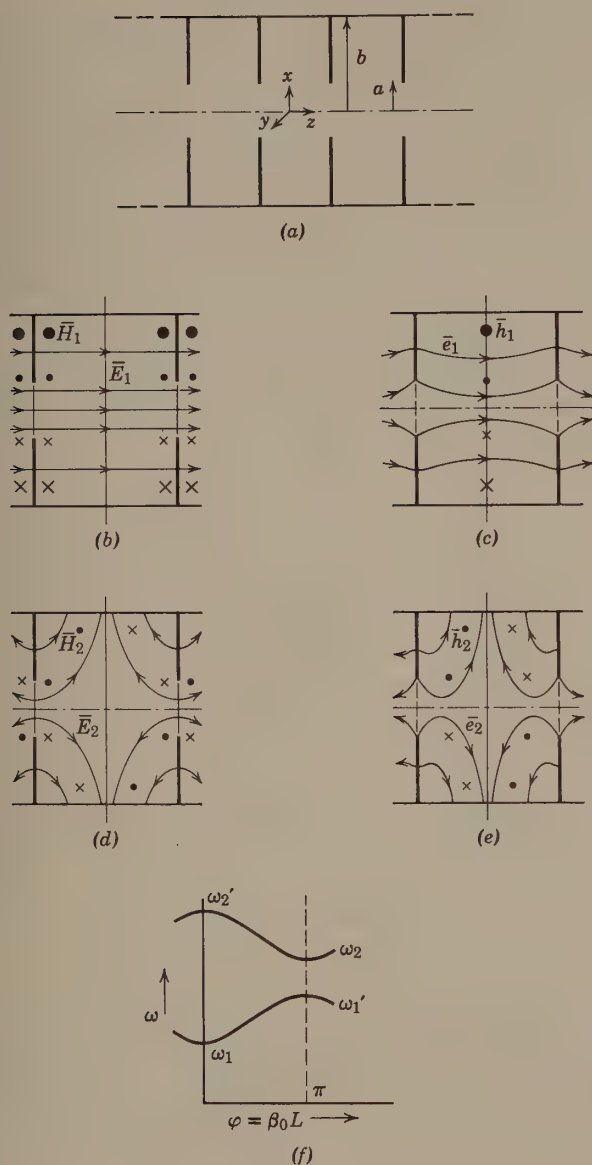


Fig. 1-6. A centerhole-coupled cavity chain. (a) A side view of the structure, (b) mode patterns $\bar{E}_1$ and $\bar{H}_1$ at the first resonant frequency ω_1 at which the centerholes are effectively electrically shorted, (c) mode patterns $\bar{e}_1$ and $\bar{h}_1$ at the second resonant frequency ω'_1 at which the centerholes are effectively magnetically shorted, (d) patterns of the short-circuited resonant cavity mode at frequency ω_2 , (e) patterns of the open-circuit resonant cavity mode at ω'_2 , (f) dispersion curves between these resonant frequencies.

Only the real patterns $\bar{E}_1$ and $\bar{H}_1$ are sketched in Fig. 1-6b, at the first resonant frequency ω_1 .

Having defined the first (short-circuit) resonant mode within one cavity we now wish to represent the fields with it throughout the chain. We always adopt the convention of repeating all patterns from cavity to cavity with the same orientations, as shown by the lines in the neighboring cavities of Fig. 1-6. Observe that some of the lines oppose others on the centerhole surfaces, in Fig. 1-6c for example. We shall account for such anomalies in due course. In order to satisfy the periodicity condition we shall have to phase shift the *amplitudes* by $e^{-j\varphi}$ from one cavity to the next. It follows that this "straight-through" mode will represent a true field which is continuous across the coupling holes only if phase shift φ happens to be zero. Hence ω_1 is plotted at $\varphi = 0$ in Fig. 1-6f.

Let us now increase the frequency to the next resonance, which happens to occur for magnetic shorting planes on the holes, Fig. 1-6c. By definition, such a mode has tangential electric and normal magnetic field on each hole and is entirely analogous to the open-circuit mode of Fig. 1-5c. We call it an open-circuit resonant cavity mode and denote the electric field portion by $v_1\bar{e}_1$ and the magnetic field portion by $i_1\bar{h}_1$.⁸ Maxwell's equations for the mode are very similar to Eq. 1-21:

$$\nabla \times \bar{e}_1 = p_1\bar{h}_1, \quad p_1 = \omega_1'\sqrt{\mu\epsilon} \quad (1-22a)$$

$$\nabla \times \bar{h}_1 = p_1\bar{e}_1, \quad \bar{e}_1 \times \bar{n} = 0 \text{ on walls; } \bar{e}_1 \cdot \bar{n} = 0 \text{ on coupling holes}$$

$$p_1v_1 = -j\omega_1'\mu i_1, \quad p_1i_1 = j\omega_1'\epsilon v_1 \quad (1-22b)$$

We repeat the patterns from cavity to cavity as indicated in Fig. 1-6c and observe that they reverse directions upon crossing a hole. Therefore the total fields $v_1\bar{e}_1$ and $i_1\bar{h}_1$ will be continuous at their resonant frequency only if amplitudes v_1 and i_1 also change sign across a hole. In other words, this mode represents a true field only at frequency ω_1' with π phase shift of amplitudes per cavity. Hence ω_1' is plotted at $\varphi = \pi$ in Fig. 1-6f.

The next two resonant cavity modes are shown in Fig. 1-6d,e; the first is a $\varphi = \pi$ mode resonating at frequency ω_2 and the second is a $\varphi = 0$ mode resonating at $\omega_2' > \omega_2$. Knowledge of the first four resonant frequencies of the chain, plus the fact that the structure cannot propagate at very low frequencies, enables us to sketch the ω - φ curves as shown in Fig. 1-6f.

⁸ Where resonant cavity modes are concerned, capital letters will always denote short-circuit modes which resonate within a cavity or periodic cell which is closed by electric shorting planes on the coupling surfaces. Small letters will always denote the patterns and amplitudes of open-circuit modes which resonate within a cavity closed by *magnetic* shorting planes on the coupling surfaces.

Such considerations do not tell us, however, the best manner of representing the fields in order to simplify the solution of Maxwell's equations for the precise shapes of the dispersion curves. We realize from the discussion of the periodically loaded transmission line that we can represent the fields in basically two different ways, one of them very efficient for small fins (heavy coupling between cavities) and the other more efficient for large fins (light coupling). In order to contrast these two basic field representations we shall first outline the Marcuvitz-Schwinger technique [9] for studying the propagation of one or two waveguide modes when small obstacles or fins are present, at frequencies such that the rest of the waveguide modes are cut off (decaying exponentially with distance). We use this technique to determine equivalent circuit elements which represent the obstacle or fin in the equivalent transmission line for each propagating mode.

B. Analysis by the Marcuvitz-Schwinger Technique Using Waveguide Modes⁹

Marcuvitz and Schwinger assert that we may solve Maxwell's equations in a waveguide loaded periodically by fins or obstacles by employing the waveguide modes defined *without* the fins or obstacles but excited by appropriate surface currents. We must therefore define a complete set of TM and TE waveguide modes. The i th such TM or E mode in a waveguide with constant cross section and uniform parameters μ and ϵ is generated from a scalar function $\phi_i(x, y)$ defined in a cross-sectional plane bounded by surface S of the guide:

$$\begin{aligned} \nabla_T^2 \phi_i(x, y) + k_i'^2 \phi_i &= 0 & \phi_i &= 0 \text{ on } S \text{ if } k_i' \neq 0 \\ \nabla_T \phi_0 \times \bar{n} &= 0 \text{ on } S & \text{if } k_0' &= 0, \text{ TEM mode} \end{aligned} \quad (1-23a)$$

Here $\bar{n}$ is the normal direction out of the waveguide cross section, $\nabla_T = \bar{i}_x \partial / \partial x + \bar{i}_y \partial / \partial y$ in the cross section, and k_i' is the wavenumber of the normalized cutoff frequency. The field *patterns* of the mode are obtained as¹⁰

$$\begin{aligned} \bar{e}_i(x, y) &= -\nabla_T \phi_i(x, y) \\ \bar{h}_i(x, y) &= \bar{i}_z \times \bar{e}_i, \quad \text{with } \bar{i}_z \text{ the unit axial vector} \end{aligned} \quad (1-23b)$$

⁹ This section can be omitted or scanned in the first reading. The first portion is a review of the derivation of TM and TE waveguide modes.

¹⁰ Marcuvitz and Schwinger put an additional factor k_i' or k_i'' on the right side of Eqs. 1-23b and 1-25b. This somewhat obscures the interpretation of ϕ_i and ψ_i as potentials.

We define these convenient parameters for the mode

$$\kappa_i' = +(k^2 - k_i'^2)^{1/2}, \quad Y_i' = \frac{1}{Z_i'} = \frac{\omega \epsilon}{\kappa_i'}; \quad k^2 = \omega^2 \mu \epsilon \quad (1-24)$$

The i th TE or H mode is generated from a scalar potential function $\psi_i(x, y)$ defined in the cross-sectional plane by

$$\nabla_T^2 \psi_i(x, y) + k_i''^2 \psi_i = 0, \quad \frac{\partial \psi_i}{\partial n} = 0 \text{ on } S \quad (1-25a)$$

Here k_i'' is the normalized cutoff frequency of this mode. The *patterns* are

$$\begin{aligned} \bar{e}_i(x, y) &= -\bar{i}_z \times \bar{h}_i(x, y) \\ \bar{h}_i(x, y) &= -\nabla_T \psi_i \end{aligned} \quad (1-25b)$$

Convenient parameters for this TE mode are¹¹

$$\kappa_i'' = +(k^2 - k_i''^2)^{1/2}, \quad Z_i'' = \frac{1}{Y_i''} = \frac{\omega \mu}{\kappa_i''} \quad (1-26)$$

The transverse waveguide fields may be expanded with these modes as

$$\begin{aligned} \bar{E}_i(x, y, z) &= \sum_n V_n(z) \bar{e}_n(x, y) \\ \bar{H}_i(x, y, z) &= \sum_n I_n(z) \bar{h}_n(x, y) \end{aligned} \quad \begin{array}{l} \text{including both TM and TE modes} \\ (1-27a) \end{array}$$

whereupon Maxwell's equations determine the axial fields as

$$\begin{aligned} j\omega \epsilon \bar{E}_z(x, y, z) &= \bar{i}_z \sum_n (k_n')^2 I_n(z) \phi_n(x, y) \\ j\omega \mu \bar{H}_z(x, y, z) &= \bar{i}_z \sum_n (k_n'')^2 V_n(z) \psi_n(x, y). \end{aligned} \quad (1-27b)$$

The problem now is to determine amplitudes $V_n(z)$ and $I_n(z)$ when obstacles or apertures exist within the waveguide or on its surface. Marcuvitz and Schwinger show that the driven transmission line equations for V_n and I_n of either the n th TE or TM mode are

$$\begin{aligned} \frac{dV_n(z)}{dz} &= -j\kappa_n Z_n I_n(z) - v_n(z) \\ \frac{dI_n(z)}{dz} &= -j\kappa_n Y_n V_n(z) - i_n(z) \end{aligned} \quad (1-28)$$

in which the appropriate κ_n and $Z_n = I/Y_n$ are taken according to whether the mode is TE or TM. The excitations $v_n(z)$ and $i_n(z)$ may be written in

¹¹ If normalized operating frequency k is less than k_i'' , κ_i'' is written $-j(k_i''^2 - k^2)^{1/2}$, so that the cutoff mode decays as $e^{j\omega t - \alpha z}$ with $\alpha = +(k_i''^2 - k^2)^{1/2}$. Similar remarks apply if $k < k_i'$ of the TM mode.

the completely general forms

$$\begin{aligned} v_n(z) &= \oint_C \bar{n} \times \hat{E}(\bar{\rho}, z) \cdot \bar{h}_n(\bar{\rho}) dC - Z_n \oint_C \hat{H}(\bar{\rho}, z) \times \bar{n} \cdot \bar{e}_{zn}(\bar{\rho}) dC \\ i_n(z) &= \oint_C \hat{H}(\bar{\rho}, z) \times \bar{n} \cdot \bar{e}_{zn}(\bar{\rho}) dC - Y_n \oint_C \bar{n} \times \hat{E}(\bar{\rho}, z) \cdot \bar{h}_{zn}(\bar{\rho}) dC \quad (1-29) \end{aligned}$$

in which $Z_n \bar{e}_{zn} = \bar{i}_z (k_n')^2 \phi_n / j\omega\epsilon$; $Y_n \bar{h}_{zn} = \bar{i}_z (k_n'')^2 \psi_n / j\omega\mu$. (Note the definitions of $\bar{e}_i$ and $\bar{h}_i$ in Eq. 1-23b and 1-25b.) Here $\bar{\rho}$ is the transverse position vector in the cross-sectional plane at z , and $\bar{n}$ and contour C are defined in Fig. 1-7. Contour C includes not only the intersection curve of a cross-sectional plane with the waveguide boundary, which may include the aperture shown, but also the intersection curve(s) of this plane with the obstacle(s) within.

Equations 1-28 and 1-29 may be plausibly obtained as follows. Dot-multiply the first Maxwell equation by $\bar{h}_n$, thus

$$\bar{h}_n \cdot \left(\nabla_T \times \hat{E} + \bar{i}_z \times \frac{\partial \hat{E}}{\partial z} + j\omega\mu \hat{H} \right) = 0 \quad (1-30a)$$

and add to it the second Maxwell equation dot-multiplied by $\bar{i}_z Z_n e_{zn}$, as well as the expressions

$$\begin{aligned} -\hat{E} \cdot [\nabla_T \times \bar{h}_n - \bar{i}_z j\omega\epsilon Z_n e_{zn}] &= 0 \\ -\hat{H} \cdot \left[\nabla_T \times (\bar{i}_z Z_n e_{zn}) - \frac{k_n'^2 \bar{h}_n}{j\omega\epsilon} \right] &= 0 \end{aligned} \quad (1-30b)$$

Upon integration of these expressions over the open cross section bounded

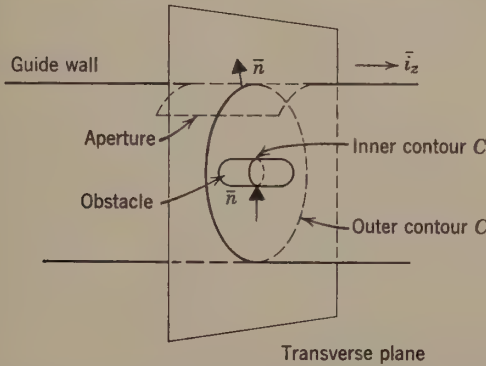


Fig. 1-7. Section of waveguide with an obstacle and aperture. The transverse plane intersects the waveguide wall and obstacle on contour C , with $\bar{n}$ the unit normal pointing out of the guide region.

by the C contours and utilization of the orthogonality properties of the modes,

$$\int_A \bar{e}_n \times \bar{h}_m \cdot \bar{i}_z dA = 0 \text{ if } n \neq m, \quad 1 \text{ if } n = m \text{ (normalization)} \quad (1-31)$$

we obtain Eq. 1-28a. Its mate is obtained analogously.

To obtain the Marcuvitz-Schwinger form of Eq. 1-29 we first realize that on an *obstacle* $\hat{E}_{\tan} \equiv 0$, even though a finite number of waveguide modes may not make this exactly so, whereupon the $\bar{n} \times \hat{E}$ integrals vanish in Eq. 1-29. On an *aperture in the wall* both tangential $\hat{E}$ and $\hat{H}$ exist, in general, as established by external excitation. But the waveguide modes are conveniently defined by the preceding equations with the aperture electrically shorted, so that $\bar{e}_{i, \tan} = \bar{e}_{iz} = 0$ on it. It follows that the $\hat{H}$ integrals in Eq. 1-29 are nil on an aperture. We therefore obtain the Marcuvitz-Schwinger result,

$$\begin{aligned} v_n(z) &= \oint_{\text{ap}} \bar{n} \times \hat{E}(\bar{\rho}, z) \cdot \bar{h}_n(\bar{\rho}) dC - Z_n \oint_{\text{obs}} \hat{H}(\bar{\rho}, z) \times \bar{n} \cdot \bar{e}_{zn}(\bar{\rho}) dC \\ i_n(z) &= \oint_{\text{obs}} \hat{H}(\bar{\rho}, z) \times \bar{n} \cdot \bar{e}_n(\bar{\rho}) dC - Y_n \oint_{\text{ap}} \bar{n} \times \hat{E}(\bar{\rho}, z) \cdot \bar{h}_{zn}(\bar{\rho}) dC \end{aligned} \quad (1-32)$$

Now let us apply the technique to a description of the propagation in a fin-loaded waveguide, for example. Consider the guide of Fig. 1-8a at a frequency such that only the TE_{10} mode propagates past the infinitesimally thin fins. We can represent the behavior of its amplitudes $V_1(z)$ and $I_1(z)$ by Eq. 1-28 with $n = 1$. The v and i generators are given by Eq. 1-29, in which contour C may be understood more clearly with reference to the distorted fin of Fig. 1-8b. As the sides of the fin steepen, the C contours “bunch” together. On the fins (obstacles) $\hat{E}_{\tan} \equiv 0$ and $\bar{n}$ becomes $\pm \bar{i}_z$, so that $v_1(z) = 0$ and $i_1(z)$ is, on the left side of the fin,

$$i_1(z < 0) = \delta(z) \int_{\text{fin}} \hat{H}(\bar{\rho}, 0) \times \bar{i}_z \cdot \bar{e}_1(\bar{\rho}) dS \quad (1-33)$$

Here δ is the Dirac delta function. But this current generator will be canceled by $i_1(z > 0)$ on the other side of the fin, leaving no net excitation of the TE_{10} mode unless we account for the fact that the TE mode component of $\hat{H}$ does have a finite change of phase across the fin surface.

At this point the complexion of the problem changes because of the necessity of introducing the phase shift of the TE_{10} mode amplitudes across the fin. The simplest way to do this is by exciting a section of waveguide which contains only one fin, first by antisymmetric excitation, depicted in Fig. 1-8c, and then by symmetric excitation depicted in Fig. 1-8d.

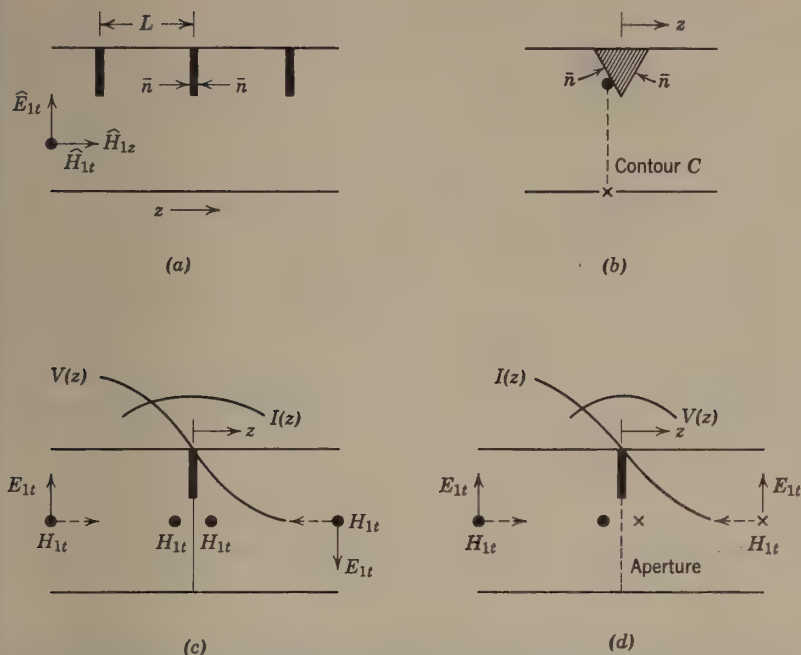


Fig. 1-8. TE_{10} mode propagation in a rectangular waveguide. (a) Physical structure, showing the transverse and z -directed fields, (b) geometry of a distorted fin, (c) excitations and boundary condition at $z = 0$ for antisymmetrical excitation, (d) excitations and fields on the aperture for symmetrical excitation.

The reference terminal pairs may be considered to be any number of half guide wavelengths away from the fin. Solution of the equations for these distinctly different boundary-value problems will yield the equivalent circuit elements shown in Fig. 1-9 for this propagating guide mode. These equivalent circuit elements enable us to compute the periodic phase shift $\varphi = \beta_0 L$ at any frequency, in a range to be specified in a moment.

First we apply the antisymmetric excitation shown in Fig. 1-8c, such that $\hat{E}_{1t}(z) = -\hat{E}_{1t}(-z)$ and therefore $\hat{E}_{1t} = 0$ at $z = 0$. The surface at $z = 0$ has therefore been electrically shorted as shown, and this corresponds to a short of the equivalent circuit of Fig. 1-9a so as to yield Fig. 1-9b. It is very convenient to close surfaces this way, because now we need only consider the $z < 0$ region of Fig. 1-8c in order to determine element Z_1 in Fig. 1-9b.

Now let an incident TE_{10} mode with amplitudes V_1^i and I_1^i travel in the $+z$ direction and impinge on the $z = 0$ plane. Obviously the boundary conditions are satisfied by a reflected mode with amplitudes $V_1^r = -V_1^i$ and $I_1^r = I_1^i$ such that the total electric field $(V_1^i + V_1^r)\bar{e}_{1t} \equiv 0$ at $z = 0$.

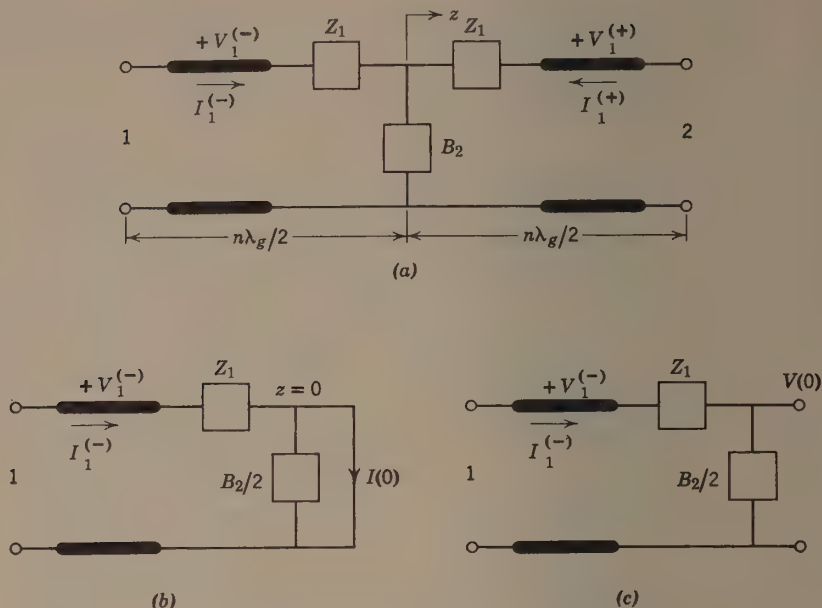


Fig. 1-9. Equivalent circuit for the effect of the fins on the TE_{10} mode in Fig. 1-8. (a) The circuit between terminal pairs 1 and 2 a distance $n\lambda_g/2$ from a fin, (b) the circuit from terminal pair 1 when antisymmetrical excitation is applied, (c) the circuit from terminal pair 1 when symmetrical excitation is applied.

We can verify this conclusion from the transmission line equations for all the modes, Eqs. 1-28 and 1-33. We interpret these equations at the end of a waveguide, $z = 0^-$, by integrating them through an infinitesimal distance to $z = 0$ (see Prob. 7). The result is

$$\begin{aligned} V_n(0^-) &= V_n^i + V_n^r = 0 \\ I_n(0^-) &= I_n^i + I_n^r = + \int_{z=0} \hat{H}(\bar{\rho}, 0^-) \times \bar{i}_z \cdot \bar{e}_n(\bar{\rho}) dS \end{aligned} \quad (1-34)$$

These are both satisfied by $V_1^i = -V_1^r$ and $I_1^i = I_1^r$, with $\hat{H}(\bar{\rho}, 0^-) = (I_1^i + I_1^r)\bar{h}_1(\bar{\rho})$, because $\oint \bar{h}_1 \times \bar{i}_z \cdot \bar{e}_n dS = \delta_{1n}$ by the orthogonality properties. As a result, we may deduce that the equivalent circuit element Z_1 is zero in Fig. 1-9b for this infinitesimally thin fin.

To find the circuit element B_2 we apply the symmetrical excitation $V_1(z) = +V_1(-z)$ shown in Fig. 1-8d, for which $\hat{H}_{1t}(z) = -\hat{H}_{1t}(-z)$, $\hat{H}_{1t}(0) = 0$, and the aperture at $z = 0$ is effectively magnetically shorted. The equivalent circuit is open-circuited at $z = 0$ so as to yield Fig. 1-9c for the $z < 0$ region. To determine element B_2 from the transmission line equations we must first interpret Eqs. 1-28 and 1-29 at $z = 0^-$. By

integrating Eq. 1-28 through an infinitesimal distance to $z = 0$ we obtain, with $\vec{n} = +\vec{i}_z$, $\vec{n} \times \vec{E} = 0$ on the fin and $\vec{n} \times \vec{H} = 0$ on the aperture,

$$\begin{aligned} V_n(0-) &= + \int_{\text{ap}} \vec{i}_z \times \vec{E}(\vec{\rho}, 0-) \cdot \vec{h}_n(\vec{\rho}) dS \\ I_n(0-) &= + \int_{\text{fin}} \vec{H}(\vec{\rho}, 0-) \times \vec{i}_z \cdot \vec{e}_n(\vec{\rho}) dS \end{aligned} \quad (1-35)$$

These agree with the result obtained by using Eq. 1-32, although the latter were derived for an aperture in the wall. The element B_2 is therefore determined by allowing an incident TE_{10} mode of reference amplitudes V_1^i and I_1^i to impinge on the magnetically shorted aperture from the left and then solving the set of simultaneous Eqs. 1-35:

$$V_1(0-) = (V_1^i + V_1^r) = \int_{\text{ap}} \vec{i}_z \times [\sum V_m(0-)\vec{e}_m(\vec{\rho})] \cdot \vec{h}_1(\vec{\rho}) dS \quad (1-36a)$$

$$I_1(0-) = (I_1^i + I_1^r) = \int_{\text{fin}} [\sum I_m(0-)\vec{h}_m(\vec{\rho})] \times \vec{i}_z \cdot \vec{e}_1(\vec{\rho}) dS \quad (1-36b)$$

$$V_n(0-) = V_n^r = \int_{\text{ap}} \vec{i}_z \times [\sum V_m(0-)\vec{e}_m] \cdot \vec{h}_n dS \quad (1-36c)$$

$$I_n(0-) = I_n^r = \int_{\text{fin}} [\sum I_m(0-)\vec{h}_m] \times \vec{i}_z \cdot \vec{e}_n dS \quad (1-36d)$$

$n > 1$

so as to obtain $jB_2/2 = (I_1^i + I_1^r)/(V_1^i + V_1^r)$ in Fig. 1-9c. Actually it is only necessary to solve one of the first two equations together with one of the last two for each value of $n > 1$ because the amplitudes are related by $V_n^i = I_n^i Z_n$ and $V_n^r = -I_n^r Z_n$ according to previous equations (see Prob. 7 for an investigation of consistency).

Strictly speaking, the cutoff modes $n > 1$ excited at adjacent fins will influence the fields at the fin under consideration, but the correction will be negligible if $\epsilon^{(-\alpha_2 L)} \ll 1$, where α_2 is the decay constant of the first cutoff mode. The other cutoff modes have even larger decay constants.

With B_2 known in the equivalent circuit of Fig. 1-9 and $Z_1 = 0$ we can study the TE_{10} mode periodic phase shift as a function of frequency in a range far enough from the cutoff frequency of the next waveguide mode with decay constant α_2 such that $\epsilon^{-\alpha_2 L} \ll 1$. The Marcuvitz-Schwinger method for calculating equivalent circuit elements of waveguide obstacles is often more convenient to use than a Green's function technique and is particularly appropriate when the obstacles are small so as to perturb only slightly the propagation of the first TE or TM waveguide mode.

The agreement we can expect between the measured dispersion curve

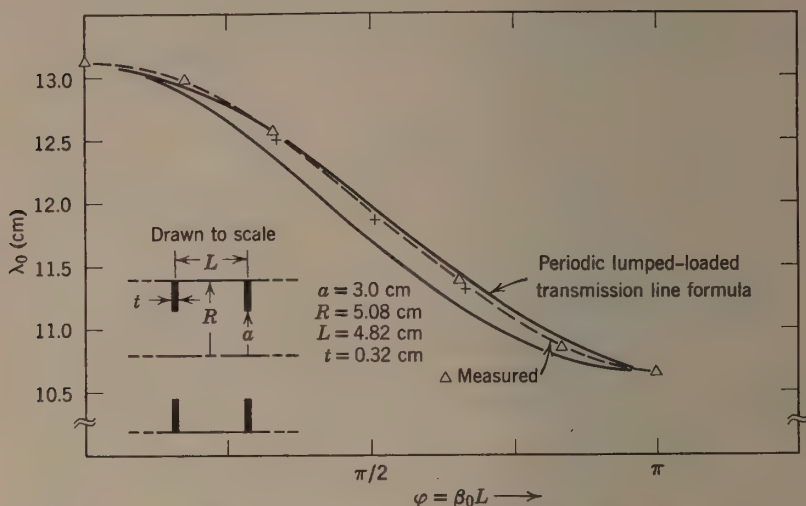


Fig. 1-10. First dispersion curve of the centerhole-coupled chain shown. The upper solid curve has been calculated by Slater's equation 1-37. The lower solid curve is calculated by the single mode-pair analysis of Chapter 4. The points marked with pluses are calculated by the two mode-pair analysis of Section 5-5. λ_0 is the free-space wavelength.

of a lightly loaded waveguide and the calculated curve on the basis of equivalent circuit elements is depicted in Fig. 1-10. The calculated curve is based upon Slater's relation¹²

$$\cos \varphi = \cos \varphi_0 - a\varphi_0 \sin \varphi_0 \quad (1-37)$$

in which φ is the phase shift per period L with the fins present, φ_0 is the phase shift in the absence of fins, $2\pi L(1/\lambda_0^2 - 1/\lambda_c^2)^{1/2}$, where λ_0 is the free-space wavelength and λ_c is the cutoff wavelength, and a is a parameter chosen to "fit" the measured π point.

C. Analysis in Terms of the Resonant Cavity Modes

Now let us consider the other extreme case of a periodic cavity chain—the chain coupled through very small holes. If the holes in the Fig. 1-2 structure were completely closed, the ω - φ curves would consist of a series of horizontal lines at the various cavity resonant frequencies. The small holes merely distort those lines somewhat, so that the $\varphi = 0$ cutoff frequency of a passband differs slightly from its $\varphi = \pi$ cutoff frequency. Clearly, a given passband is characterized very well by only one resonant

¹² J. C. Slater [6, Section 8.4].

configuration, either the short-circuit mode resonating at one end of the passband or the open-circuit mode resonating at the other end of the passband. For instance, the first passband of Fig. 1-6*a* is characterized by the modes of Fig. 1-6*b,c*, which look very much alike if the holes are small. The second passband is characterized by the modes of Fig. 1-6*d,e*. In analogy to Eq. 1-19 for the heavily lumped-loaded transmission line we can expand the electric field, for example, at a given frequency with the short-circuit resonant cavity modes defined as in Eq. 1-21:

$$\hat{E} \text{ (within a cavity)} = \sum V_n E_n(r) \quad (1-38)$$

in which the relative amplitudes V_n depend upon the operating frequency in relation to the various mode resonant frequencies. This statement will be amplified in Chapters 4 and 5. The advantage expansion 1-38 has over the ordinary waveguide mode expansion lies in the fact that only *one* resonant cavity mode will predominate: the one with a resonant frequency very near the operating frequency of interest. We can ignore, in the first approximation, the contributions of the other resonant cavity modes to the field at the operating frequency. If the operating frequency lies in a passband where many waveguide mode-harmonics are propagating, we thereby circumvent the problem of determining all their relative amplitudes by choosing that combination of mode-harmonics which comprises the resonant cavity mode associated with that passband. As we discover in Chapter 4 we can say a great deal about the propagation in this passband in terms of a few measurable parameters without knowing the precise structure of the relevant resonant cavity mode.

If we employ Eq. 1-38 for the electric field expansion we must observe the fact that it is incapable of representing tangential electric field on the holes because each $\bar{E}_n$ was defined to be perpendicular to them. We always assume, however, that the expansion is complete for total electric field $\hat{E}$ *within* the cavity.

In analogy to Eq. 1-20 for the heavily loaded transmission line, we may write another rapidly converging expansion which *does* represent tangential electric field on the holes. We employ the open-circuit resonant cavity modes defined as in Eq. 1-22, two of which are shown in Fig. 1-6*c,e*:

$$\hat{E} \text{ (within a cavity)} = \sum v_n e_n(r) \quad (1-39)$$

At a frequency within a given passband only that mode predominates which resonates at one end of the passband; this mode is also a linear combination of waveguide mode-harmonics. We also assume that expansion 1-39 is complete for the fields within a cavity and for the tangential electric field on the coupling surfaces—at least complete enough for practical purposes.

The $\hat{E}$ field in the next cavity at $(x, y, z + L)$ is the field at (x, y, z) times $e^{-j\varphi}$ by either Eq. 1-38 or Eq. 1-39. All the patterns are repeated from cavity to cavity and the amplitudes are phase shifted by this factor.

The point of this discussion is that we may conveniently represent the electric field at frequencies within a narrow passband of the cavity chain *either* by rapidly converging expansion 1-38 *or* by expansion 1-39 instead of by expansion 1-8 in terms of the waveguide mode-harmonics. We shall see in Chapter 4 that we actually need both, because the tangential electric field on the holes as given by Eq. 1-39 excites the short-circuit modes in Eq. 1-38.

We have discussed in this section efficient methods of expanding the fields in two extreme cases of periodic cavity chains: the heavily coupled (small-fin) variety and the lightly coupled (small-hole) variety. A practical intermediate case of the ridged waveguide will be reviewed in Section 2-6 and treated again in Section 6-6 by a very useful variational method.

Problems

1. Consider an infinitely long ladder slow wave structure, which is a strip of metal in which transverse slots are cut with period L . Show that e_z (and h_z) may be expanded in a set of mode-harmonics as

$$e_z = \sum_{\ell mn} A_{\ell mn} e^{-j\beta_{\ell} x - j\beta_m y - j\beta_n z}, \quad \beta_n = \beta_0 + \frac{2n\pi}{L}$$

such that $\beta_{\ell}^2 + \beta_m^2 + \beta_n^2 = k^2 = \omega^2 \mu \epsilon$

2. Is this statement precise? "In a periodic fin-loaded waveguide the complete field at each point is a linear combination of the waveguide modes." If not, sharpen the statement.
3. From the energy theorem verify the power flow-stored energy-group velocity relation when attenuation is present, provided $\partial \alpha / \partial \omega = 0$ where α is the attenuation constant.
4. In order to appreciate the nonuniqueness of the representation for a voltage (or current) within a periodic cell, $0 < z < L$, assume the true voltage is given by $\sin(\pi z/L)$. Choose the amplitudes of the first few $\cos(n\pi z/L)$ harmonics so as to approximate $\sin(\pi z/L)$. Do the Fourier amplitudes minimize the mean square error in the cosine series over the cell length?
5. Obtain the first three resonant frequencies, as measured by kL , for the periodic lumped-loaded transmission line of Fig. 1-5a, but with an additional series lumped inductance of L_L at $z = L/2$. Take the parameters $\omega_1 C_L / Y_0 = 1$, $\omega_1 L_L = (2Y_0)^{-1}$, Y_0 being the characteristic admittance $\sqrt{C_0/L_0}$, and ω_1 the first π cutoff frequency. A Smith Chart is useful. Sketch the voltage and current patterns of each resonant mode, as well as the kL - φ curves. Verify them by the exact dispersion relation.

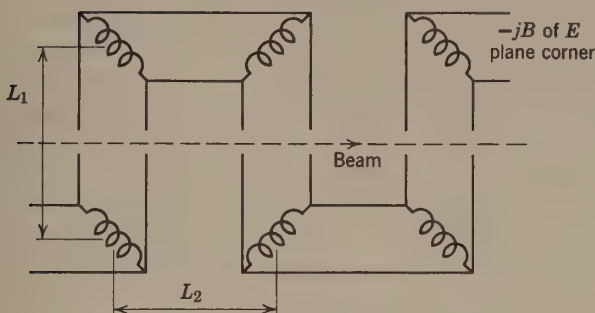


Fig. 1-11.

6. Reference [7, p. 738] outlines a variational method which we can modify and use to prove the set of ϕ_i functions defined by Eq. 1-23a is complete for $\hat{e}_z$ in a cross-sectional plane of a waveguide. Use the function

$$\Omega(\phi) = \int |\nabla_t \phi|^2 dS$$

and develop the proof of completeness in a mean square sense.

7. Suppose a propagating waveguide mode n is incident on a transverse surface in a waveguide which consists of a fin and a magnetically shorted aperture. Show by integrating Eqs. 1-28 and 1-29 that

$$V_n(0+) - V_n(0-) = -V_n(0-) = - \int_{\text{ap}} \vec{i}_z \times \hat{E}(0-) \cdot \vec{h}_n dS$$

$$I_n(0-) = \int_{\text{fin}} \hat{H}(0-) \times \vec{i}_z \cdot \vec{e}_n dS$$

Verify the interpretation that $V_n(0+) = I_n(0+) = 0$ by first multiplying Eq. 1-28a by I_n^* , and the conjugate of Eq. 1-28b by V_n . Then integrate from $z = 0-$ to $z = 0+$ and sum over the modes so as to form a power theorem.

8. In Chapter 5 of Ref. [10] choose an infinitesimally thin obstacle in a rectangular waveguide, compute the TE_{10} mode susceptance by the Marcuvitz-Schwinger method, and check the result against the appropriate curve in the reference.
9. Consider a meander line built with rectangular waveguide, shown in Fig. 1-11. Sketch the phase shift of axial electric field vs. frequency over a "periodic length" L_2 as seen by an electron beam. Let the waveguide propagation constant be $\beta_g = \sqrt{k^2 - k_c^2}$, k_c being the guide cutoff frequency, and first neglect the equivalent inductance at each corner. What is the qualitative effect of this inductance upon the dispersion curve?

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Chapter II

Conventional Analyses of Helices and Loaded Waveguides

2-1. Synopsis

We review the Pierce analysis of the sheath helix and the Sensiper analysis of the tape helix as preparation for the variational analysis in Section 6-3. The sheath helix, as a circular waveguide on which currents can flow only in helical paths, supports an infinite number of waves.¹ The n th such wave is composed of two modes, both varying as the I_n modified Bessel function inside the sheath and as the K_n function outside the sheath. These two modes form the TM and TE portions of the field. The ω - β curves for the various waves are essentially a series of straight lines which serve as guide lines for the curves of the tape helix. The $n = 0$ mode of the tape helix, important for traveling wave tubes, propagates with a phase velocity very nearly $c \tan \psi$ in the frequency range $ka \cot \psi \geq 2$.

The tape helix exhibits periodicity in both the angular and the axial directions and nonuniform boundary conditions around the circumference at the tape radius. As a result, all the modes as defined for a sheath helix couple together to form a wave with a particular fundamental propagation constant. The nonradiation condition determines certain forbidden zones on the ω - β diagram. We can sketch the ω - β curves so that they follow the sheath helix lines up to the boundaries of the forbidden zones and then become asymptotic to those boundaries. At a given frequency below the critical value of $k_c = \pi/p$, p being the periodic pitch, many independent waves exist, each with the propagation constant

¹ See the Introduction for the definitions of wave, mode, and harmonic.

of its predominant mode within the fundamental region $-\pi/p \leq \beta_n \leq \pi/p$. There are no stopbands in this frequency range. For frequencies $k > \pi/p$ no free waves exist on a radially unbounded helix. We review Sensiper's derivation of the determinantal equation for ω vs. β_0 of each wave. His results compared to measurement indicate that, within a small percentage of error, the width and shape of the tape and the assumed current distribution on it are unimportant.

Multifilar helices are developed from the single-tape helix by adding more windings so that the infinite-filar helix would be just the sheath helix. Each independent wave on an N -filar helix has a certain transverse phase pattern and behaves far from the cutoff regions like that unifilar helix wave of the same predominant mode component. Only a certain ordered infinity of modes belongs to each N -filar wave; as a result, its ω - β diagram has fewer forbidden regions. As the number of windings N goes to infinity the number of forbidden regions decreases, so in the limit the sheath helix has one such region. Each helix has associated with it an impedance parameter which measures the effectiveness of the circuit interaction with an electron beam. Measurements show that the impedances of the bifilar helix waves of predominant $n = 0$ and 1 mode components tend to be about half those of the corresponding sheath waves.

A survey of the analysis of tape ladder lines is timely, for certain modified forms of them, the "comb" structures, have been employed in traveling wave masers, and A. Karp has proposed other varieties for millimeter wave amplifiers. We review P. N. Butcher's analysis of the single- and double-ridge and the single- and double- T structures, whereby he matched the field expressions in various portions of each structure to obtain the dispersion relations. The analyses show that the ridge ladder structures support a forward wave of power flow in the direction of the phase velocity, whereas the T structures support backward waves of power flow in the opposite direction. We also quote the accurate expressions of A. A. Oliner and W. Rotman for the dispersion in a corrugated trough waveguide.

We develop in some detail the dispersion relation for a particular ridged waveguide, with arbitrary phase shift per period, in order to show by dispersion curve plots that such a structure can be fairly broadband in its first passband.

2-2. The Sheath Helix

The sheath helix is a fictitious model of a tape helix which has dispersion or ω - β curves resembling closely those of the tape helix over much of the

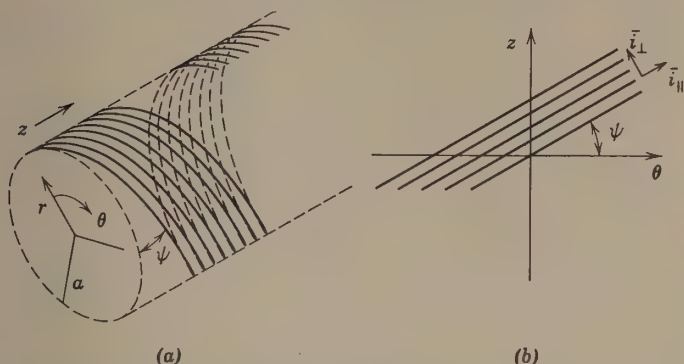


Fig. 2-1. Sheath helix geometry, showing radius a and pitch angle ψ . In (b) unit vectors are shown parallel and perpendicular to the direction of current flow along the wires.

ω - β diagram. The sheath model is pictured in Fig. 2-1. Current on the sheath surface at $r = a$ is constrained to flow at the pitch angle ψ along the lines shown. The sheath surface is treated as anisotropic in the sense that it supports components $\hat{E}_\perp$, $\hat{H}_\parallel$, and $H_\perp$ (in the directions of unit vectors $\hat{i}_\perp$ and $\hat{i}_\parallel$) but shorts out $\hat{E}_\parallel$. Pierce² has analyzed the positive group velocity wave³ that travels at nearly the phase velocity of light in the parallel direction because this is the wave most often utilized in traveling wave tubes. Sensiper [2] extended the analysis to the n th such wave. We shall outline the derivation of the determinantal equation for ω vs. β_n of the n th wave, with some minor changes in notation.

In accord with Eq. 1-6 let us represent the n th mode components of axial electric and magnetic field inside the sheath, $r \leq a$, by

$$\begin{aligned} \hat{e}_{zn} &= (B_3)_n I_n(\gamma_n r) e^{jn\theta - j\beta_n z} \\ \hat{h}_{zn} &= (B_1)_n I_n(\gamma_n r) e^{jn\theta - j\beta_n z} \end{aligned} \quad (2-1)$$

with $\gamma_n = (\beta_n^2 - k^2)^{1/2}$, $k^2 = \omega^2 \mu \epsilon$. It turns out that we can satisfy the boundary conditions on the sheath surface with the field components of these two n th modes alone; the other modes are "decoupled." Hence there is no need to relate the propagation constant β_n of this wave to a " β_0 " as in Eq. 1-6. These two field components generate, through

² J. R. Pierce [1, Appendix II].

³ We say sheath wave whereas most workers say sheath "mode." This wave is composed of two modes, one satisfying Maxwell's equations for the TM portion of the field and one satisfying those equations for the TE portion. Each of these modes is defined for "open" boundary conditions on the sheath surface. According to the Introduction we call the proper linear combination of these two modes a sheath wave.

Maxwell's equations, the other field components inside the sheath:

$$\left. \begin{aligned} \hat{e}_r &= (B_3)_n(j\beta_n/\gamma_n)I_n' - (B_1)_n(n\omega\mu/\gamma_n^2 r)I_n \\ \hat{e}_\theta &= -(B_3)_n(n\beta_n/\gamma_n^2 r)I_n - (B_1)_n(j\omega\mu/\gamma_n)I_n' \\ \hat{h}_r &= (B_3)_n(n\omega\varepsilon/\gamma_n^2 r)I_n + (B_1)_n(j\beta_n/\gamma_n)I_n' \\ \hat{h}_\theta &= (B_3)_n(j\omega\varepsilon/\gamma_n)I_n' - (B_1)_n(n\beta_n/\gamma_n^2 r)I_n \end{aligned} \right\} e^{jn\theta - j\beta_n z} \quad (2-2)$$

The prime denotes differentiation with respect to the entire argument $\gamma_n r$, of the I_n modified Bessel function.

Outside the sheath, $r \geq a$, we represent the z components of the fields in terms of two new amplitudes $(B_4)_n$ and $(B_2)_n$. We employ the K_n modified Bessel function of argument $\gamma_n r$; the total fields are denoted by capital letters in the outer region:

$$\begin{aligned} \hat{E}_{zn} &= (B_4)_n K_n(\gamma_n r) e^{jn\theta - j\beta_n z} \\ \hat{H}_{zn} &= (B_2)_n K_n(\gamma_n r) e^{jn\theta - j\beta_n z} \end{aligned} \quad (2-3)$$

with $\gamma_n = (\beta_n^2 - k^2)^{1/2}$, $k^2 = \omega^2 \mu \varepsilon$.

These generate the other external field components:

$$\left. \begin{aligned} \hat{E}_r &= (B_4)_n(j\beta_n/\gamma_n)K_n' - (B_2)_n(n\omega\mu/\gamma_n^2 r)K_n \\ \hat{E}_\theta &= -(B_4)_n(n\beta_n/\gamma_n^2 r)K_n - (B_2)_n(j\omega\mu/\gamma_n)K_n' \\ \hat{H}_r &= (B_4)_n(n\omega\varepsilon/\gamma_n^2 r)K_n + (B_2)_n(j\beta_n/\gamma_n)K_n' \\ \hat{H}_\theta &= (B_4)_n(j\omega\varepsilon/\gamma_n)K_n' - (B_2)_n(n\beta_n/\gamma_n^2 r)K_n \end{aligned} \right\} e^{jn\theta - j\beta_n z} \quad (2-4)$$

To determine the relative amplitudes $(B_1)_n, \dots, (B_4)_n$ we invoke the boundary conditions at $r = a$:

$$\hat{e}_1(\gamma a) - \hat{E}_1(\gamma a) = (\hat{e}_{zn} \cos \psi - \hat{e}_{\theta n} \sin \psi - \hat{E}_{zn} \cos \psi + \hat{E}_{\theta n} \sin \psi)_{r=a} = 0 \quad (2-5a)$$

$$\hat{e}_{||}(\gamma a) = \hat{E}_{||}(\gamma a) = 0$$

which is

$$(\hat{e}_{zn} \sin \psi + \hat{e}_{\theta n} \cos \psi - \hat{E}_{zn} \sin \psi + \hat{E}_{\theta n} \cos \psi)_{r=a} = 0 \quad (2-5b)$$

and also

$$\begin{aligned} \hat{h}_{||}(\gamma a) - \hat{H}_{||}(\gamma a) \\ = (\hat{h}_{zn} \sin \psi + \hat{h}_{\theta n} \cos \psi - \hat{H}_{zn} \sin \psi - \hat{H}_{\theta n} \cos \psi)_{r=a} = 0 \end{aligned} \quad (2-5c)$$

These relations yield one equation in terms of one amplitude, which, for a nonzero amplitude, furnishes the determinantal equation for β_n in

terms of $k = \omega\sqrt{\mu\epsilon}$ as

$$\left[\frac{(\gamma_n a)^2 - n\beta_n a \cot \psi}{(\gamma_n a)(ka) \cot \psi} \right]^2 = - \frac{I_n'(\gamma_n a) K_n'(\gamma_n a)}{I_n(\gamma_n a) K_n(\gamma_n a)} \quad (2-6)$$

The most useful wave is the $n = 0$ wave for which Eq. 2-6 simplifies to⁴

$$(\gamma_0 a)^2 \frac{I_0(\gamma_0 a) K_0(\gamma_0 a)}{I_1(\gamma_0 a) K_1(\gamma_0 a)} = (ka \cot \psi)^2 \quad (2-7)$$

A given value of $ka \cot \psi$ therefore determines $\gamma_0 a$ and the quantity $ka \cot \psi / \gamma_0 a$ which is plotted for the $n = 0$ wave in Fig. 2-2. As Pierce observes, at relatively large frequencies we have

$$ka \cot \psi \gg 2, \quad (k/\gamma) \cot \psi \simeq 1.$$

This means that the axial phase velocity v_p of this wave is very much less than that of light, c . To see this recall that

$$v_p/c = k/\beta_0 = \tan \psi (\gamma/\beta_0) [(k/\gamma) \cot \psi].$$

The quantity $(k/\gamma) \cot \psi$ is $\simeq 1$ and $\tan \psi$ is a small number—usually less than 0.3—and so k/γ is a small number. This means $\gamma/\beta_0 = \gamma/\sqrt{\gamma^2 + k^2}$ is very nearly 1. Thus, $v_p/c \simeq \tan \psi$, a small number, over the frequency range $ka \cot \psi \gg 2$.

For the n th wave we can solve Eq. 2-6 approximately by saying

$$\frac{I_n' K_n'}{I_n K_n} = - \frac{n^2/a^2 + \gamma_n^2}{\gamma_n^2} \quad \begin{array}{l} \text{for } \gamma_n > 0 \text{ and not } \simeq 0 \\ \text{and also } |n| \geq 1 \end{array} \quad (2-8)$$

Pierce and Tien [3] observe this approximation to be a good one for the ranges of γ_n and n indicated. Then the determinantal equation is approximately

$$[(\gamma_n a)^2 - n\beta_n a \cot \psi]^2 \simeq [n^2 + (\gamma_n a)^2](ka \cot \psi)^2 \quad (2-9)$$

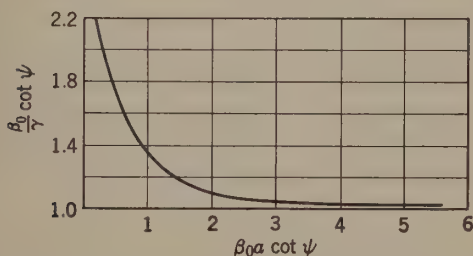


Fig. 2-2. ω - β_0 relation for the sheath helix (from Pierce [1]).

⁴ Pierce's β_0 is our $k = \omega\sqrt{\mu\epsilon}$ and his β is our β_0 .

The n^2 term on the right is much smaller than one of the terms on the left according to the inequality $\beta_n^2 \gg k^2$ for slow waves. Hence we take the square root of Eq. 2-9 and obtain

$$\beta_n a \simeq (n \pm ka) \cot \psi, \quad \beta_n \simeq \gamma_n \quad (2-10)$$

This relation determines a number of dashed straight lines for the various integral values of n , parallel to one or the other of the $\beta_0 = \pm ka \cot \psi$ lines emanating from the origin in Fig. 2-4. We shall see that these lines are very nearly the ω - β curves of the tape helix modes away from the so-called forbidden zones (crosshatched regions). The sheath helix has only the one forbidden zone emanating from the origin.

For later reference we quote Sensiper's⁵ expressions for the amplitudes $(B_4)_n$ and $(B_2)_n$ in the external region (with his n changed to $-n$).

$$\begin{aligned} (B_4)_n &= j\omega\mu a \frac{(\gamma_n a)^2 - n\beta_n a \cot \psi}{(ka)^2} I_n(\gamma_n a) \sin \psi \\ (B_2)_n &= -\gamma_n a I_n'(\gamma_n a) \cos \psi \end{aligned} \quad (2-11)$$

The amplitudes $(B_3)_n$ and $(B_1)_n$ for the inner region are obtained, respectively, by replacing I_n by K_n and I_n' by K_n' in these expressions.

2-3. The Tape Helix

The single-tape helix shown in Fig. 2-3 differs from the sheath helix in that the former has periodicity p in the z direction and supports both θ and z components of electric field in the gaps at $r = a$. Because of the

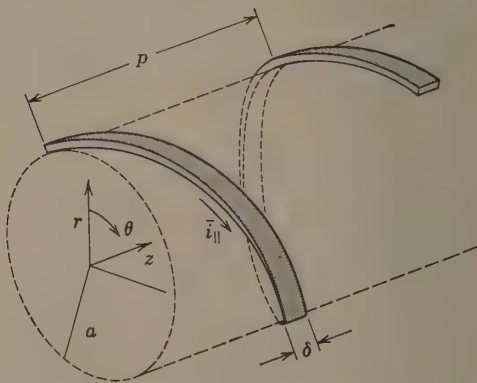


Fig. 2-3. Geometry of the single-tape helix. p is the pitch and δ is the tape width.

⁵ S. Sensiper [2, p. 8, Eqs. (51) and (54)].

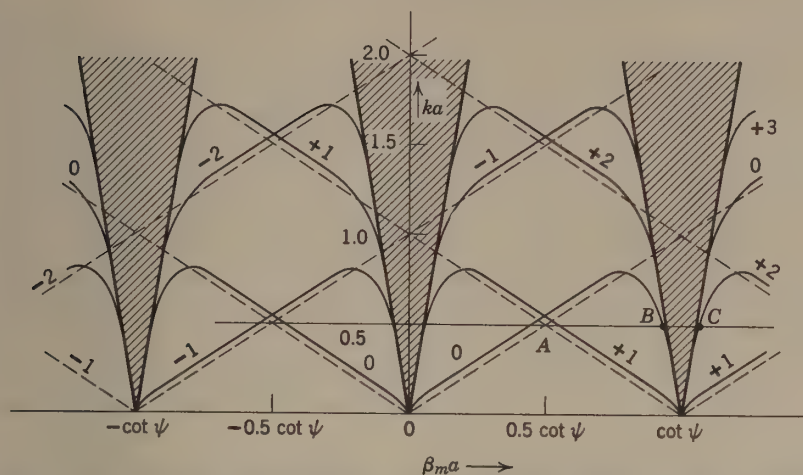


Fig. 2-4. ω - β diagram of the single-tape helix. The shaded regions are the forbidden regions which the curves do not enter. The dashed lines are solutions of Eq. 2-10 for the sheath helix. The dispersion curves for the various sheath helix waves (Eq. 2-6) avoid the forbidden zone emanating from the origin but go through the other zones. Numbers on the curves indicate the mode belonging to the wave at a given frequency.

nonuniform boundary conditions at $r = a$ in any cross-sectional plane, all the modes in Eqs. 2-1 through 2-4 contribute to each wave, each one related to a fundamental propagation constant β_0 and of relative amplitude to satisfy the boundary conditions on the tape. The central problem of the tape helix is to determine the value of β_0 for each wave and the amplitudes of its predominant constituent modes.

Before reviewing Sensiper's solution of the problem let us deduce the presence of certain forbidden regions on the ω - β diagram. In Section 1-2 we observed that any one of the β_n 's in the mode expansion of a particular helix wave may lie within the fundamental region $-\pi/p \leq \beta_n \leq \pi/p$. Suppose β_{-1} of the $n = -1$ mode lies within this region. In order to have a physical nonradiating field which decays to zero properly at $r = \infty$ we must have a real argument of the $K_1(\gamma_{-1}r)$ Bessel function in Eq. 2-3. This means that $|\beta_{-1}| \geq k$, which implies the ω - β_{-1} curve for the two $m = -1$ tape modes, one for the TM portion of the field and one for the TE portion (Fig. 2-4), will follow the dashed line of the $m = -1$ sheath wave right up to the $\beta = -k$ line and then sharply deflect downward to approach the $\beta = -k$ line asymptotically. Detailed calculation verifies this behavior. At the same time the ω - β curve for the $m = 0$ component of this same helix wave, the propagation constant β_0 of which is related to β_{-1} as $\beta_0 = \beta_{-1} + 2\pi/p$, will deflect downward toward the $\beta - 2\pi/p = -k$ line bordering the next crosshatched region. By similar

reasoning we deduce that if the β_n of any wave lies within the region $(-\pi/p, \pi/p)$ in Fig. 2-4, then as ω or k rises β_n will at first approach the first crosshatched region at $\beta a = 0$ along one of the dashed lines, then reach an inflection point and subsequently approach $\beta_n = 0$ along one of the $\beta = \pm k$ lines only as ω approaches zero.⁶ This first crosshatched region is the first forbidden zone; the other zones are implied by the symmetry of the ω - β curves embodied in the relation $\beta_n = \beta_0 + 2n\pi/p$.

With the forbidden zones crosshatched on Fig. 2-4 it is a simple matter to distort the dashed lines representing sheath helix waves so that they never enter the zones. In this way we obtain the qualitative ω - β curves shown for the tape helix.

Since the crosshatched regions merge at a value of $ka = 0.5 \cot \psi = \pi a/p$ ($\cot \psi$ can be defined as $2\pi a/p$ without developing the helix) we see that above the critical frequency $k_c = \pi/p$ no free nonradiating waves exist. Below k_c no stopbands exist; in fact, many independent waves exist at each frequency. Suppose we compare one of those waves with all the independent waves on the sheath helix. If the n th ω - β curve of this tape helix wave nearly coincides with the dashed line of the n th sheath helix wave, all the other ω - β curves of that tape wave nearly coincide with the lines of other sheath helix waves. The tape couples all the sheath waves into one tape helix wave of nearly the same fundamental propagation constant.

Let us examine the wave picture on the tape helix at a particular frequency in order to be quite clear about the multitude of waves which actually do exist. Let us take $ka = 0.5$ in Fig. 2-4, for example. The ordinary TWT (traveling wave tube) wave without the beam at this frequency has a propagation constant of its two $n = 0$ modes—which comprise the TM and TE portions of the field—such that

$$\beta_0 a \simeq 0.5 \cot \psi = \pi a/p$$

near point *A*. Any n th mode in this wave has a propagation constant $\beta_n = \beta_0 + 2n\pi/p$. Another independent backward wave exists at this frequency, with $\beta_{+1} a \simeq 0.5 \cot \psi$ and β_n related to β_0 and β_1 in the usual way. A third and fourth independent wave also exists as denoted by points *B* and *C*. The former has a value of $\beta_0 a$ for its two $n = 0$ modes somewhat less than $\cot \psi = 2\pi a/p$; the latter, a value of $\beta_{+2} a$ of its two $n = 2$ modes somewhat greater than $\cot \psi$.

If we look again we can find still more independent waves, one with propagation constant β_{-1} of its $n = -1$ modes on the curve labeled -1 very near point *B*, another with propagation constant β_{+2} on the curve

⁶ If the helix is surrounded by a conducting pipe the Bessel function arguments can become imaginary and the ω - β curves can enter the forbidden regions. See [4].

labeled $+2$ very near point B , and so on. These branches of the curves near point B essentially coincide with the boundary of the forbidden zone. There is still another infinite set of waves, one with its β_0 on the curve labeled 0 very near point C , another with its β_{+3} on the curve labeled $+3$ very near point C , etc. These branches of the curves near point C are also essentially asymptotic to the forbidden-zone boundary. Each of these waves has, in principle, an infinity of modes in addition to the ones we have mentioned. Any one of these waves will have one of its $\beta_{n'}$ $\simeq k$ on the edge of the first forbidden zone and the n' mode fields will be predominant because they are nearly uniform with radius r inside the helix and decay as r^{-2} outside it ($\gamma_{n'}$ is nearly zero). Usually a TWT is designed so that the electron beam is synchronous with the TM mode of only one wave at the operating frequency, in which case we can neglect the other weakly interacting waves.

We can tell the direction of the phase velocity of a wave by the sign of β_0 of the $n = 0$ modes alone. For example, $\hat{E}_z$ will vary as

$$\epsilon^{j\omega t - j\beta_0 z} \sum_n (B_n)_n K_n(\gamma_n r) \epsilon^{jn\theta - j2\pi n z/p}$$

outside the tape according to Eq. 2-3. We follow the wavefront of a wave with positive β_0 by moving along the tape (so that $\theta - 2\pi z/p$ remains constant) by an axial distance $z = \omega t / \beta_0$ in time t . Therefore this wave follows the tape clockwise in the $+z$ direction with axial phase velocity ω / β_0 . A wave of negative β_0 follows the tape backwards in the course of time with axial phase velocity $-\omega / |\beta_0|$.

We now outline Sensiper's derivation⁷ of the ω - β_0 relation for the narrow tape helix; more of the details and computed curves will be found in Hutter's book.⁸ With net current $\hat{K}$ flowing on the tape surface at $r = a$ the boundary conditions to be satisfied by a wave are continuity of $\hat{E}_{z,\theta}$ and discontinuity of $\hat{H}_{z,\theta}$ according to $\hat{K}$. Thus, in our notation,

$$\begin{aligned} \hat{e}_{\theta,z} &= \hat{E}_{\theta,z} \\ \hat{K}_\theta &= \hat{h}_z - \hat{H}_z \quad \text{at } r = a \\ \hat{K}_z &= \hat{H}_\theta - \hat{h}_\theta \end{aligned} \quad (2-12a)$$

The current $\hat{K}$, too, may be expanded in harmonics of the same functional form as the modes of Eq. 2-3,

$$\hat{K} = \sum \hat{K}_n \epsilon^{jn\theta - j\beta_n z}, \quad \hat{K}_n = i_\theta \kappa_{\theta n} + i_z \kappa_{zn} \quad (2-12b)$$

By orthogonality of the modes in the angular direction he satisfies Eqs. 2-12a for each n th component independently. When this is done, the

⁷ Sensiper [2, p. 54]. His spacial harmonics of a wave are our modes of that wave.

⁸ R. G. E. Hutter [5, p. 121].

amplitudes $(B_1)_n, \dots, (B_4)_n$ of Eqs. 2-1 through 2-4 are determined by $\kappa_{\theta n}$ and κ_{zn} , the Fourier coefficients of tape current. In order to determine these κ he assumes one of two situations: either $\hat{K}_{\parallel}$ is uniform across the tape width δ except for a noninfluential phase shift, or it approaches infinity along with $\hat{E}_{\perp}$ in an inverse square root fashion at the tape edges. The final dispersion relation is very insensitive to either of these assumptions. With the current $\hat{K}$ known he proceeds to obtain the Fourier coefficients of tape current $\kappa_{\theta n}$ and κ_{zn} from Eq. 2-12b. The dispersion relation obtains by setting $\hat{E}_{\parallel} = 0$ along the central line of the tape and has the form

$$\sum_n \left\{ \left[\beta_0^2 - k^2 + k^2 \left(\frac{n \cot \psi}{\gamma_n a} \right)^2 \right] I_n(\gamma_n a) K_n(\gamma_n a) + (k \cot \psi)^2 I_n' K_n' \right\} D_n \simeq 0$$

$$= \frac{\partial}{\partial(\arg)}, \quad \gamma_n^2 = \beta_n^2 - k^2 \quad (2-13)$$

where D_n is a factor depending upon the assumed current distribution on the tape. The same relation obtains by setting $\hat{e}_{\parallel}$ of the internal helix region equal to zero along a central line of the tape. By adding and subtracting known series to improve the convergence and by making large-argument approximations such as Eq. 2-8, Sensiper converts Eq. 2-13 into manageable form. We do not quote the final form here because we shall obtain a dispersion relation more straightforwardly and neatly in Section 6-3; it will be the determinantal equation for the sheath helix $n = 0$ mode plus correction terms. This form as well as Sensiper's equation indicates that the ω - β_0 curve for the $n = 0$ mode portion of the ordinary tape helix wave will follow closely the sheath helix line in the region near point *A* of Fig. 2-4.

Sensiper's calculated ω - β_0 curves for a tape helix of $\cot \psi = 5.7$ and $\pi\delta/p = 0.1$ verify the first three branches along the $n = 0$ dashed line of positive slope in Fig. 2-4. Only the first branch and a portion of the second branch are shown. The wave with this $n = 0$ mode component is the dominant one excited at low frequencies, $ka < 0.5$. In this range the analysis predicts a phase velocity lower than that of the $n = 0$ sheath helix wave. Measurements on a round-wire helix indicate that the phase velocity is about the same or only slightly less than the sheath wave velocity on a helix of radius equal to the mean wire radius. The discrepancy is resolved, however, by increasing the tape width so as to correspond more closely to the wire diameter. Although his analysis is developed for a narrow tape, the results also agree quite well with measurements taken on the zero-mode branches shown in Fig. 2-4 for a frequency range $ka \geq 0.5$. Therefore, not only is the theory quite insensitive to the precise width of the narrow tape but it also describes well the round-wire helix.

2-4. Multifilar Helices

An N -filar helix is developed from a unifilar or single-tape helix by inserting extra windings rotated by angles $2\pi/N, 4\pi/N, \dots, (N-1)\pi/N$ with respect to the original winding. Figure 2-5 shows a bifilar helix ($N=2$). The geometry of an N -filar helix remains invariant if we rotate by any angle $(n/N)2\pi$, n being an integer. However, the field may phase shift by $2\pi/N, \dots, (N-1)2\pi/N$ radians per winding in the cross-sectional plane irrespective of the fundamental propagation constant along any one of the wires. In fact, the total fields at a given frequency can have any one or more of N independent phase configurations in the cross-sectional planes, some of which are shown in Fig. 2-6. There q denotes a wave with $2\pi q/N$ rotational phase shift between successive tapes. The $q=0$ wave is built up with only those sheath helix modes of Eqs. 2-1 and 2-3 with zero phase shift per winding; that is, the $n = \dots, -2N, -N, 0, N, 2N, \dots$ components. The $q=1$ wave is built up from the $n = \dots, -2N+1, -N+1, 1, N+1, 2N+1, \dots$ components, and so on up to the $N-1$ wave with the remaining $\dots, -N-1, -1, N-1, \dots$ components.

For example, the $q=0$ wave of a quadrifilar helix ($N=4$) would have its inner $\hat{e}_z$ field represented as

$$\hat{e}_{z0} = e^{-j\beta_0 z} \sum_{\dots, -4, 0, 4, \dots} (B_3)_n I_n(\gamma_n r) e^{jn(\theta - 2\pi z/p)} \quad (q=0) \quad (2-14)$$

where we have, as usual,

$$-\gamma_n^2 + \beta_n^2 = k^2, \quad \beta_n = \beta_0 + \frac{2n\pi}{p}.$$

The $q=1$ wave would have the representation

$$\hat{e}_{z1} = e^{-j(\beta_1 - 2\pi/p)z} \sum_{\dots, -3, 1, 5, \dots} (B_3)_n I_n(\gamma_n r) e^{jn(\theta - 2\pi z/p)} \quad (2-15)$$

$$\beta_n = \beta_1 + \frac{(n-1)2\pi}{p} \quad (q=1)$$

Here we write the phase factor of the entire wave as $\beta_1 - 2\pi/p$ so that β_1 will be precisely the complete propagation constant of the $n = +1$ mode

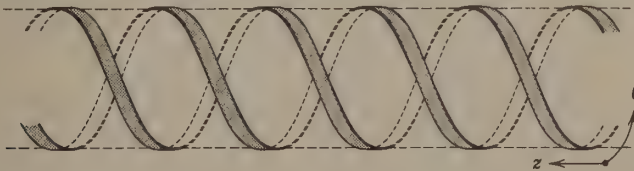


Fig. 2-5. Bifilar helix geometry. Each winding is displaced from the other by π radians in the angular direction.

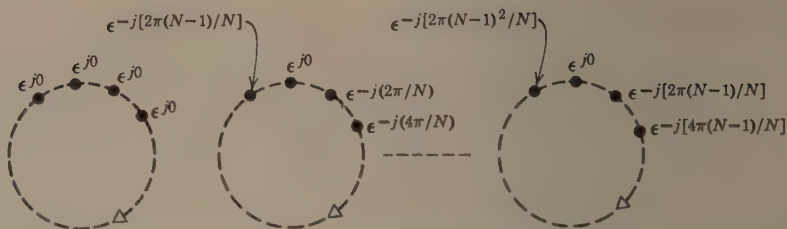


Fig. 2-6. The phase configurations in the transverse plane of the N independent N -filar helix waves. In a given wave the phase shift of current from one winding to the next is the same around the helix.

of this wave. The expressions for $\hat{e}_{z2,3}$ of the remaining $q = 2, 3$ waves are similar, and each has its own fundamental propagation constant β_2 or β_3 .

In the region of the ω - β diagram for the tape helix, Fig. 2-4, away from the forbidden regions the ω - β_n curve of a single tape mode corresponds closely with the curve for the same mode which helps compose one and only one of the multifilar helix waves. The tape helix discussion of the preceding section also applies to the waves of the multifilar helix with the understanding that "wave" now signifies " q -wave," which has only the component modes $n = \dots, q - N, q, q + N, \dots$ of Eqs. 2-1 and 2-3.

In microwave tube work an impedance parameter introduced by Pierce represents the effectiveness with which the circuit can extract r.f. power from an electron beam. This impedance is $E_z^2/2\beta_e^2 P_{av}$, where E_z^2 is the effective square of the circuit axial electric field in the beam region, $\beta_e = \omega/(\text{d.c. beam velocity})$ is the electronic propagation constant, and P_{av} is the "cold" circuit power flow in the wave which will interact with the beam. Now, as the number of tapes in a multifilar helix increases, the impedances of the important waves with predominant $n = 0$ and 1 components both rise rather strikingly. We certainly expect them to approach the impedances of the corresponding sheath helix waves since the sheath helix is just an infinite-filar helix (see Fig. 2-1). Even the conversion from unifilar to bifilar helix raises the impedance for the wave with predominant $n = 1$ mode from zero near zero frequency to about 15 ohms, as compared to about 30 ohms in the $n = 1$ sheath wave. Johnson et al.⁹ have reported measurements of the dispersion characteristics on a bifilar helix of $\tan \psi \simeq 0.10$. They studied the lowest-frequency $n = 0$ mode branch of one $q = 0$ wave near $0.5 \cot \psi$ and the lowest-frequency $n = +2$ branch of another $q = 0$ wave in the range $\cot \psi \leq \beta_n a \leq 2 \cot \psi$ shown on our Fig. 2-4, as well as the lowest-frequency $n = +1$ branches of two $q = 1$ waves. Their measurement

⁹ Johnson, Everhart, and Siegmán [6]. We use the mode number n in place of $-m$ in this article.

technique, which involved a thin hollow electron beam at radius $0.82a$, is described in Appendix B. The results verify the preceding remarks. They also measured the impedance of the bifilar forward wave with predominant $n = 0$ mode as about half or more of the $n = 0$ sheath wave impedance in the range $0 < ka \leq 0.6$. The measured impedance of the backward bifilar wave with predominant $n = +1$ mode component (in the range $0 \leq \beta_n a \leq \cot \psi$ on Fig. 2-4) was also more than half that of the $n = +1$ sheath wave, in agreement with the qualitative theory.

2-5. Tape Ladder Lines

End views of various tape ladder lines appear in Fig. 2-7. Butcher [7] has analyzed these varieties of ladder lines by expanding the fields appropriately in each subregion of the structure and matching them on the common boundaries, with the proviso that the ladder rungs conduct current only transversely. Typical dispersion curves from his analysis appear in Fig. 2-8; observe that the abscissa measures $\beta_0 B$ rather than the periodic phase shift $\beta_0 D$. The analyses were actually performed with $\beta_0 D = 0$ because the dispersion relations plotted in Fig. 2-8 are only slightly affected by a finite phase shift per period.

In the "easitron" structure of Fig. 2-8a each tape resonates uncoupled from its neighbors at a frequency such that its length A is half a free-space wavelength. Each tape is uncoupled from its neighbors because the fields are TEM (transverse electric and magnetic) in the x direction. Thus, $\hat{E}$ and $\hat{H}$ carry transverse power in a cross-sectional plane perpendicular to the z axis between two tapes (see the sketches of the $\beta_0 D = 0$ and $\beta_0 D = \pi$ modes of the tape). However, coupling between neighbors is only effected through $\int_S \hat{E} \times \hat{H} \cdot \hat{i}_z dS$ on this plane, as will be clarified in Section 4-2. This coupling integral is identically zero in the easitron.

Now consider the ridge structures of Fig. 2-7b,c. The central portions of one or both horizontal faces have been pushed in, thus displacing more electric energy than magnetic energy in the central region. This lowers the zero phase shift cutoff frequency as shown in Fig. 2-8. The π phase shift frequency remains little affected because the electric fields of the π mode are concentrated near the wires; see Fig. 2-8a. Dual remarks apply to the T -structures of Fig. 2-8d,e in which the end portions of one or both horizontal faces have been pushed in. This displaces more magnetic than electric stored energy and raises the $\beta_0 D = 0$ cutoff frequency above the easitron frequency. The magnetic field at $\beta_0 D = \pi$ clings to the tapes and is little affected by this perturbation, so the π cutoff frequency rises very little.

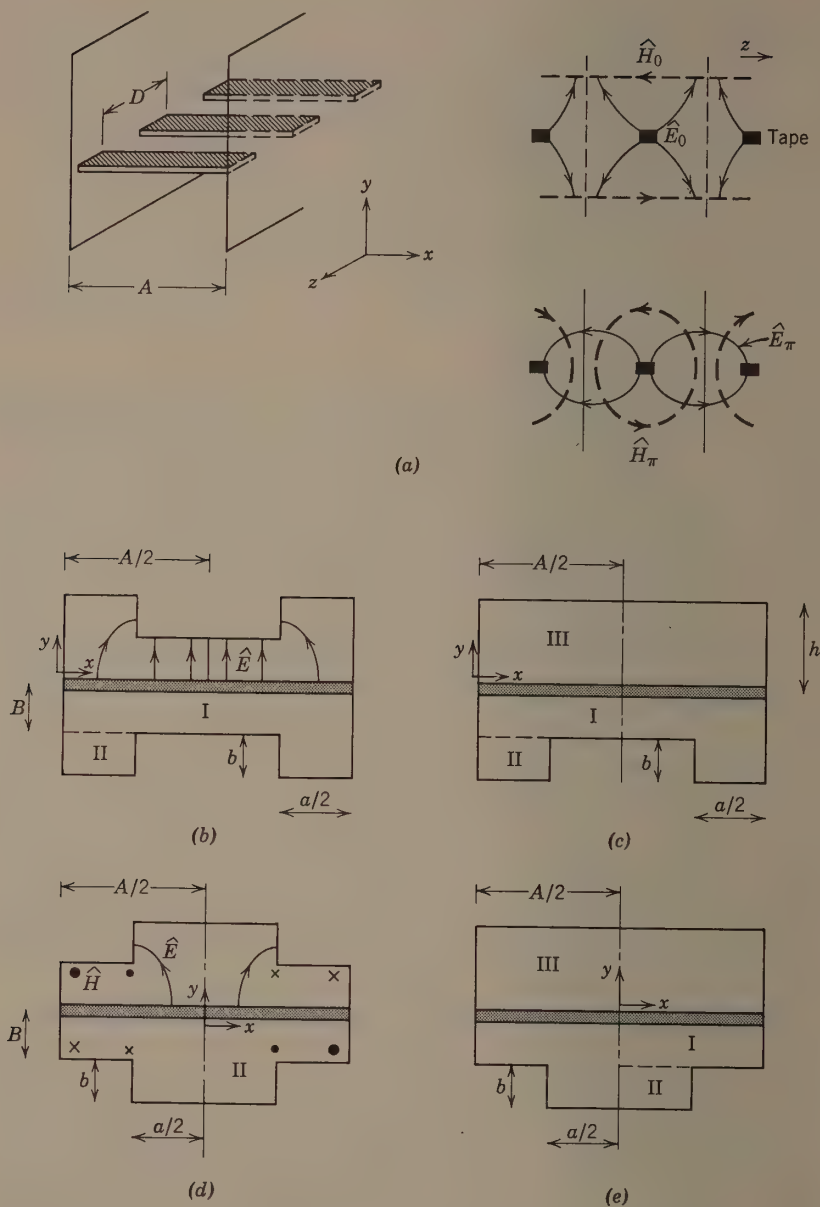


Fig. 2-7. Various tape ladder lines. (a) Basic "easitron" structure, with resonant modes. The $\beta_0 D = 0$ mode resonates between transverse magnetic shorting planes midway between the tapes while the $\beta_0 D = \pi$ mode resonates between electric shorting planes. (b) The double-ridge structure, in which electric field is strongest at the center, $x = A/2$. (c) The single-ridge structure. (d) The double-T line, in which magnetic field is strongest near the edges, $x = 0$ and A . (e) The single-T line.

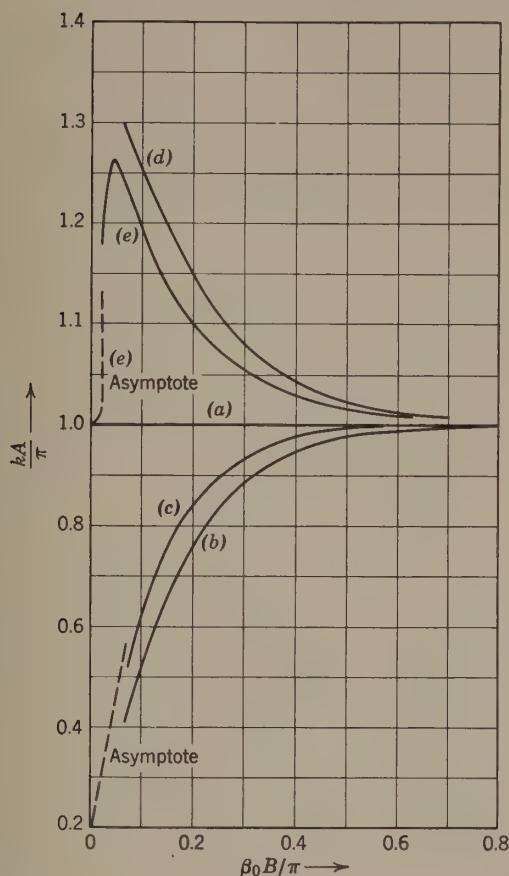


Fig. 2-8. Dispersion curves for the tape ladder lines of Fig. 2-7 (from Butcher [7]). Letters identify each curve with the appropriate structure of Fig. 2-7. Note that the abscissa is proportional to $\beta_0 B$ whereas the periodic phase shift is $\beta_0 D$.

Now we summarize Butcher's analysis for the double and single ridge structures. The fields, all assumed to propagate as $e^{-j\beta_0 z}$, are developed from transverse $U(x, y)$ and $V(x, y)$ functions analogous to the φ and ψ functions, respectively, of Eqs. 1-23 and 1-25:

$$\begin{aligned}
 E_x &= -j\left(\beta_0 \frac{\partial U}{\partial x} + k \frac{\partial V}{\partial y}\right) & Z_0 H_x &= -j\left(\beta_0 \frac{\partial V}{\partial x} - k \frac{\partial U}{\partial y}\right) \\
 E_y &= -j\left(\beta_0 \frac{\partial U}{\partial y} - k \frac{\partial V}{\partial x}\right) & Z_0 H_y &= -j\left(\beta_0 \frac{\partial V}{\partial y} + k \frac{\partial U}{\partial x}\right) \\
 E_z &= (k^2 - \beta_0^2)U & Z_0 H_z &= (k^2 - \beta_0^2)V
 \end{aligned} \quad (2-16)$$

$$Z_0 = \sqrt{\frac{\mu}{\epsilon}}, \quad k = \frac{\omega}{c}$$

U and V each satisfy the equation $\nabla_T^2 U, V + (k^2 - \beta_0^2)U, V = 0$. In region I the potentials are expanded as

$$\begin{aligned} U_I(x, y) &= \sum_r (U_r^e \cosh \gamma_r y + U_r^o \sinh \gamma_r y) \sin \frac{r\pi x}{A} \\ V_I(x, y) &= \sum_r (V_r^e \cosh \gamma_r y + V_r^o \sinh \gamma_r y) \cos \frac{r\pi x}{A} \\ \gamma_r &= \sqrt{\beta_0^2 - k^2 + (r\pi/A)^2} \end{aligned} \quad (2-17)$$

Here r takes the values 1, 3, 5, . . . corresponding to the even symmetry of E_y and odd symmetry of H_z about the $x = A/2$ plane, and U_r^e , etc., are complex amplitudes to be evaluated. In region II the potentials are

$$\begin{aligned} U_{II}(x, y) &= \sum_m U_m' \sinh \gamma_m'(y + B + b) \sin \frac{m\pi x}{a} \\ V_{II}(x, y) &= \sum_m V_m' \cosh \gamma_m'(y + B + b) \cos \frac{m\pi x}{a} \\ \gamma_m' &= \sqrt{\beta_0^2 - k^2 + (m\pi/a)^2} \end{aligned} \quad (2-18)$$

and m takes the values 0, 2, 4, . . . in order that $E_y = E_z = H_x = 0$ on both of the side walls.

The boundary conditions on the plane $y = -B$ demand that both E_{xI} and E_{zI} vanish on the ridge and that E_{xI} and E_{zI} join E_{xII} and E_{zII} on the common boundary, where H_x and H_z must also match. All these conditions stipulate two relations among the preceding coefficients, for which we refer the reader to Butcher for details.

The remaining boundary conditions on the $y = 0$ plane determine the remaining relations which yield a determinantal equation for $k(\beta_0)$. In the *double-ridge* structure, $E_z(y) = E_z(-y)$ while $H_z(y) = -H_z(-y)$, and so U and V must be even and odd functions of y , respectively. Thus, from Eq. 2-16, E_z is continuous at $y = 0$ while $H_x = 0$ there. Butcher's simplifying assumption that $E_x = 0$ at $y = 0$ (better for small $\beta_0 D$), together with the fact that $H_x = 0$ there determine additional relations among the U_r^e and other coefficients. Then a set of homogeneous equations is formed in terms of all the amplitudes and the 2×2 or 4×4 determinant of the matrix of coefficients set equal to zero yields the requisite dispersion relation for this double-ridge structure.

In the *single-ridge* structure the fields of region III of Fig. 2-7c must be matched differently to those of region I. In the former region U and V can be expanded as

$$\begin{aligned} U_{III}(x, y) &= \sum_r U_r'' \sinh \gamma_r(y - h) \sin \frac{r\pi x}{A} \\ V_{III}(x, y) &= \sum_r V_r'' \cosh \gamma_r(y - h) \cos \frac{r\pi x}{A} \end{aligned} \quad (2-19)$$

The term γ_r is given in Eq. 2-17b and $r = 1, 3, 5, \dots$ by the symmetry of E_y about the $x = A/2$ plane. Now we substitute U and V from Eqs. 2-17 and 2-19 into Eq. 2-16, equate to zero individual terms in the series for E_{xI} and E_{xIII} at $y = 0$ and also match term by term the series for E_z and H_x at $y = 0$. The resultant coefficient relations together with the previous ones obtained at the $y = -B$ plane yield a new dispersion relation for this single-ridge structure.

The analysis of the T structures is trivially different, but the peculiar shape of the single- T dispersion curve in Fig. 2-8 warrants especial comment. All these curves are for open-sided structures such that $b = h = \infty$, with no radiation in the y direction. This means $\gamma_0' = \sqrt{\beta_0^2 - k^2}$ of Eq. 2-18b must stay real in the open region II, and so the dispersion curves of both ridged waveguides approach the origin along the $\beta_0 = k$ line. In the *double-T* structure the smallest value of γ_m' in the open region II is

$$\gamma_1' = \sqrt{\beta_0^2 - k^2 + (\pi/a)^2}$$

so that the asymptote $k^2 = \beta_0^2 + (\pi/a)^2$ defines the forbidden region, the $\beta_0 D = 0$ point of which is off scale in Fig. 2-8. In the *single-T* structure, however, it turns out that the smallest transverse propagation constant is

$$\gamma_1 = \sqrt{\beta_0^2 - k^2 + (\pi/A)^2} \quad \text{in region III}$$

This is zero for $\beta_0 = 0$ at $k = \pi/A$, the easitron frequency. As k increases, frequency rises along the asymptote $k^2 = \beta_0^2 + (\pi/A)^2$, then levels off at the transverse resonance point before dipping toward the easitron frequency at $\beta_0 D = \pi$. As the dimensions b and h are decreased from infinity, with the sides closed, the $\beta_0 D = 0$ cutoff point rises steadily until the whole dispersion curve for this structure has negative slope over the range $0 \leq \beta_0 D \leq \pi$.

In conclusion we observe Butcher's remark that the "apparently formidable determinantal equations obtained . . . are surprisingly easy to handle numerically." He also points out that the convergence of the successive approximations to the dispersion relation is extremely rapid and the differences between the simplest 2×2 determinantal equation and the 4×4 equation amount to only several percent. It seems, therefore, that the next improvement in the analysis would come from removing the approximation of $E_x \equiv 0$ between the tapes and including the axial harmonics which propagate as $e^{-j(\beta_0 - 2\pi n/D)z}$. It turns out, however, that each dispersion curve is very slightly affected by these corrections over most of the range $0 \leq \beta_0 B \leq \pi$, provided $D < B$. This statement is plausible because even if D is comparable to B (which it is in practical structures) Fig. 2-8 indicates that the group velocity will be zero at $\beta_0 D = \pi$, as it should be.

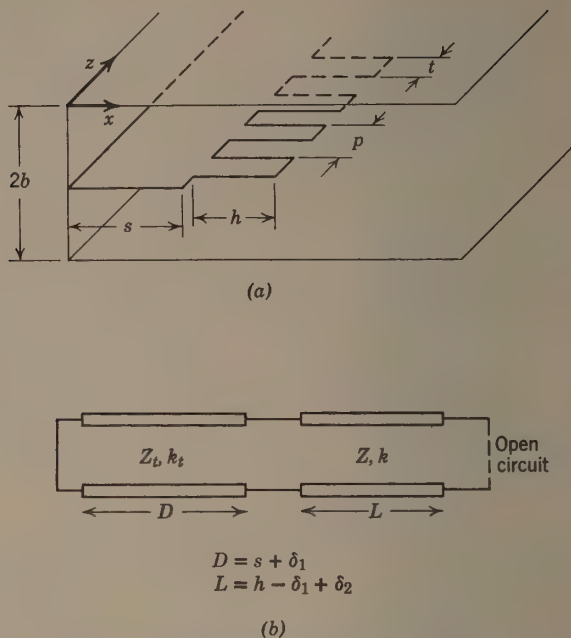


Fig. 2-9. A serrated ridge waveguide (from Oliner and Rotman [8]). (a) Physical structure, open at $x = \infty$, (b) Equivalent circuit, in which D is the electrical distance s and L is the electrical height of the ridges.

The problem of obtaining a compact dispersion relation is much more difficult if the tapes do not extend the full width of the guide. For such trough waveguides, as shown in Fig. 2-9, Oliner and Rotman¹⁰ have obtained a simple yet surprisingly accurate dispersion relation of great utility. In the equivalent circuit shown they equate the TE_{10} mode transverse impedance looking in the $-x$ direction to the impedance looking toward the open end of the guide. The characteristic impedance of the TEM tooth fields in this latter direction is estimated by computing the effective enlargement of the transverse cross section due to the teeth. The resultant dispersion relation is

$$\tan k_t D = \frac{k_t}{k} \left(1 + \frac{p}{\pi b} \ln \csc \frac{\pi t}{2p} \right) \cot kL \quad (2-20)$$

Transverse wave number $k_t = (k^2 - \beta_0^2)^{1/2}$ is real for a fast wave and imaginary for a slow one. When it is imaginary, $\tan k_t D$ is replaced by $\tanh |k_t| D$ and k_t by $|k_t|$. From the figure, D is the dimension s corrected by a term δ_1 and L is the tooth height h corrected by $\delta_2 - \delta_1$. Both δ_1

¹⁰ Oliner and Rotman [8, p. 138].

and $\delta_2 - \delta_1$ are virtually independent of dimensions s , h , and t/p and can be determined from the extremes of one measured dispersion curve. Equation 2-20, then, describes very well the dispersion for a wide range of relative dimensions s/b , h/b , and t/p , as indicated by curves plotted in the reference.

Kirschbaum and Tsu [9] have analyzed a similar structure which they call a serrated-ridge waveguide with all four sides closed. Their field-matching method based upon a variational expression leads to a somewhat longer dispersion relation, which we shall not quote here. It predicts extremely well the relative phase velocity or index of refraction measured in a typical structure.

2-6. The Ridged Waveguide

We wish to analyze the dispersion in the first passband of the ridged waveguide shown in Fig. 2-10 in order to show that such a structure can have a nearly uniform phase velocity over a wide frequency range. The

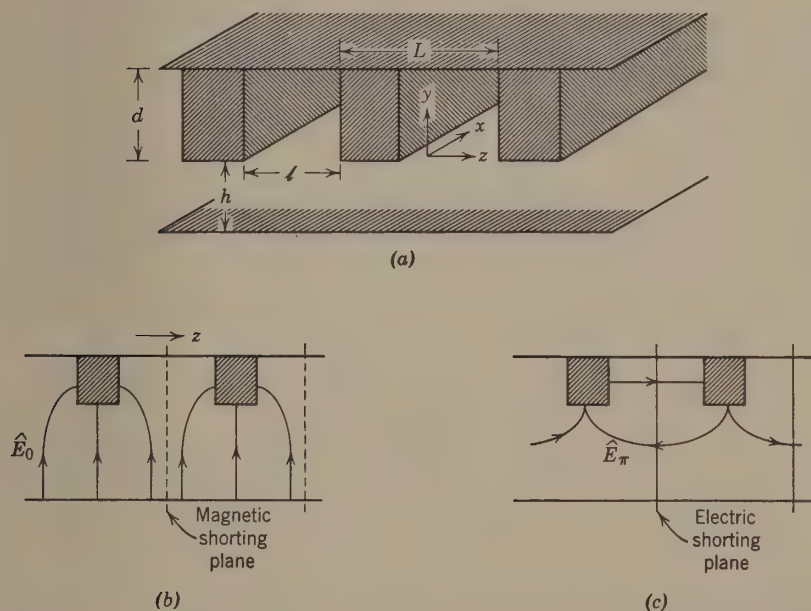


Fig. 2-10. A ridged waveguide open on the sides, $x = \pm\infty$. (a) Physical structure; (b) E field at the zero phase shift frequency, at which each cell resonates between magnetic shorting planes; (c) E field at the π phase shift frequency, at which each cell resonates between electric shorting planes.

fields propagate in the z direction perpendicular to the ridges. The structure will be assumed to extend to infinity in the $\pm x$ directions; we can easily modify the following analysis to account for the presence of the side walls. The modification of the analysis for structures of circular cross section is also straightforward; the very common circular waveguide periodically loaded by circular fins is treated in Section 6-6.

We first take the electric field in the ridged region, $-\ell/2 \leq z \leq \ell/2$, $0 \leq y \leq d$, to be uniformly perpendicular to the teeth,

$$\hat{E}_z = C \sin k(d - y) = C', \quad 0 \leq y \leq d \quad (2-21a)$$

whereupon $\hat{H}$ is purely transverse and varies as

$$\hat{H}_x = C \frac{k}{j\omega\mu} \cos k(d - y) \quad (2-21b)$$

The assumption of constant $\hat{E}_z$ from the edge of one tooth to the next is not a particularly good one, since the field must approach infinity as the field point approaches an edge. However, Eq. 2-21 will suffice for a start and in Section 6-5 we shall account for more of the axial harmonics.

Within the open portion of the structure, $-h \leq y \leq 0$, the ridges can support electric field between their edges and the lower plate at $y = -h$ right down to zero frequency since the structure is open on the sides. The first resonant cavity mode exists at $k = 0$ with zero phase shift per period, between magnetic shorting planes at $z = \pm L/2$, as shown in Fig. 2-10b. As frequency rises, β_0 increases until the structures resonates in its first π mode of Fig. 2-10c, between electric shorting planes at $z = \pm L/2$. Obviously $\hat{E}_y$ and $\hat{E}_z$ fields exist throughout the passband, but $\hat{E}_x$ and therefore $\hat{H}_y \propto \partial \hat{E}_x / \partial z - \partial \hat{E}_z / \partial x$ are both identically zero. It follows from the divergence relation for the magnetic field that $\hat{H}_z$ is also zero. Therefore the fields within the open region of the guide may be expanded with a set of TM harmonics as

$$\begin{aligned} \hat{E}_z &= \sum_{n=-\infty}^{\infty} A_n \sinh \gamma_n(y + h) \epsilon^{-j\beta_n z} \\ -h \leq y \leq 0, \quad \hat{H}_x &= -\sum_n A_n \frac{j\omega\epsilon}{\gamma_n} \cosh \gamma_n(y + h) \epsilon^{-j\beta_n z} \\ \hat{E}_y &= \sum_n A_n \frac{j\beta_n}{\gamma_n} \cosh \gamma_n(y + h) \epsilon^{-j\beta_n z} \end{aligned} \quad (2-22)$$

where $\beta_n = \beta_0 + 2n\pi/L$ and $\gamma_n = +(\beta_n^2 - k^2)^{1/2}$, real for slow waves. These fields obey the boundary conditions on the lower plate at $y = -h$.

We must now match the fields along the periodic surface $-L/2 \leq z \leq L/2$ at $y = 0$. Since we know $E_z = C'$, constant, over $-\ell/2 \leq z \leq \ell/2$ and is zero on the tooth surfaces, we can obtain the A_n coefficients by

equating the two expressions for E_z :

$$\sum_n A_n \sinh \gamma_n h \epsilon^{-j\beta_n z} = \begin{cases} C' & \text{over } -\ell/2 \leq z \leq \ell/2 \\ 0 & \text{on tooth surfaces} \end{cases} \quad (2-23)$$

Multiplying both sides of this equation by $\epsilon^{j\beta_n z}$, integrating from $-L/2$ to $L/2$, and utilizing the orthogonality of the axial harmonics, we obtain A_n as

$$A_n = \frac{C'}{\sinh \gamma_n h} \frac{\sin(\beta_n \ell/2)}{\beta_n L/2} \quad (2-24)$$

Now we must match $\hat{H}_x$ on the surface $-\ell/2 \leq z \leq \ell/2$ and $y = 0$. But this is difficult to do exactly, so we match the complex power flow

$$\int_{-\ell/2}^{\ell/2} \hat{E}_z \times \hat{H}_x^* \cdot \hat{i}_y dS$$

instead. In Section 6-5 we show that the relative error introduced into the frequency, as calculated for a specified value of β_0 , tends to be smaller than the relative errors in $\hat{E}_z$ and $\hat{H}_x$. When we equate power flows we obtain

$$\ell \sin kd \frac{jk}{\omega \mu} \cos kd = j\omega \epsilon L \sin^2 kd \sum_n \frac{1}{\sinh^2(\gamma_n h)} \left[\frac{\sin(\beta_n \ell/2)}{\beta_n L/2} \right]^2 \cdot \sinh \gamma_n h \cosh \gamma_n h \frac{1}{\gamma_n}$$

$$\text{or} \quad \frac{L}{\ell} \left(\frac{\cot kd}{k} \right) = \sum_n \frac{\coth \gamma_n h}{\gamma_n} \left[\frac{\sin(\beta_n \ell/2)}{\beta_n \ell/2} \right]^2 \quad (2-25)$$

We now describe the properties of this determinantal equation for the first passband of the Fig. 2-10 structure, which is infinite in the $\pm x$ direction. As k , β_0 , and γ_0 approach zero only the $n = 0$ term is important on the right side, and we obtain

$$\left(\frac{\partial k}{\partial \beta_0} \right)_{\omega=0} = \left(\frac{Lh}{\ell d + Lh} \right)^{1/2} \quad (2-26)$$

If this slope changes slowly as frequency increases the structure will be inherently broadband. At a frequency such that $0 < \beta_0 L < \pi$ we can solve Eq. 2-25 approximately by letting $\coth \gamma_n h = 1$, $\gamma_n = \beta_n$ for large n and by utilizing this identity:

$$\sum_{n=-\infty}^{\infty} \frac{1}{\beta_n \ell/2} \left[\frac{\sin(\beta_n \ell/2)}{\beta_n \ell/2} \right]^2 \equiv \frac{L/\ell}{\tan(\beta_0 L/2)}, \quad \beta_n = \beta_0 + \frac{2n\pi}{L} \quad (2-27)$$

This identity may be derived by integration in the complex θ plane of the function

$$f(\theta) = \frac{1}{\beta_0 \ell/2 + \theta \ell/L} \left[\frac{\sin(\beta_0 \ell/2 + \theta \ell/L)}{\beta_0 \ell/2 + \theta \ell/L} \right]^2 \frac{1}{\tan \theta} \quad (2-28)$$

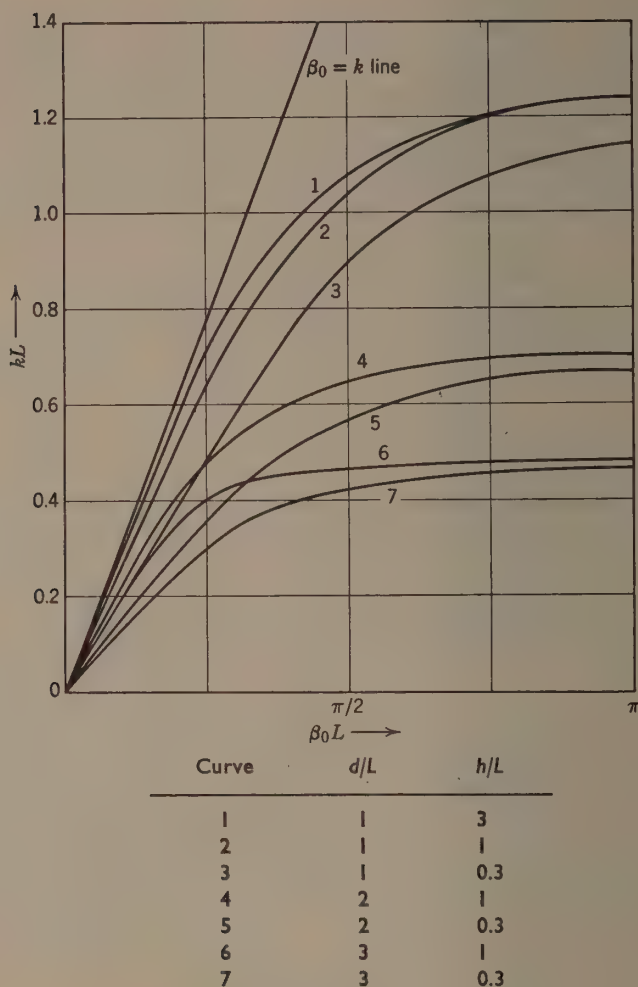


Fig. 2-11. Dispersion curves for the open-sided ridged waveguide of Fig. 2-10. The low-frequency asymptotes are given by Eq. 2-26, and the accurate dispersion relation is Eq. 2-29. Here $\ell/L = \frac{1}{2}$.

which has simple poles at $n\pi$ for all n and also at $-\beta_0 L/2$. To illustrate the use of Eq. 2-27 suppose we wish to write the $n = -1, 0, +1$ terms of Eq. 2-25 exactly and approximate the rest. Then our dispersion equation is

$$\frac{2L}{\ell^2} \left(\frac{\cot kd}{k} \right) = \sum_{s=-1,0,1} \left(\frac{\coth \gamma_s h}{\gamma_s \ell/2} - \frac{1}{\beta_s \ell/2} \right) \left[\frac{\sin(\beta_s \ell/2)}{\beta_s \ell/2} \right]^2 + \frac{L/\ell}{\tan(\beta_0 L/2)} \quad (2-29)$$

Hutter presents a nomograph¹¹ by which we have calculated and plotted the approximate values of k_0L vs. β_0L in Fig. 2-11. These curves have the asymptotes of Eq. 2-26 and are very nearly the curves we would compute from Eq. 2-29. The curves are all for a gap-to-periodic length spacing ℓ/L of one-half. According to them, the smaller the tooth height d on Fig. 2-10 the greater the bandwidth. For a given tooth height d , the smaller the open spacing h the less the phase velocity over the nearly linear portion of the curve. These two parameters can be adjusted nearly independently of each other.

We shall not discuss this ridged waveguide analysis any further now because it does neglect the effect of side walls in Fig. 2-10. However, the essentials of this elementary analysis should be clear. It is apparent that such a structure can be designed for nearly constant phase velocity over an appreciable fraction of its passband width and over a considerable range of phase velocities. The higher phase velocity curves are the more linear ones and require, of course, synchronous beams of d.c. velocities which may be appreciable compared to light velocity.

Problems

1. Verify the fact that each n th mode of Eq. 2-2 obeys Maxwell's equations and that e_{zn} and h_{zn} each obeys the vector wave equation.
2. Obtain the asymptotic expressions for the fields of the sheath helix $n = 0$ and $n = 1$ waves, as $ka \rightarrow 0$. Use Eq. 2-11 after verifying that the fields have the proper continuities at $r = a$.
3. Verify an impedance of about 45 ohms for the $n = +1$ sheath wave at $ka = 0.5$ near point A of Fig. 2-4. Take $\cot \psi = 10$ so $\gamma a \simeq \beta a \simeq 5$. The field e_z is measured at $r = a$. The integrals in the power flow expression can be evaluated with the aid of Table 6-1. The impedance depends upon $\omega\mu a$; assume it has the value 150.
4. Discuss the reinterpretation of Eq. 1-4 and 1-6 and the dispersion curves of Fig. 2-4 if an extra winding is inserted into a unifilar helix so as to make it a bifilar helix of the same pitch angle ψ .
5. Derive a dispersion relation for the unifilar tape helix wave of predominant $n = 0$ mode by assuming a current $\hat{K}_{||}$ parallel to and uniform across the narrow tape. Let the current determine the relative amplitudes of the $n = -1$, 0, and $+1$ modes, whereupon the dispersion relation obtains by setting $\hat{e}_{||} = 0$ on the centerline of the tape. The boundary conditions at $r = a$ include $-\hat{H}_{\perp} + \hat{h}_{\perp} = \hat{K}_{||}$, which yields the Fourier components of $\hat{K}_{||}$ in terms of the fields.
6. Consider the axial impedance of the $q = 0$ wave with predominant $n = 0$ mode in a multifilar helix. Show that the impedance as given by the $n = 0$

¹¹ R. G. E. Hutter [5, p. 104].

mode alone is precisely that of the $n = 0$ sheath wave, provided the dispersion curve of the $q = 0$ wave is essentially that of the sheath wave. The measured reduction in impedance is due to the presence of the $n = \dots, -2N, -N, N, 2N, \dots$ modes in this wave.

7. Verify the qualitative nature of the ω - β curve for the single- T structure of Fig. 2-8e by deriving and studying the dispersion relation.
8. In later chapters we shall show that the passband width for the easitron structure of Fig. 2-7a is proportional to

$$\int_S \mathbf{E}_0 \times \mathbf{H}_\pi \cdot \hat{\mathbf{i}}_z dS$$

on any one of the transverse bisecting planes shown. Show that the passband width is identically zero. Why does shortening adjacent ends of the rods create finite passband width?

9. Verify curve 3, for example, of Fig. 2-11, for the ridged waveguide of Fig. 2-10 from dispersion Eq. 2-29.

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Chapter III

Radiation and Cavity Coupling through Small Holes and Apertures

3-1. Synopsis¹

The solution to Maxwell's equations within a cavity and the dispersion relation for the ω - β_0 curves depend upon the tangential electric or magnetic field on the coupling surfaces. In this chapter we obtain expressions for these tangential fields and their integrals over circular and elliptical holes and apertures of various shapes.

We first review the Bethe theory of diffraction through round holes which are small in the sense that $(ka)^2 \ll 1$, a being the hole radius. Then we develop more accurate expressions for $\hat{E}_{\text{tan}}$ appropriate for holes of $ka \simeq 1$ on the axis of a centerhole-coupled cavity chain. The more general case of arbitrary excitation impinging on a round hole of $ka \lesssim 1$ has been treated by W. Eggimann; we quote the expressions for $\hat{E}_{\text{tan}}$ derivable from his work. Rather than extend the Bethe theory for elliptical holes we present expressions derived by R. Collin for the polarizabilities. N. Marcuvitz' work is directly applicable to a study of the aperture coupling between two cavities composed of sections of waveguide. He has presented expressions for the integral

$$\int_{\text{ap}} \hat{E}_{\text{tan}} \times \hat{h}_{1t} \cdot \hat{i}_z dS$$

for a variety of apertures, where $\hat{h}_{1t}$ is the first TE or TM waveguide mode transverse magnetic field pattern and $\hat{E}_{\text{tan}}$ is the electric field in the

¹ The reader who is uninterested in the details of calculating tangential electric and magnetic fields on a coupling surface between cavities may skip immediately to Section 3-9, which deals with the solution of Maxwell's equations within a cavity.

plane of the aperture. We shall see that this integral is closely related to the coupling integral descriptive of the first passband of a slow wave structure composed of a chain of aperture-coupled cavities.

After obtaining expressions for $\hat{E}_{\text{tan}}$ on coupling holes and apertures we proceed to derive the field equations for two cavities coupled through a hole or aperture. These equations are cast into a form in which the short-circuit resonant cavity mode (defined with the holes electrically shorted) is excited within each cavity by the amplitudes of an *open*-circuit resonant mode (defined with the holes *magnetically* shorted) on the coupling surface. This latter resonant mode was introduced in Section 1-7 (see Eq. 1-22). It represents the physical field which resonates at the other resonant frequency of the two-cavity system.

The equation for the two new resonant frequencies of two coupled cavities involves a coupling integral which can be evaluated by means of the expressions for $\hat{E}_{\text{tan}}$ and

$$\int_{\text{ap}} \hat{E}_{\text{tan}} \times \hat{h}_{1t} \cdot \hat{i}_z dS$$

obtained for various holes and apertures in this chapter. The equation for resonant frequencies is cast into a form which will be generalized in Chapter 4 to describe cavities coupled to their neighbors across coupling surfaces of *arbitrary* shapes in a periodic chain. The significance of the coupling integral lies in the fact that it is very nearly the width of a narrow passband in units of $k = 2\pi/\lambda$.

3-2. Objectives

In Section 1-7 we showed how we can predict qualitatively the shapes of the ω - β curves for a centerhole-coupled cavity chain from knowledge of the frequencies of the cavity resonances at zero and at π phase shift per cavity. One set of resonant cavity modes was defined with electric shorting planes on the coupling holes and the other set—the so-called “open-circuit” modes of which Fig. 1-6c is an example—was defined with magnetic shorting planes on the holes. In order to determine the dispersion relation for a narrow passband of any slow wave structure (Chapter 4) we shall have to know the tangential electric field on each coupling hole. If the hole is small in the sense that $k(\text{hole radius}) < 1$ and if it does not lie too near the waveguide wall, we may proceed to calculate this tangential field from the perturbation of one of the short-circuit modes. That is, when a small hole is cut in the presence of normal electric and tangential magnetic field of a short-circuit mode the field fringes and creates some

tangential electric field on the hole. The work in Section 3-9 will show us, however, that we are actually calculating the tangential electric field of an open-circuit mode by such a perturbation procedure. Once we realize this we may calculate the tangential electric field pattern of this *resonant* mode by any procedure we please, both for large and small holes.

We first assume that a small round hole of radius a is cut in an infinitesimally thin screen between two cavities, Fig. 3-1. We discard Bethe's coordinate system in favor of the more conventional one but retain his cgs units and $\epsilon^{-i\omega t}$ time dependence for Section 3-3 only. The hole does not, in general, lie on the axis of a pair of cylindrical cavities, and the shape of the cavities is irrelevant to the discussion of the near-hole fields. However, we assume along with Bethe that the error fields on the waveguide walls are negligible; that is, the screen acts as if it were of infinite cross section. Only the near-fields $\hat{E}_{0\text{norm}}$ and $\hat{H}_{0\text{tan}}$ in cavity (1) which are present before the hole is cut determine the tangential electric field on the hole *after* it is cut. We are considering small holes of $ka < 1$; for example, at 3 kMc a hole with an 0.5-cm radius has a ka of 0.31.

After the review of Bethe's method of computing tangential electric field we shall extend it for centerholes of cylindrical cavities so as to include the next higher-order terms in $(ka)^2$. Then we return to the general problem of determining the tangential electric field on a hole from arbitrary incident fields. We can consider two cases of the general problem: that of symmetrical $\hat{H}_{\text{tan}}$ and antisymmetrical $\hat{E}_{\text{norm}}$ across the hole, and the dual case. Eggimann has treated both these cases for us.

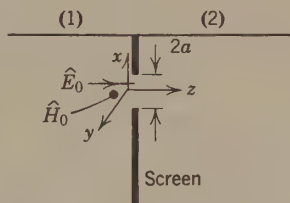


Fig. 3-1. Incident fields $\hat{E}_0$ and $\hat{H}_0$ impinging upon a hole in a waveguide partition.

3-3. Review of the Bethe Theory

Bethe² begins by assuming the presence of incident fields $\hat{E}_0$ and $\hat{H}_0$ on the hole in cavity (1) of Fig. 3-1 before the hole is cut. No fields are initially present in cavity (2). For us, $\hat{E}_0$ and $\hat{H}_0$ are the fields of a short-circuit resonant cavity mode. Either $\hat{E}_0$ or $\hat{H}_0$ or both are taken to be uniform over the hole; we neglect that portion of $\hat{E}_0$ whose curl generates part of $\hat{H}_0$ and that portion of $\hat{H}_0$ whose curl generates part of $\hat{E}_0$. After

² H. A. Bethe [1]. We retain his cgs units and $\epsilon^{-i\omega t}$ time-dependence in order to facilitate a review of this basic article. The general formulas for conversion to mks units with $\epsilon^{j\omega t}$ dependence are on page 71.

the hole is cut the total fields in the cavities are denoted by

$$\begin{aligned} \hat{E}_0 + \hat{E}_1 \quad \text{and} \quad \hat{H}_0 + \hat{H}_1 \quad \text{in cavity (1),} \quad z \leq 0 \\ \hat{E}_2 \quad \text{and} \quad \hat{H}_2 \quad \text{in cavity (2),} \quad z \geq 0 \end{aligned} \quad (3-1)$$

Here $\hat{E}_1$ and $\hat{H}_1$ are the reflected fields; $\hat{E}_2$ and $\hat{H}_2$ are the diffracted fields. The fields must satisfy the boundary conditions

$$\begin{aligned} \hat{E}_{1 \tan} &= \hat{E}_{2 \tan} && \text{on hole surface} \\ \hat{H}_0 + \hat{H}_{1 \tan} &= \hat{H}_{2 \tan} && \text{on hole surface} \\ \hat{E}_{1 \tan} &= \hat{E}_{2 \tan} = 0 && \text{on screen surface} \end{aligned} \quad (3-2)$$

The total fields represent a solution to Maxwell's equations (at a frequency essentially that of the short-circuit mode) in which transverse $\hat{E}_{x,y}$ and normal $\hat{H}_z$ are symmetrical about the plane of the screen. Bethe deduces, therefore,

$$\begin{aligned} \hat{E}_{2 \text{ norm}} &= \hat{E}_{2z} = \frac{1}{2} \hat{E}_0 && \text{on } +z \text{ side of hole} \\ \hat{H}_{2 \tan} &= \frac{1}{2} \hat{H}_0; \\ \hat{E}_{1z} &= -\frac{1}{2} \hat{E}_0 && \text{on } -z \text{ side of hole} \\ \hat{H}_{1 \tan} &= -\frac{1}{2} \hat{H}_0 \end{aligned} \quad (3-3)$$

We can see in Fig. 3-2 that despite the reversals of the reflected and diffracted fields across the hole the total fields, Eq. 3-1, are continuous there. With the normal electric and tangential magnetic fields completely known on the hole, we must now specify a surface magnetic current and charge distribution which will create the known $\hat{H}_{2 \tan} = \frac{1}{2} \hat{H}_0$ on the $+z$ side of the hole. Once the magnetic current is known it specifies $\hat{E}_{2 \tan}$ directly. It will be a consequence of the formulas that the reflected fields $\hat{E}_1$ and $\hat{H}_1$ in the $z < 0$ region [cavity (1)] are determined by the negatives

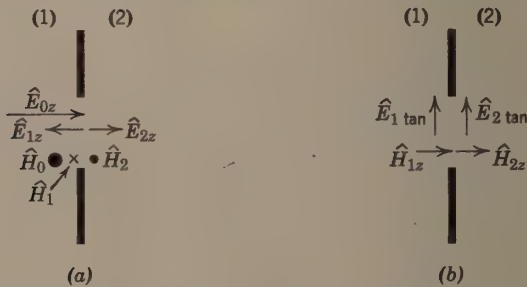


Fig. 3-2. Incident, reflected, and refracted fields at the hole. (a) Normal $\hat{E}$ and tangential $\hat{H}$ components which imply continuity of the total fields, (b) tangential $\hat{E}$ and normal $\hat{H}$ components which imply continuity of the total fields.

of the surface current and charge just specified. However, $\hat{E}_{1 \tan} = \hat{E}_{2 \tan}$, Fig. 3-2b, as of course it should be. From now on we omit the irrelevant subscript to $\hat{E}_{\tan}$.

The mathematical determination of the proper magnetic current and charge on the hole proceeds as follows. Maxwell's equations for cavity (2) in the cgs system of units are

$$\nabla \times \hat{H}_2 = \frac{1}{c} \frac{\partial \hat{E}_2}{\partial t} = -ik\hat{E}_2 \quad (3-4a)$$

$$\nabla \cdot \hat{E}_2 = 0 \quad (3-4b)$$

$$\nabla \times \hat{E}_2 + \frac{1}{c} \frac{\partial \hat{H}_2}{\partial t} = \nabla \times \hat{E}_2 - ik\hat{H}_2 = -4\pi\hat{K} \quad (3-4c)$$

$$\nabla \cdot \hat{H}_2 = 4\pi\eta, \quad k = \frac{\omega}{c} \quad (3-4d)$$

where Bethe takes the $\epsilon^{-i\omega t}$ time dependence, and magnetic surface current $\hat{K}$ is related to charge density η as $\nabla \cdot \hat{K} = ik\eta$. The first two Maxwell equations are implied by the definitions of $\hat{E}_2$ and $\hat{H}_2$ relative to scalar potential ψ and vector potential $\hat{F}$ as

$$\begin{aligned} \hat{E}_2 &= \nabla \times \hat{F} \\ \hat{H}_2 &= \frac{1}{c} \frac{\partial \hat{F}}{\partial t} - \nabla\psi = -ik\hat{F} - \nabla\psi \end{aligned} \quad (3-5)$$

But now the latter two of Eqs. 3-4 in terms of $\hat{F}$ and ψ are entirely analogous to the equations for the ordinary magnetic vector potential and electric scalar potential so the solutions must be

$$\begin{aligned} \hat{F}(\bar{r}) &= - \int_{\text{hole}} \hat{K}(\bar{r}') G |\bar{r} - \bar{r}'| dS' \\ \psi(\bar{r}) &= \int_{\text{hole}} \eta(\bar{r}') G |\bar{r} - \bar{r}'| dS' \end{aligned} \quad (3-6)$$

where $\bar{r}$ is the vector position of a field point in cavity (2) as measured from the center of the hole, $\bar{r}'$ is a source point on the hole, and G is the Green's function

$$G |\bar{r} - \bar{r}'| = \frac{\exp(ik |\bar{r} - \bar{r}'|)}{|\bar{r} - \bar{r}'|}$$

We observe that $\hat{F}(\bar{r})$ at a point on the screen is entirely in the plane of the screen so that $\hat{E}_2$ is normal to the screen as it must be.

The problem, then, is to specify $\hat{K}(\bar{r}')$ and $\eta(\bar{r}')$ on the hole so that

$$\begin{aligned} \hat{E}_{2z}(\bar{r}_{\text{hole}}) &= \frac{1}{2} \hat{E}_0 = \nabla_r \times \hat{F}(\bar{r}_{\text{hole}}) \\ &= + \int_{\text{hole}} \hat{K}(\bar{r}') \times \nabla_r G |\bar{r}_{\text{hole}} - \bar{r}'| dS' \end{aligned} \quad (3-7a)$$

and

$$\hat{H}_{2 \tan}(\bar{r}_{\text{hole}}) = \frac{1}{2} \hat{H}_0 = ik \int_{\text{hole}} \hat{K}(\bar{r}') G dS' - \nabla_r \int_{\text{hole}} \eta(r') G dS' \quad (3-7b)$$

The subscript r on ∇ indicates a gradient taken at the field point $\bar{r}_{\text{hole}}$ on the hole. Observe that the right sides of Eqs. 3-7 are actually normal and tangential, respectively, to the hole. There is no contribution to Eqs. 3-7 from the fields on the screen.

Bethe next chooses a component of current $\hat{K}_H$ with related magnetic charge density $\eta = \nabla \cdot \hat{K}_H / ik$, in order to satisfy Eq. 3-7b. The first integral is about $(ka)^2$ of the second since $\hat{K} \approx \eta ka$ and $\nabla_r G \approx G/a$. Then if we neglect the retardation in the hole we have, from Eq. 3-7b,

$$\frac{1}{2} \hat{H}_0 = -\nabla_r \int_{\text{hole}} \frac{\eta(r') dS'}{|\bar{r}_{\text{hole}} - \bar{r}'|} \quad (3-8)$$

This equation is satisfied at each point $\bar{r}_{\text{hole}}$ provided

$$\eta(\bar{r}') = -\frac{1}{\pi^2} \hat{H}_0 \cdot \bar{r}' (a^2 - r'^2)^{-1/2} \quad (3-9a)$$

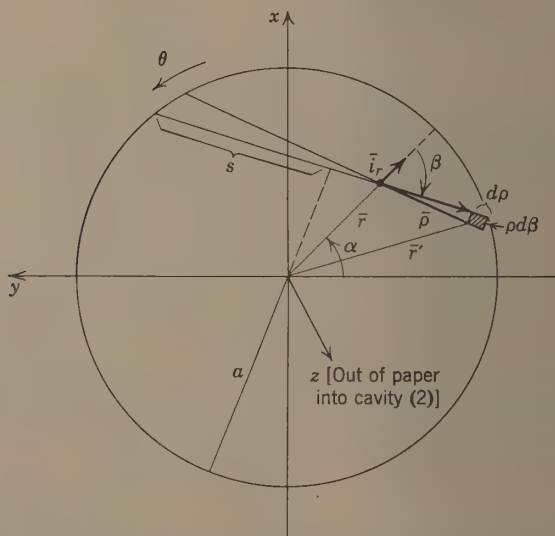


Fig. 3-3. Geometry and coordinates for calculations of field components on the hole. Bethe's original axes have been relabeled so that z is perpendicular to the hole. Here

$$\begin{aligned} \bar{\rho} &= \bar{r} - \bar{r}', & \rho &= \chi - r \cos \beta \\ a^2 - r'^2 &= s^2 - \chi^2, & s^2 &= a^2 - r^2 \sin^2 \beta \\ y' &= r \sin \alpha + \rho \sin(\alpha - \beta), & -z &= r \cos \alpha + \rho \cos(\alpha - \beta) \\ \bar{i}_0 r &= \bar{i}_y x - \bar{i}_x y, & \bar{r}' \times \bar{E}_{0z} &= (\bar{i}_x y' - \bar{i}_y x') \bar{E}_{0z} \end{aligned}$$

so that the related current may be taken to be

$$\hat{K}_H(\bar{r}') = \frac{ik}{\pi^2} (a^2 - r'^2)^{1/2} \hat{H}_0 \quad (3-9b)$$

Bethe proves that Eq. 3-9 is correct for Eq. 3-8 by means of an ingenious coordinate system which removes the troublesome $|\bar{r} - \bar{r}'|^{-1}$ factor from the integrands. Since we have used this coordinate system in the extension of Bethe's work we show it in Fig. 3-3. An explanation follows.

We observe the hole in Fig. 3-3 from the cavity (2) or $+z$ side of Fig. 3-1. The field point on the hole is $\bar{r}$ (which stands for $\bar{r}_{\text{hole}}$) at an angle α defined as positive in the direction shown. A typical source point lies at $\bar{r}'$. The rule for defining the coordinates of $\bar{r}'$ relative to $\bar{r}$ is this: Rotate the unit vector $\bar{i}_r$ clockwise around point $\bar{r}$ until it points at $\bar{r}'$ and then move along $\bar{i}_r$ in its rotated position away from $\bar{r}$ a distance ρ which lies in the range $-(s + r \cos \beta) \leq \rho \leq s - r \cos \beta$ to point $\bar{r}'$. Coordinate $\rho + r \cos \beta$ is conveniently defined as χ (not to be confused with coordinate x in the hole plane), and we see that χ varies between limits $-s, s$, where $s^2 = a^2 - r^2 \sin^2 \beta$. Other coordinate relations appear in the legend of Fig. 3-3; $\bar{r}$ and $\bar{r}'$ may each lie anywhere on the hole.

In this coordinate system we see that the integral in Eq. 3-8, with Eq. 3-9a, evaluates as

$$\begin{aligned} & \int_{\text{hole}} \frac{\eta(\bar{r}') dS'}{|\bar{r} - \bar{r}'|} \\ &= -\frac{1}{\pi^2} \int_0^\pi d\beta \int_{-s(\beta)}^{s(\beta)} \frac{\hat{H}_0 \cdot \bar{r} + H_0 \rho \cos(\alpha - \beta)}{(a^2 - r'^2)^{1/2}} \frac{1}{|\rho|} |\rho| d\rho \\ &= -\frac{1}{\pi^2} \int_0^\pi d\beta \int_{-s(\beta)}^{s(\beta)} \frac{\hat{H}_0 \cdot \bar{r} + H_0(\chi - r \cos \beta) \cos(\alpha - \beta)}{(s^2 - \chi^2)^{1/2}} d\chi \quad (3-10) \end{aligned}$$

We must add that $\hat{H}_0$ is regarded as positive in the $-y$ direction on Fig. 3-3. Evaluation involves integrals no worse than

$$\int_{-s}^s d\chi (s^2 - \chi^2)^{-1/2} = \pi \quad \text{and} \quad \int_0^\pi \cos^2 \beta d\beta = \frac{\pi}{2}$$

The truth of Eq. 3-8 follows.

In this way Bethe chooses a magnetic charge density on the hole to create the proper $\hat{H}_{2 \tan}$ on the $+z$ side of the hole. The correct $\hat{K}_H$ creates a term in Eq. 3-7a on the order of $(ka)^2 \hat{H}_0$. This is *presumed* negligible as compared to a large uniform component of $\hat{E}_{2z} = \frac{1}{2} \hat{E}_{0z}$ on the hole. Therefore he chooses another magnetic current $\hat{K}_E$ to establish the proper $\hat{E}_{2z}$ on the hole without disturbing the proper $\hat{H}_{2 \tan}$ given by

Eq. 3-7b. This is accomplished by means of a circulating current with no divergence and hence no charge density:

$$\hat{K}_E(\bar{r}') = \frac{1}{2\pi^2} (a^2 - r'^2)^{-1/2} \bar{r}' \times \hat{E}_0 \quad (3-11)$$

We may substitute Eq. 3-11 into Eq. 3-7a, evaluate the integral by means of the coordinate relations shown with Fig. 3-3, and verify that Eq. 3-7a is satisfied. The error term $ik \int \hat{K}_E(\bar{r}') G dS'$ in Eq. 3-7b may be shown to be on the order of $(ka)^2 E_{0z}$ which is *presumed* much smaller than $\frac{1}{2} \hat{H}_{0 \tan}$.

The total magnetic surface current $\hat{K}_H + \hat{K}_E$ on the hole is now given by Eqs. 3-9 and 3-11. A question arises whether contour currents or charges are required on the hole edge. Meixner² has shown that none are required if the magnetic current is parallel to the hole edge. Bethe says that the tangential electric field on the hole, $\hat{E}_{\tan}$, is given by $2\pi \bar{i}_z \times \hat{K}_{\text{tot}}$ which is precisely the limit of $\nabla \times \hat{F}$, with $\hat{F}$ given by Eq. 3-6, as $\bar{r} \rightarrow \bar{r}_{\text{hole}}$. Let us compare this result with that obtained from Stratton's formulas³ (with his $\bar{r}$, $\bar{r}'$, $\bar{n}$, and ∇ changed to $\bar{r}'$, $\bar{r}$, $-\bar{i}_z$, and $-\nabla_r$, respectively, and with $\omega\mu$ and $\omega\epsilon$ changed to k) for the diffracted fields at a point $\bar{r}$ in cavity (2). In the limit as $\bar{r} \rightarrow \bar{r}_{\text{hole}}$ these formulas yield directly,⁴ for the tangential components,

$$\hat{E}_2(\bar{r}_{\text{hole}}) = \frac{1}{4\pi} \int_{\text{sheet} + \text{hole}} [ik \bar{i}_z \times \hat{H}(\bar{r}') G |\bar{r}_{\text{hole}} - \bar{r}'| - \bar{i}_z \cdot \hat{E}(\bar{r}') \nabla_r G] dS' + \frac{1}{2} \hat{E}_{\tan}(\bar{r}_{\text{hole}}) \quad (3-12a)$$

$$\hat{H}_2(\bar{r}_{\text{hole}}) = \frac{1}{2} \hat{H}_{\tan}(\bar{r}_{\text{hole}}) - \frac{1}{4\pi} \int_{\text{sheet} + \text{hole}} [ik \bar{i}_z \times \hat{E}(\bar{r}') G + (\bar{i}_z \cdot \hat{H}(\bar{r}')) \nabla_r G] dS' \quad (3-12b)$$

Therefore we obtain the interesting result that the sheet fields and hole fields away from a given point on the hole create precisely $\frac{1}{2} \hat{E}_{\tan}$ and $\frac{1}{2} \hat{H}_{\tan}$ at that point.

The expression

$$\hat{E}_{\tan}(\bar{r}_{\text{hole}}) = 2\pi \bar{i}_z \times \hat{K}(\bar{r}_{\text{hole}}) \quad (3-13)$$

implies a discontinuity in $\hat{E}_{\tan}$ across the hole, as evidenced by Fig. 3-4, in order that $\nabla \times \hat{E}_{\tan} = -4\pi(\hat{K}_H + \hat{K}_E)$ be obeyed. We recall that the equation $\nabla \times \hat{E}_{2z} - ik \hat{H}_{2 \tan} = 0$ has already been satisfied by the

³ J. A. Stratton [3, p. 466].

⁴ We can say that the magnetic current and charge which establish the correct $\hat{H}_{\tan}$ alone on the hole determine uniquely the fields in cavity (2); in particular, $E_{\tan}$ on the hole. The author acknowledges interesting discussions with Professor W. J. Welch of the Electrical Engineering Department, University of California, Berkeley, on this point.

incident fields in Eq. 3-7. Therefore the complete Eq. 3-4c is satisfied on the hole as well as inside cavity (2). Lastly, Eq. 3-4d is true on the hole in the sense that charge density η establishes the correct $\hat{H}_{2z}$ there. The $\hat{E}_{2 \tan}$ shown in cavity (1) of Fig. 3-4 has no physical meaning; the true $\hat{E}_{\tan}$ is continuous across the hole.

To obtain, now, the true reflected fields $\hat{E}_1$ and $\hat{H}_1$ in cavity (1) we have only to reverse $\hat{K}_H + \hat{K}_E$ and change the sign of η in Eq. 3-7, according to Eq. 3-3. However, $\hat{E}_{1 \tan} = \hat{E}_{2 \tan}$ so we have already found the correct $\hat{E}_{\tan}$. It is

$$\begin{aligned} \hat{E}_{\tan}(\bar{r}_{\text{hole}}) &= 2\pi \bar{i}_z \times \hat{K}_{\text{tot}} = \frac{1}{\pi} \bar{r}_{\text{hole}}(a^2 - r_{\text{hole}}^2)^{-1/2} \hat{E}_{0z} \\ &+ \frac{2ik}{\pi} (a^2 - r_{\text{hole}}^2)^{1/2} \bar{i}_z \times \hat{H}_{0 \tan} \quad (3-14) \end{aligned}$$

Equation 3-14 is Bethe's expression for a small round hole, as due to incident fields $\hat{E}_{0z}$ and $\hat{H}_{0 \tan}$ in cavity (1) of Fig. 3-1.

To convert Bethe's expressions in the cgs system into the mks system we make the substitutions

$$\begin{aligned} \hat{H}_{\text{cgs}} &= -\sqrt{\mu} \hat{H}_{\text{mks}}, & \text{Magnetic } \eta_{\text{cgs}} &= -\frac{1}{4\pi\sqrt{\mu}} \eta_{\text{mks}} \\ \hat{E}_{\text{cgs}} &= \sqrt{\epsilon} \hat{E}_{\text{mks}}, & \text{Electric } \eta_{\text{cgs}} &= +\frac{1}{4\pi\sqrt{\epsilon}} \eta_{\text{mks}} \\ \text{Magnetic } \hat{K}_{\text{cgs}} &= \frac{\sqrt{\epsilon}}{4\pi} \hat{K}_{\text{mks}}, & \mu &= 4\pi \times 10^{-7} \\ \text{Electric } \hat{K}_{\text{cgs}} &= -\frac{\sqrt{\mu}}{4\pi} \hat{K}_{\text{mks}}, & \epsilon &= \frac{1}{36\pi} \times 10^{-9} \end{aligned} \quad (3-15)$$

When these are substituted into Maxwell's and the continuity equation in cgs units with $\epsilon^{-i\omega t}$ dependence and with k equal to ω/c they go over to the corresponding equations in mks units with $\epsilon^{i\omega t}$ dependence. For example, Eq. 3-14 in mks units is now

$$\begin{aligned} \sqrt{\epsilon} \hat{E}_{\tan, \text{mks}}(\bar{r}_{\text{hole}}) &= \frac{1}{\pi} \frac{\bar{r}_{\text{hole}}}{\sqrt{a^2 - r_{\text{hole}}^2}} \sqrt{\epsilon} \hat{E}_{0 \text{mks}} \\ &- \frac{2ik}{\pi} (a^2 - r_{\text{hole}}^2)^{1/2} \bar{i}_z \times \sqrt{\mu} \hat{H}_{\tan, \text{mks}} \quad (3-16) \end{aligned}$$

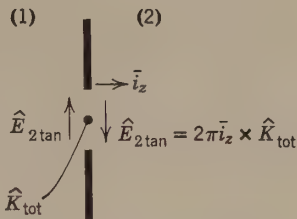


Fig. 3-4. Tangential $\hat{E}$ field in relation to magnetic surface current K on the hole.

In retrospect, it is apparent that Bethe is endeavoring to solve two problems at once. Let us break up each diffracted field into two portions such that $\hat{E}_2$ and $\hat{H}_2$ are the uniform components and $\hat{E}_2'$, $\hat{H}_2'$ the linearly-varying portions. Charge density η and current $\hat{K}_H$ were chosen to satisfy the equation $\nabla \times \hat{E}_{2z}' - ik\hat{H}_{2\tan} = 0$ (with $\nabla \times \hat{E}_{2\tan} = -4\pi\hat{K}_H$ satisfied indirectly) and with $\nabla \times \hat{H}_2 \equiv 0$ because $\frac{1}{2}\hat{H}_0$ is uniform over the hole. Circulating current $\hat{K}_E$ was selected to satisfy $\nabla \times \hat{H}_2' = -ik\hat{E}_2$ on the hole, with $\nabla \times \hat{E}_{2z} \equiv 0$ (and $\nabla \times \hat{E}_{2\tan} = -4\pi\hat{K}_E$ satisfied indirectly) because $\frac{1}{2}\hat{E}_0$ is uniform over the hole. If we wish to account carefully for the corrections to $\hat{K}_{\text{tot}}$ on the order of $(ka)^2$ of Bethe's terms we should consider two separate cases: that of uniform $\hat{H}_0$ and inflecting $\hat{E}_0$ and that of uniform $\hat{E}_0$ and inflecting $\hat{H}_0$. We shall discuss these cases after considering again the very common centerhole-coupled cavity chain.

3-4. The Centerhole-Coupled Cavity Chain

We first extend Bethe's work for the common centerhole-coupled chain with the hole on the axis of a cylindrical cavity in Fig. 3-1. Expressions in this section and the remainder of the book are in mks units, unless otherwise stated, with the $e^{j\omega t}$ time dependence. Let the incident fields in cavity (1) be those of the lowest TM circular waveguide mode (the "straight-through" mode of Fig. 1-6*b*); no fields are initially present in cavity (2), and we again neglect the fields on the waveguide walls. The component $\hat{E}_0$ is proportional to the zero-order Bessel function $J_0(k_c r)$ and $\hat{H}_0$ to $J_1(k_c r)$, k_c being the cutoff or resonant frequency of this mode.⁵ We approximate them in a narrow frequency range just above k_c as

$$\begin{aligned}\hat{E}_0 &= i_z V_0 \bar{E}_0, & \bar{E}_0 &= \sqrt{3.71} J_0(k_c r) \simeq \sqrt{3.71} \left[1 - \frac{(k_c r)^2}{4} \right] \\ \hat{H}_0 &= i_\theta I_0 \bar{H}_0, & I_0 &= \frac{j\omega \varepsilon V_0}{k_c}, \\ \bar{H}_0 &= \sqrt{3.71} J_1(k_c r) \simeq \sqrt{3.71} \frac{1}{2} k_c r \left[1 - \frac{(k_c r)^2}{8} \right]\end{aligned}\tag{3-17}$$

Mode patterns $\bar{E}_0$ and $\bar{H}_0$ are written this way so as to be normalized:

$$\int_V \bar{E}_0^2 dV = \int_V \bar{H}_0^2 dV = \tau, \quad \text{cavity volume}$$

We wish to place a magnetic current sheet on the hole to establish $\hat{H}_{2\tan} = \frac{1}{2}\hat{H}_0$ in the angular direction.

⁵ In subsequent work we denote k_c by P_1 , the capital P_1 referring to the first short-circuit resonant cavity mode.

Since a circulating current with no divergence creates only θ -directed magnetic field by Eq. 3-7b, let us specify a $\hat{K}_E$ at point $\bar{r}$ on the hole of

$$\hat{K}_E(\bar{r}) = \frac{4}{\pi} (a^2 - r^2)^{-1/2} (\bar{r} \times \hat{E}_{0z}) [A + B(k_c r)^2] \quad (3-18a)$$

which determines $\hat{E}_{\tan}$ due to the fields in cavity (1) alone of

$$\hat{E}_{\tan}(\bar{r}) = \frac{1}{2} \hat{i}_z \times \hat{K}_E(\bar{r}) \quad (3-18b)$$

A conceivable $(\bar{r} \times \hat{E}_0)(a^2 - r^2)^{1/2}$ variation is included as a linear combination of the A and B terms. According to Eq. 3-11 (converted to the mks system) $A = \frac{1}{2}$ if we neglect the r^3 portion of $\hat{H}_0$. The first integral of Eq. 3-7b consists of three portions. The A term neglecting retardation is evaluated analogous to Eq. 3-10. With Green's function G expanded as

$$G(\rho) = \rho^{-1} - jk_c + \frac{(jk_c)^2 \rho}{2!}, \quad \rho = |\bar{r} - \bar{r}'| \quad (3-19)$$

we see that the jk_c term of G contributes nothing to $\hat{H}_{2 \tan}$ by virtue of the antisymmetry of the entire $\hat{K}_E$. But the $(jk_c)^2$ term contributes an A term on the order of $(ka)^2 \hat{H}_0$ which we must now include. However, with the coordinate relations of Fig. 3-3 we can evaluate this term with integrals no worse than

$$\int_{-s}^s \chi^2 (s^2 - \chi^2)^{-1/2} d\chi \quad \text{and} \quad \int_0^\pi \cos^4 \beta d\beta$$

for example. The third contribution to the right side of Eq. 3-7b is the B term with retardation neglected.

The expressions for A and B that match $\hat{H}_2$ to $\frac{1}{2} \hat{H}_0$ in Eq. 3-7b are

$$A = \frac{1 + (k_c a)^2/6}{2[1 + (k_c a)^2/3]}, \quad B = -\frac{1 + (k_c a)^2/2}{12[1 + (k_c a)^2/3]} \quad (3-20)$$

For a calculation of the width of the first passband the B term in Eq. 3-20 is negligible for $k_c a < 1$. As we presently see, the passband width depends upon

$$\begin{aligned} & \int_{\text{hole}} \hat{E}_{\tan} \times \bar{H}_0 \cdot \hat{i}_z dS \\ & = 2V_0 k_c (3.71) a^3 \left\{ A \left[\frac{2}{3} - \frac{(k_c a)^2}{15} \right] + B \left[\frac{8}{15} (k_c a)^2 - \frac{2}{35} (k_c a)^4 \right] \right\} \end{aligned} \quad (3-21)$$

in which the B term is less than 5% of the A term, even for $k_c a = 1$.

In Section 3-9 we verify Eq. 3-21 by calculating the π cutoff frequency, k_{π} , of a pair of centerhole-coupled cavities treated by the very accurate Chu-Schwinger variational method (Section 6-6). We chose a structure

with a rather large hole of $k_c a = 1$ and calculated by means of Eq. 3-21, $(k_\pi b)^2 - (k_c b)^2 = 0.69$, where b is the outer radius of the cavity, which is known to be somewhat low, whereas the Chu-Schwinger calculation yields 0.74. Our expression for $\hat{E}_{\tan}$ on the centerhole therefore seems to be accurate enough in the range $k_c a < 1$ even though we have neglected the effect of the cavity walls and the $(jk_c)^3$ retardation term in the Green's function. We therefore believe that the error created by this term is negligible.

3-5. Nearly Uniform $\hat{H}_0$ and Inflecting $\hat{E}_0$ on the Hole

We now want to determine the magnetic surface current and tangential electric field on a hole due to incident fields of this particular symmetry. We take the incident field in cavity (1) of Fig. 3-1 [with no initial fields in cavity (2)] to be

$$\begin{aligned}\hat{H}_0(r) &= \hat{i}_y H_0 \left[1 - \frac{(k_c x)^2}{2} \right], & E_0 &= +j\omega\mu H_0 \quad \text{in mks units} \\ \hat{E}_0(r) &= \hat{i}_z E_0 x & & \text{with } e^{j\omega t} \text{ time dependence}\end{aligned} \quad (3-22)$$

which satisfy Maxwell's equations 3-4 in mks units at the normalized resonant frequency k_c . These fields are pictured on the hole in Fig. 3-5a; the induced tangential electric field will resemble that shown in Fig. 3-5b. The incident fields are assumed not to vary in the z direction.

Now, we could construct an expression for $\hat{K}$ appropriate to the incident field symmetry and proceed to choose the coefficients of the various terms so as to satisfy Eq. 3-7b. The integrals are straightforward, though

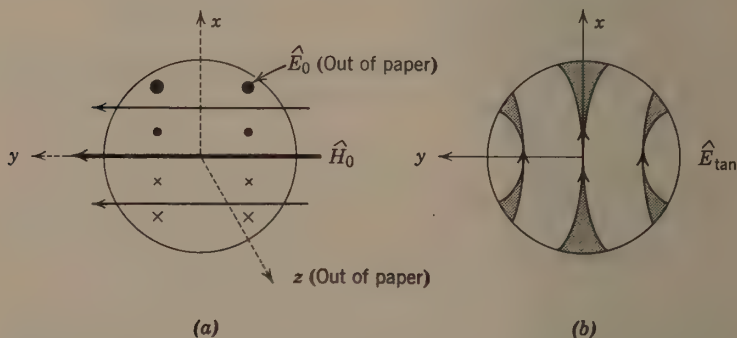


Fig. 3-5. The field components for nearly uniform incident $\hat{H}_0$ and inflecting incident $\hat{E}_0$ on the hole. (a) $\hat{E}_0$ reverses direction from one side of the hole to the other in the x direction, (b) $\hat{E}_{\tan}$ approaches infinity at the edges of the hole.

tediously evaluated, and the predominant current terms turn out to be correct if we include terms through order $(ka)^3$ (but neglect the $(jka)^3$ retardation terms). However, the smaller terms in the expression for $\hat{K}$ do not quite evaluate correctly, according to the more exact calculations of Eggimann [4]⁶. Therefore we shall obtain the proper expression for $\hat{K}_m$ from his work.

Eggimann treats the situation dual to ours, in which an incident field impinges upon a perfectly conducting round disk. To use his results we need only take the incident fields to be $\hat{E}_0 = \hat{i}_y E_0 [1 - (k_c x)^2/2]$ and $\hat{H}_0 = \hat{i}_z H_0 x$ in the coordinate system of Fig. 3-1, with amplitude relation $H_0 = -j\omega\epsilon E_0$, corresponding to a magnetic shorting plane in the plane of the disk. Results are in the mks system with $e^{j\omega t}$ time dependence. The final expression for electric current $\hat{J}_e$ on the disk is then converted to $-\hat{K}_m$ with reference to Babinet's principle.

The procedure is outlined as follows. The initially unknown electric current density on the disk is first expanded as

$$\frac{\mu_0}{4\pi} \hat{J}_e(x, y) = \frac{\hat{f}}{\pi^2(a^2 - \rho^2)^{1/2}}, \quad \rho^2 = x^2 + y^2 \quad (3-23)$$

where

$$\begin{aligned} \hat{f}_x &= a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + \dots \\ \hat{f}_y &= b_0 + b_1 y + b_2 x + b_3 y^2 + b_4 yx + \dots \end{aligned}$$

and each coefficient a_i, b_i is expanded in powers of frequency $k = \omega/c$. Then, by evaluating the integral for vector magnetic potential $\hat{A}$,

$$\hat{A}(\bar{r}) = \frac{\mu_0}{4\pi} \int_{\text{disk}} \hat{J}_e(x', y') G|\bar{r} - \bar{r}'| dS' \quad (3-24)$$

the a_i and b_i coefficients are related to those of $\hat{A}$. The coefficients of the latter can, in turn be related to those of the incident fields, denoted by superscript i , through Maxwell's equations:

$$\begin{aligned} (\nabla_{xy}^2 + k^2)A_x &= \mu_0 \frac{\partial \hat{H}_y^i}{\partial z} \\ (\nabla_{xy}^2 + k^2)A_y &= -\mu_0 \frac{\partial \hat{H}_x^i}{\partial z} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} &= -\mu_0 \hat{H}_z^i \end{aligned} \quad (3-25)$$

The result is a set of expressions for each a_i and b_i in the current expansion as a linear combination of the moment coefficients of the incident field.

⁶ The basic expressions for the amplitudes of the electric current components on the disk in terms of the incident fields are on his pp. 30-34. See also Ref. [9].

The evaluation of Eggimann's expressions for each a_i and b_i , with the incident fields dual to those of Eq. 3-22, results in the following expression for $\hat{J}_e$ in mks units with $e^{j\omega t}$ time dependence:

$$\begin{aligned} \hat{J}_e(x, y)_{\text{on disk}} = & + \frac{4}{\pi} \frac{H_0}{(a^2 - \rho^2)^{1/2}} \frac{1}{315} \\ & \times \{ \bar{i}_x [xy(420 - 12(ka)^2) + x^3y \cdot 150k^2 + xy^3 \cdot 102k^2] \\ & - \bar{i}_y [a^2(210 + 168(ka)^2) - y^2(210 + 195(ka)^2) \\ & + x^2(210 - 15(ka)^2) + y^4 \cdot 27k^2 \\ & - x^2y^2 \cdot 36k^2 - x^4 \cdot 15k^2] \} \end{aligned} \quad (3-26)$$

The $(ka)^3$ terms have been dropped from this expression. To obtain our expression for $\hat{K}_m$ on the hole we must first divide it by 2, since it was derived so that the fields determined by $\hat{A}$ would cancel the incident fields,

$$\begin{aligned} \hat{E}_{x,y} &= -\hat{E}_{x,y}^i \\ \hat{H}_z &= -\hat{H}_z^i \end{aligned} \quad (3-27)$$

whereas we want to match them to $-\frac{1}{2}\hat{E}^i$ and $-\frac{1}{2}\hat{H}^i$ on the left side of the disk as in the Bethe theory. In addition, we convert the answer for the disk in a magnetic shorting plane into the answer for the hole cut in a perfectly conducting plane by changing $\hat{J}_e \rightarrow -\hat{K}_m$ and $H_0 \rightarrow -E_0$.

The desired expression for $\hat{K}_m$ appropriate for the incident fields of Eq. 3-22 on one side of the hole is, then,

$$\hat{K}_{m \text{ mks}} = \frac{2}{\pi} \frac{+E_0 \text{ mks}}{(a^2 - \rho^2)^{1/2}} \frac{1}{315} \{ \quad \} \quad (3-28)$$

where the quantity within braces appears in Eq. 3-26. The dominant terms in $\hat{K}_m$ were verified independently by extending Bethe's calculations. The value of $\hat{E}_{\tan}$ is then given by $\frac{1}{2}\bar{i}_z \times \hat{K}_m$ on the hole.

3-6. Nearly Uniform $\hat{E}_0$ and Inflecting $\hat{H}_0$ on the Hole

We now take the incident fields of cavity (1) in Fig. 3-1 to have the other symmetry specified by

$$\begin{aligned} \hat{H}_0(\bar{r}) &= \bar{i}_y H_0 \left[x - \frac{(k_c x)^3}{6} \right] \\ \hat{E}_0(r) &= \bar{i}_z E_0 \left[1 - \frac{(k_c x)^2}{2} \right], \quad H_0 = j\omega \epsilon E_0 \end{aligned} \quad (3-29)$$

which obey Maxwell's equations at the resonant frequency k_c . No incident

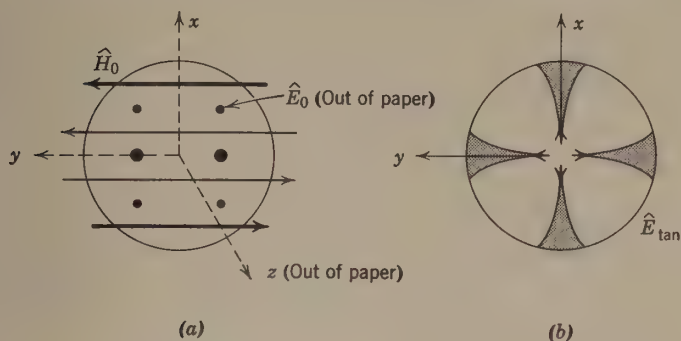


Fig. 3-6. The field components for nearly uniform incident $\hat{E}_0$ and inflecting incident $\hat{H}_0$ on the hole. (a) $\hat{H}_0$ reverses direction from one side of the hole to the other in the x direction, (b) $\hat{E}_{tan}$ approaches infinity at the edges of the hole.

fields exist in cavity (2). These fields are illustrated in Fig. 3-6a; the resultant tangential electric field is shown in Fig. 3-6d.

By reinterpreting Eggimann's results, as described in the preceding section, we obtain this expression for $\hat{K}_m$ on the hole:

$$\hat{K}_{mks} = -\frac{2}{\pi} \frac{E_{0mks}}{\sqrt{a^2 - \rho^2}} \frac{1}{30} \{ \hat{i}_x [y(-30 + (ka)^2) + x^2 y \cdot 17k^2 + y^3 k^2] + \hat{i}_y [x(30 - 9(ka)^2) + y^2 x \cdot 7k^2 - x^3 \cdot 9k^2] \} \quad (3-30)$$

in the coordinate system of Fig. 3-1. By extending Bethe's calculations we verified the predominant terms of Eq. 3-30 exactly and the small terms almost exactly. Equation 3-30 is accurate enough for all considerations of cavity coupling that neglect the effects of the cavity walls upon the physical parameter of interest, $\hat{E}_{tan} = \frac{1}{2} \hat{i}_z \times \hat{K}_m$, on the hole.

3-7. Elliptical Hole in a Plane Sheet

Bethe suggests an extension of these methods to the calculation of $\hat{E}_{tan}$ on an elliptical hole, which might very well approximate a long rectangular hole. He derives no formulas, however, and we shall merely point out how the integrals may be reevaluated. In connection with Eq. 3-10, for example, and the coordinate system of Fig. 3-3, we must now write the limits for χ not as $\pm s(\beta)$ but as more complicated expressions dependent upon the equation for an ellipse. Nevertheless, these limits can again be expressed in terms of r , α , and β so that integration can be taken first over all χ and then over β . If the length of the long axis k is ≈ 1 retardation terms will most certainly have to be included.

However, Collin⁷ has developed expressions for the dipole moments of an elliptical aperture, neglecting the retardation terms and higher variations of incident fields over the hole. We can use them to evaluate in the first approximation the cavity-coupling integral

$$\int_{\text{hole}} \hat{E}_{\text{tan}} \times \bar{H}_1 \cdot \bar{i}_z dS$$

where $\bar{H}_1$ is the normalized magnetic field pattern of the short-circuit cavity mode defined by Eq. 1-21 and incident on the hole in cavity (1). This integral will assume fundamental significance in Section 3-9.

To evaluate it we follow Collin's Eqs. (57-61) and derive

$$\int_{\text{hole}} \bar{H}_1 \cdot \hat{K}_m dS \simeq \bar{H}_1(0) \cdot \int \hat{K}_m dS + \int (\bar{r} \cdot \nabla \bar{H}_1) \cdot \hat{K}_m dS \quad (3-31)$$

in which magnetic current $\hat{K}_m = -\bar{i}_z \times 2\hat{E}_{\text{tan}}$ stands for Collin's $\hat{J}_m$ and magnetic charge density η_m stands for his ρ_m . To evaluate the first integral we develop an equivalent expression for it from the fact that there is no net divergence of $\hat{K}_m$ from the hole:

$$\int_{\text{hole}} \nabla_T \cdot (x \hat{K}_m) dS = \int \bar{i}_x \cdot \hat{K}_m dS + \int x \nabla_T \cdot \hat{K}_m dS = 0 \quad (3-32)$$

that is,

$$\int \hat{K}_{mx} dS = - \int x \nabla_T \cdot \hat{K}_m dS$$

Since the same is true when x is replaced by y we obtain a vector expression for $\int \hat{K}_m dS$ which can be evaluated in terms of charge density η :

$$\int_{\text{hole}} \hat{K}_m dS = - \int \bar{r} \nabla_T \cdot \hat{K}_m dS = +j\omega \int \bar{r} \eta_m dS \triangleq +j\omega \mu_0 \mathbf{M} \quad (3-33)$$

in mks units with $e^{j\omega t}$ time dependence according to the *definition* of $\mathbf{M}$. The second term of Eq. 3-31 evaluates as

$$\int_{\text{hole}} (\bar{r} \cdot \nabla \bar{H}_1) \cdot \hat{K}_m dS = \int [\tfrac{1}{2}x \hat{K}_{my} (\nabla \times \bar{H}_1)_z - \tfrac{1}{2}y \hat{K}_{mx} (\nabla \times \bar{H}_1)_z] dS + Q \quad (3-34)$$

in which the various terms in quadrupole term Q may be rearranged into

$$Q = \int \tfrac{1}{2} [\bar{r} \cdot (\hat{K}_m \cdot \nabla_T) \bar{H} + \hat{K}_m \cdot (\bar{r} \cdot \nabla_T) \bar{H}] dS$$

This term is usually small because it involves the transverse gradient of pattern $\bar{H}$ across the hole rather than its curl. So we neglect it and employ

⁷ R. E. Collin [5, Chapter 7].

the definition of the patterns, $\nabla \times \bar{H}_1 = P_1 \bar{E}_1$ (Eq. 1-21) to obtain

$$\int_{\text{hole}} (\bar{r} \cdot \nabla \bar{H}_1) \cdot \hat{K}_m dS \simeq - \frac{P_1}{\epsilon_0} \bar{E}_1(0) \cdot \left(-\epsilon_0 \int_{\text{hole}} \frac{1}{2} \bar{r} \times \hat{K}_m dS \right) \triangleq - \frac{P_1}{\epsilon_0} \mathbf{P} \cdot \bar{E}_1(0) \quad (3-35)$$

according to the *definition* of $\mathbf{P}$, the electric dipole moment for magnetic current $\hat{K}_m$.

Therefore our coupling integral is

$$\int_{\text{hole}} \bar{H}_1 \cdot \hat{K}_m dS = j\omega\mu_0 \mathbf{M} \cdot \bar{H}_1(0) - \frac{P_1}{\epsilon_0} \mathbf{P} \cdot \bar{E}_1(0) \quad (3-36)$$

For dipole moments $\mathbf{M}$ and $\mathbf{P}$ we use Collin's Eq. (68), with all the signs changed because we are computing $\hat{K}_m = -\hat{i}_z \times 2\hat{E}_{\text{tan}}$ in the diffracted region 2, and also halved because $\hat{K}_m$ should correspond to half the incident fields as in the Bethe theory. Then

$$\begin{aligned} \mathbf{P} = \mathbf{P}_z &= + \frac{2\pi\ell_1^3(1-e^2)}{3E(e)} \epsilon_0 \bar{E}_1 \\ \mathbf{M}_x &= - \frac{2\pi\ell_1^3 e^2}{3[K(e) - E(e)]} \hat{H}_{1x} \\ \mathbf{M}_y &= - \frac{2\pi\ell_1^3 e^2(1-e^2)}{3[E(e) - (1-e^2)K(e)]} \hat{H}_{1y} \end{aligned} \quad (3-37)$$

in which ℓ_1 and ℓ_2 are the major and minor semiaxes of the ellipse along the x and y directions, respectively, eccentricity $e = (1 - \ell_2^2/\ell_1^2)^{1/2}$, and K and E are elliptic integrals of argument e .

With incident fields $E_1 = V_1 \bar{E}_1$ and $H_1 = I_1 \bar{H}_1$, the proper relation between V_1 and I_1 is $P_1 V_1 = -j\omega\mu_0 I_1$. Then we can write Eq. 3-36 in terms of $\hat{E}_{\text{tan}} = \frac{1}{2} \hat{i}_z \times \hat{K}_m$ as

$$\left(- \int_{\text{hole}} \frac{2\hat{E}_{\text{tan}}}{V_1} \times \bar{H}_1 \cdot \hat{i}_z dS \right) \simeq + P_1 \bar{H}_1(0) \cdot (\hat{i}_x \mathbf{m}_x \bar{H}_{1x} + \hat{i}_y \mathbf{m}_y \bar{H}_{1y}) - P_1 \mathbf{p} \bar{E}_1(0)^2$$

with

$$\begin{aligned} \mathbf{m}_x &= + \frac{2\pi\ell_1^3 e^2}{3[K(e) - E(e)]} \\ \mathbf{m}_y &= + \frac{2\pi\ell_1^3 e^2(1-e^2)}{3[E(e) - (1-e^2)K(e)]} \\ \mathbf{p} &= + \frac{2\pi\ell_1^3(1-e^2)}{3E(e)} \end{aligned} \quad (3-38)$$

The approximation sign means we have neglected the quadrupole term.

The significance of this particular integral in the cavity-coupling analysis will be evident in Section 3-9.

3-8. Fields and Coupling on Waveguide Apertures

In preceding sections we calculated the tangential electric fields induced on small round holes by incident magnetic and electric fields $\hat{H}_0$ and $\hat{E}_0$. It is difficult to make the calculations for holes of various sizes and to account for the presence of the waveguide walls. Now it will be clear in the next section that the cavity-coupling parameter which determines the first TM or TE passband of a cavity chain composed of sections of waveguide is not $\hat{E}_{\text{tan}}$ but rather

$$\int_{\text{hole}} \hat{E}_{\text{tan}} \times \bar{H}_1 \cdot \bar{i}_z dS$$

$\bar{H}_1$ being the pattern of the short-circuit mode resonating at one of the cutoff frequencies of the passband. For many of these cavity chains we can evaluate this coupling integral with Marcuvitz' results [6].⁸

For example, suppose we wish to study the first TE passband bounded at the π resonant frequency by the TE_{10} mode of rectangular waveguide or the TE_{01} mode of circular guide. Ordinarily we are interested in a passband within which the fields have no angular variation, so we assume either that the TE_{11} mode in circular guide is not excited or that it is suppressed. The amplitudes of the mode are denoted by $V_1^w(z)$ for the electric field and $I_1^w(z)$ for the magnetic field, the superscript w reminding us that these are amplitudes of a waveguide mode and not a resonant cavity mode as we shall define it shortly. The amplitudes V_1^w and I_1^w vary with z as shown in Fig. 3-7a. The precise variation of the fields within the cavity at this π resonant frequency are, from Eqs. 1-25-1-28,

$$\begin{aligned} \hat{E} &= V_1^w \sin \frac{\pi z}{L} \bar{e}_{1t}(x) \\ \hat{H} &= I_1^w \left(\cos \frac{\pi z}{L} \bar{h}_{1t}(x) - \bar{i}_z k_c^2 \frac{L}{\pi} \psi_1 \sin \frac{\pi z}{L} \right) \end{aligned} \quad (3-39)$$

with $k_c = \pi/a$, a being the width of the waveguide. Here $\bar{e}_{1t}$ and $\bar{h}_{1t} = \bar{i}_z \times \bar{e}_{1t}$ are the transverse field patterns of the TE_{10} mode which have been normalized over the cross-sectional area as

$$\int_{\text{area}} \bar{e}_{1t}^2 dS = \int \bar{h}_{1t}^2 dS = 1 \quad (3-40)$$

⁸ Unfortunately Ref. [6] does not present equivalent circuits for the first TM waveguide mode impinging on various apertures. There are a few such equivalent circuit elements in [7].

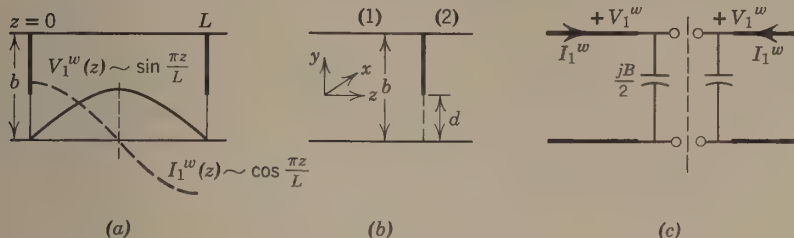


Fig. 3-7. TE₁₀ mode excitation of aperture fields for periodic phase shift of π . (a) The amplitude variations of electric and magnetic field incident on the electrically shorted apertures, (b) aperture geometry between two cavities (1) and (2), (c) equivalent circuit representation of the TE₁₀ mode amplitudes on the aperture with symmetrical excitation applied.

The normalized resonant frequency of this configuration is

$$P_1 = \left[k_o^2 + \left(\frac{\pi}{L} \right)^2 \right]^{1/2}$$

and we see that it corresponds to electrically shorted apertures.

If now we magnetically short the apertures, the tangential magnetic field on them will excite many TE_{1_n} cutoff waveguide modes of the same symmetry along the y axis so as to match the boundary conditions and reduce $\hat{H}_{\text{tan}}$ on the aperture to zero. These cutoff modes determine $\hat{E}_{\text{tan}}$ in the apertures, and the coupling integral

$$\int_{\text{ap}} \hat{E}_{\text{tan}} \times \hat{h}_{1t} \cdot i_z dS$$

Let us study how Marcuvitz evaluates this integral.

In the coordinate system of Fig. 3-7 Marcuvitz⁹ writes his equations in the mks system of units with $e^{j\omega t}$ time dependence. We denote his dominant TE mode quantities by subscript 1 rather than 0 in order to associate them with the short-circuit resonant cavity mode to be defined in the next section.

We now consider the right-hand aperture of Fig. 3-7a, shown in Fig. 3-7b with a magnetic shorting plane on it. From another point of view we excite the waveguide symmetrically on both sides of this aperture so as to cancel the net tangential magnetic field on it, according to the discussion of Section 1.7B pertaining to the Marcuvitz-Schwinger technique. The $\hat{H}$ field of Eq. 3-39 is incident on the left side of the aperture and generates, in a narrow frequency range near P_1 , cutoff waveguide modes which decay exponentially away from the aperture. Although Marcuvitz' work is for a semi-infinite waveguide we may use his results if the fringing fields of one aperture do not appreciably overlap the neighboring apertures.

* N. Marcuvitz [6, Chapters 3 and 5, pp. 141-143 in particular].

Two procedures exist for calculating $\hat{E}_{\text{tan}}$ on an aperture, of which the one on the *aperture basis* is important for us. This procedure starts from the condition of zero net $\hat{H}_{\text{tan}}$ on the aperture, by which we can equate the dominant mode portion $I_1^w \bar{H}_1^w$ to the transverse magnetic fields of all the cutoff modes. The magnetic field amplitude of each cutoff mode is proportional to its transverse electric field amplitude. Therefore we can equate I_1^w to a linear combination of all the cutoff V_n^w 's. But each V_n^w is obtainable as an integral of $\hat{E}_{\text{tan}}$ over the aperture. In this manner Marcuvitz obtains an integral equation for $\hat{E}_{\text{tan}}$ on the aperture. By solving this equation in one of four ways he determines $\hat{E}_{\text{tan}}$ on the aperture. By the orthogonality of the waveguide mode patterns this determines V_1^w of the dominant waveguide mode. The upshot of this procedure on the aperture basis is an expression for the transmission line susceptance of this mode:

$$\left(\frac{I_1^w}{V_1^w} \right)_{\text{ap}} = \frac{jB}{2} = \frac{I_1^w}{-\int_{\text{ap}} \hat{E}_{\text{tan}}(I_1^w) \times \hat{h}_{1t} \cdot \hat{i}_z dS} \quad (3-41)$$

independent of I_1^w . The susceptance is denoted $B/2$ in Fig. 3-7c because the symmetrical excitation on both sides of the aperture splits the total circuit element in half. The minus sign occurs in Eq. 3-41 because $\hat{E}_{\text{tan}}$ happens to lie in the $+y$ direction while $\hat{h}_{1t}$ by definition is positive along the $+x$ axis. Thus, the aperture integral by itself measures power flow in the $-z$ direction, whereas the fields propagate in the $+z$ direction.

The dual procedure calculates $(V_1/I_1)_{\text{ap}}$ on the obstacle basis by starting with the condition of zero net tangential electric field on the fin (obstacle). The result would be a relation between $B/2$ and an integral over the fin surface, which is not of direct interest. Fortunately, Marcuvitz evaluates B or B/Y_1 (Y_1 being the characteristic admittance of the dominant TE mode¹⁰) as defined by Eq. 3-41 for the apertures and holes of interest. Therefore we can use his results to evaluate the coupling integral in the denominator for the various apertures and holes in rectangular and circular waveguide. In Section 3-10 this integral is related to a coupling integral for the cavity chain.

¹⁰ In Chapter 1 we introduced the expressions for characteristic impedance and admittance of the i th TE mode as

$$Z_i = \frac{1}{Y_i} = \frac{\omega\mu(k^2 - p_i'^2)^{-1/2}}{+j\omega\mu(p_i'^2 - k^2)^{-1/2}}, \quad \begin{matrix} k \geq p_i' \\ k < p_i' \end{matrix}$$

where propagation of the i th such mode is as

$$\begin{matrix} e^{-j(k^2 - p_i'^2)^{1/2}z}, & k \geq p_i' \\ e^{-(p_i'^2 - k^2)^{1/2}z}, & k < p_i' \end{matrix}$$

Here p_i' stands for k_c in Eq. 3-39.

Some of Marcuvitz' expressions give the inverse of B/Y_1 , or X/Z_1 , but they are all calculated on the aperture basis very accurately for the first TE mode incident on the following infinitesimally thin apertures: capacitive windows (cut the full length of the x axis along which $\hat{h}_{1t}$ flows) and inductive windows (cut the full length of the y axis) in rectangular waveguide, capacitive windows in coaxial waveguide, circular and elliptical apertures in rectangular waveguide, and circular and elliptical apertures in coaxial waveguide. We observe that he calculates X/Z_1 for the TE_{01} mode impinging upon an annular window in circular waveguide in a frequency range within which the TM_{01} mode also propagates. But this latter mode defines another passband in which the fields do not have the symmetry of the TE_{01} mode. Neither does the TE_{11} mode have the symmetry of the TE_{01} mode. Therefore we can ignore the TM_{01} and TE_{11} modes in Eq. 3-41 with respect to the TE_{01} mode coupling integral. In the frequency range of validity of the X/Z_1 expression for the annular window, the window is far short of being half a free-space wavelength long. So it is not strongly resonant.

The resonant frequency $P_1 = [k_c^2 + (\pi/L)^2]^{1/2}$ of the short-circuit configuration of Fig. 3-7a must of course lie within the range of validity of Marcuvitz' expressions for B or B/Y_1 . In addition, we assume that the zero phase shift cutoff frequency belonging to a chain of such aperture-coupled cavities also lies within this frequency range, in order that the coupling integral in Eq. 3-41 have significance. Lastly, we reiterate the assumption that the fringing fields of one aperture do not appreciably overlap the neighboring apertures.

Marcuvitz describes four methods of solving the integral equation for $\hat{E}_{tan}$ on the aperture basis: the variational, integral, equivalent static, and transform, of which the first and third were the most often utilized. If we use the variational method on this basis we obtain too high a value for the B/Y_1 of capacitive windows. The dual procedure on the obstacle basis yields too low a value. It is interesting to note that a fairly good trial $\hat{E}_{tan}$ function can yield very close upper and lower bounds to

$$\int \hat{E}_{tan} \times \hat{h}_{1t} \cdot \hat{i}_z dS$$

3-9. Equations for Two Coupled Cavities

In this section we develop the equations for the fields within cavity (1) in Fig. 3-8, and cast the equations into a form suitable for generalization to cavities of quite arbitrary shapes and coupled by holes or apertures of general shape. We make the important observation that Bethe's excitation

of his cavity is *too high by a factor of 2*, even though the $\hat{E}_{\tan}$ he calculates is certainly correct within his approximations.

We again concern ourselves with the behavior of the two cavities at frequencies near the resonance of the first short-circuit cavity mode. That mode has the patterns shown in Fig. 3-8, which may be defined by Eq. 1-21:

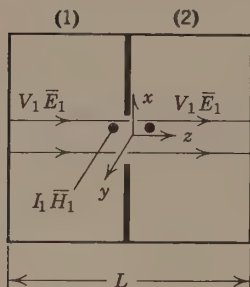


Fig. 3-8. A resonant mode in two cavities which is not disturbed by cutting a hole in the wall between them.

$$\nabla \times \bar{E}_1 = P_1 \bar{H}_1$$

$$\nabla \times \bar{H}_1 = P_1 \bar{E}_1$$

$$P_1 = \omega_1 \sqrt{\mu\epsilon}, \quad \bar{E}_1 \times \bar{n} = 0 \text{ on walls and hole}$$

where P_1 is the resonant frequency of the mode with the holes electrically shorted. Equation 1-21 also describes, with appropriate normalization factors, the dominant TM waveguide mode at its cutoff frequency, where it has only $\hat{E}_z$ and $\hat{H}_t$ components, as well as the dominant TE waveguide mode *above* its cutoff frequency, with a $\sin(\pi z/L)$ variation across the cavity (Fig.

3-7a, for example). We repeat the patterns $\bar{E}_1$ and $\bar{H}_1$ in both cavities with the same orientations, and since $\bar{E}_1$ and $\bar{H}_1$ are thereby continuous across the hole we see that this mode represents the true field at $k = P_1$ for zero phase shift of *amplitudes* between the two cavities. It is convenient to normalize both $\bar{E}_1$ and $\bar{H}_1$ to the volume τ of either cavity:

$$\int_V \bar{E}_1^2 dV = \int_V \bar{H}_1^2 dV = \tau, \quad \text{cavity volume} \quad (3-42)$$

We may verify the fact that both $\bar{E}_1$ and $\bar{H}_1$ may be so normalized by employing Green's vector theorem for two vectors $\bar{A}$ and $\bar{B}$ with continuous zero, first, and second derivatives within τ :

$$\begin{aligned} \int_V \nabla \cdot (\bar{A} \times \nabla \times \bar{B}) dV &= \int_V (\nabla \times \bar{B} \cdot \nabla \times \bar{A} - \bar{A} \cdot \nabla \times \nabla \times \bar{B}) dV \\ &= \oint_S \bar{A} \times \nabla \times \bar{B} \cdot \bar{n} dS \end{aligned} \quad (3-43a)$$

By subtracting the alternate relation with $\bar{A}$ and $\bar{B}$ interchanged we obtain another useful form,

$$\begin{aligned} \int_V (\bar{A} \cdot \nabla \times \nabla \times \bar{B} - \bar{B} \cdot \nabla \times \nabla \times \bar{A}) dV \\ = \oint_S (\bar{B} \times \nabla \times \bar{A} - \bar{A} \times \nabla \times \bar{B}) \cdot \bar{n} dS \end{aligned} \quad (3-43b)$$

Here S goes over the entire cavity surface on which $\bar{n}$ is the unit outward normal. Equation 3-43b will be referred to again and again in the text. Substitution of $\bar{A} = \bar{E}_1 = \bar{B}$ into Eq. 3-43a verifies Eq. 3-42.

We now wish to solve Maxwell's equations within cavity (1), say, at a normalized frequency $k \simeq P_1$. If the other resonant cavity modes have normalized resonant frequencies $P_2 < P_3 < \dots$, such that $P_2 - k \gg P_1 - k$, it is clear (or will be later) that only the first resonant cavity mode will be strongly excited. Let us therefore represent the fields $\hat{E}$ and $\hat{H}$ within cavity (1) at this frequency as $\hat{E} = V_1 \bar{E}_1$ and $\hat{H} = I_1 \bar{H}_1$, so that amplitudes V_1 and I_1 remain to be found. Then Maxwell's equations read

$$\nabla \times \hat{E} = -j\omega\mu(I_1 \bar{H}_1) \quad (3-44a)$$

$$\nabla \times \hat{H} = j\omega\varepsilon(V_1 \bar{E}_1) \quad (3-44b)$$

within cavity (1).

To evaluate amplitudes V_1 and I_1 we follow the procedure of Schelkunoff¹¹ which is not equivalent to Bethe's procedure [1]. We dot-multiply Eq. 3-44a by $\bar{H}_1$ and integrate it over the volume of cavity (1). We do not however, expand $\nabla \times \hat{E}$ as $V_1 \nabla \times \bar{E}_1$; we rather integrate the left side by parts, as

$$\int_V \bar{H}_1 \cdot \nabla \times \hat{E} dV = \int_V \hat{E} \cdot \nabla \times \bar{H}_1 dV + \oint_S \hat{E} \times \bar{H}_1 \cdot \bar{n} dS \quad (3-45)$$

From the definition of $\bar{E}_1$ and $\bar{H}_1$ this becomes

$$\int_V \bar{H}_1 \cdot \nabla \times \hat{E} dV = P_1 V_1 \tau + \int_{\text{hole}} \hat{E}_{\text{tan}} \times \bar{H}_1 \cdot \bar{i}_z dS, \quad (3-46)$$

and so we obtain from Eq. 3-44a the amplitude relation

$$P_1 V_1 \tau + \int_{\text{hole}} \hat{E}_{\text{tan}} \times \bar{H}_1 \cdot \bar{i}_z dS = -j\omega\mu I_1 \tau \quad (3-47)$$

Next we dot-multiply Eq. 3-44b by $\bar{E}_1$, integrate it over cavity (1), integrate the left side by parts, observe that $\oint_s \hat{H} \times \bar{E}_1 \cdot \bar{n} dS \equiv 0$ since $\bar{E}_1 \times \bar{n} \equiv 0$ on the hole, and obtain the second amplitude relation

$$P_1 I_1 = j\omega\varepsilon V_1 \quad (3-48)$$

We can combine these last two equations to obtain one equation for amplitude V_1 :

$$(P_1^2 - k^2)V_1 = -\frac{P_1}{\tau} \int_{\text{hole}} \hat{E}_{\text{tan}} \times \bar{H}_1 \cdot \bar{i}_z dS \quad (3-49)$$

¹¹ S. A. Schelkunoff [8, p. 1233].

On the other hand, Bethe derives a somewhat different expression. He expands the vector potential as $q_1 \bar{H}_1$ and excites it with $\hat{K}_m$ on the hole, obtaining, in cgs units,

$$\frac{d^2 q_1}{dt^2} + \omega_1^2 q_1 = -\frac{4\pi c^2}{\tau} \int_{\text{hole}} (-\hat{K}_m) \cdot \bar{H}_1 dS \quad (3-50)$$

With $\hat{K}_m = -\hat{i}_z \times \hat{E}_{\text{tan}}/2\pi$, $\omega_1/c = P_1$, and $V_1 \triangleq q_1 P_1$ this is

$$(P_1^2 - k^2)V_1 = -\frac{2P_1}{\tau} \int_{\text{hole}} \hat{i}_z \cdot \hat{E}_{\text{tan}} \times \bar{H}_1 dS \quad (3-50')$$

which is twice the correct excitation *because $\hat{K}$ is larger than $\hat{E}_{\text{tan}}$ by a factor of 2 in mks units*. The factor $\hat{E}_{\text{tan}}$ in *both* of these formulas is the total tangential electric field due to the fields in both cavities.

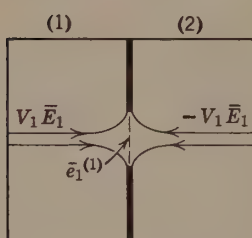


Fig. 3-9. Perturbation of the short-circuit resonant cavity mode in each of two cavities coupled by a hole which is effectively magnetically shorted. The field fringes and creates tangential $\hat{e}_1^{(1)}$ on the hole.

There will be further verification of Eq. 3-49 in Chapters 4 and 5, but right now let us verify it by calculating the first π cutoff frequency of two cylindrical cavities coupled by a centerhole, Fig. 3-8. The parameters chosen are

$$k_\pi L = \pi, \quad k_\pi a = 1 \quad (a = \text{hole radius})$$

$$P_1 b = 2.405 \quad (b = \text{outer radius})$$

In Eq. 3-49 we set $k = k_\pi$ and $\hat{E}_{\text{tan}}$ equal to twice the $\hat{E}_{\text{tan}}$ of Eq. 3-21 (with $k_c = P_1$ and not k_π) because the fields of cavity (2) also induce a $\hat{K}_m$ on the hole which doubles $\hat{E}_{\text{tan}}$ due to one cavity alone (see Fig. 3-9). Equation 3-49 then yields

$$(P_1^2 - k_\pi^2)b^2 = -0.69 \quad (3-51a)$$

known from Green's theorem to be somewhat low in magnitude. By the Chu-Schwinger method discussed in Section 6-6 there obtains $k_\pi b = 2.555 \pm 0.005$, whereupon

$$(P_1^2 - k_\pi^2)b^2 = -0.74 \quad (3-51b)$$

This verifies quite well Eq. 3-49 as opposed to Bethe's Eq. 3-50'.

Some very accurate measurements of the first passband of the Stanford linear accelerator structures¹² also verify Eq. 3-49. Moreover, if we examine Collin's expressions [5] for the effective dipole moments of apertures that couple sections of waveguide we find that he, too, measures the tangential electric field on an aperture by one-half the sum of the contributions to it on either side.

¹² Data furnished by Doctors K. Mallory and G. Loew.

We now wish to cast Eq. 3-49 into more significant form for generalization in the next chapter. This form will introduce the expressions for $\hat{E}_{\text{tan}}$ and

$$\int_{\text{hole}} \hat{E}_{\text{tan}} \times \hat{h}_{1t} \cdot \hat{i}_z dS$$

calculated or discussed in Sections 3-3 through 3-8. We write $\hat{E}_{\text{tan}}$ as

$$\hat{E}_{\text{tan on hole}} = V_1^{(1)} \bar{E}_{\text{tan}} - V_1^{(2)} \bar{E}_{\text{tan}} \quad (3-52)$$

where the first portion is due to "incident" fields in cavity (1) alone and the minus sign is taken because $\bar{E}_1$ and $\bar{H}_1$ have the same directions on both sides of the hole (Figs. 3-8 and 3-9 show this) and will tend to induce opposing components of $\hat{E}_{\text{tan}}$. Clearly this is correct for zero phase shift when there is no net $\hat{E}_{\text{tan}}$. Now, at π phase shift $V_1^{(1)} = -V_1^{(2)}$, and the normal electric and tangential magnetic fields cancel on the hole; see Fig. 3-9. At this frequency an *open-circuit mode* is resonating within both cavities, such that the electric field on the hole is purely tangential. Let us denote the pattern of this mode on the cavity (1) side of the hole as $\bar{e}_1^{(1)}$, so that

$$\bar{e}_1^{(1)} \triangleq 2\bar{E}_{\text{tan}} \quad (3-53)$$

Equation 3-49, with Eqs. 3-52 and 3-53, now reads

$$(P_1^2 - k^2)V_1^{(1)} = -\frac{P_1}{\tau} \int_{\text{hole}} \left(\frac{1}{2} V_1^{(1)} \bar{e}_1^{(1)} - \frac{1}{2} V_1^{(2)} \bar{e}_1^{(1)} \right) \times \bar{H}_1 \cdot \hat{i}_z dS \quad (3-54)$$

and *this* is the form we generalize in the next chapter.

3-10. Expressions for the Coupling Integral¹³ $-\int_{\text{hole}} \bar{e}_1^{(1)} \times \bar{H}_1^{(1)} \cdot \hat{i}_z dS$

In order to apply Eq. 3-54 to a calculation of the passband characteristics of common structures we must obtain expressions for the hole integral from Sections 3-3 through 3-8. Bethe's expression 3-14 can be written as $\hat{E}_{\text{tan}} = V_0 \bar{E}_{\text{tan}}$, where V_0 is the amplitude of the resonant cavity mode of normalized electric field pattern $\bar{E}_1$. Equation 3-53 gives us $\bar{e}_1^{(1)}$ in terms of this pattern $\bar{E}_1$. Upon integrating $\hat{E}_{\text{tan}} \times \bar{H}_1$ over the hole there obtains

$$\left(-\int_{\text{hole}} \bar{e}_1^{(1)} \times \bar{H}_1^{(1)} \cdot \hat{i}_z dS \right) = \frac{4}{3} a^3 P_1 [2 |\bar{H}_1(0)|^2 - |\bar{E}_1(0)|^2],$$

Bethe theory (3-55)

¹³ The extra superscript on $\bar{H}_1$ indicates that it is measured on the same side of the hole or aperture as $\bar{e}_1^{(1)}$.

Here a is the hole radius; $\bar{H}_1$ and $\bar{E}_1$ are the uniform portions of the magnetic and electric field patterns on the hole. Equation 3-55 has the same form in both the cgs and the mks units.

For the centerhole-coupled cavity chain $\bar{e}_1^{(1)} = 2\bar{E}_{\tan} = 2\hat{E}_{\tan}/V_0$ in Eq. 3-21 so the coupling integral is

$$\begin{aligned} & \left(- \int_{\text{hole}} \bar{e}_1^{(1)} \times \bar{H}_1^{(1)} \cdot \bar{i}_z dS \right) \\ &= -4k_c(3.71)a^3 \left\{ A \left[\frac{2}{3} - \frac{(k_c a)^2}{15} \right] + B \left[\frac{8(k_c a)^2}{15} - \frac{2(k_c a)^4}{35} \right] \right\} \quad (3-56) \\ & 3.71 = |\bar{E}_1(0)|^2 \end{aligned}$$

where A and B are given by Eq. 3-20.

For a hole of nearly uniform magnetic field $\hat{H}_0$ and inflecting $\hat{E}_0$ the coupling integral would be evaluated by expressing $\hat{E}_0$ of Eq. 3-22 in the form $V_0\bar{E}_0$ throughout the cavity volume (denoted by $\bar{i}_z E_0 x$ over the hole), whereupon $\hat{K}_m$ of Eq. 3-28 equals $-2\bar{i}_z \times (\hat{E}_{\tan}/V_0) = -\bar{i}_z \times \bar{e}_1^{(1)}$ for the integral. Similar remarks apply in case $\hat{E}_0$ is nearly uniform and $\hat{H}_0$ inflects, for which we use Eq. 3-30 for $\hat{K}_m$.

For the elliptical holes treated in Section 3-7 we have the coupling integral identically in Eq. 3-38, where it is understood the $\bar{H}_1$ and $\bar{E}_1$ are the normalized patterns.

Now the Marcuvitz procedure of Section 3-8 yields $\hat{E}_{\tan}$ on various magnetically shorted apertures, so his formulas measure the open-circuit mode patterns $\bar{e}_1^{(1)}$ directly. To obtain

$$\int_{\text{ap}} \bar{e}_1^{(1)} \times \bar{H}_1^{(1)} \cdot \bar{i}_z dS$$

from Eq. 3-41 in terms of the resonant cavity mode patterns defined by Eq. 1-21 and 3-42 we must proceed as follows. We first write Eq. 3-39 for the first TE circular- or rectangular-waveguide mode somewhat differently:

$$\begin{aligned} \hat{E} &= \left(\frac{1}{N} V_1^w \frac{\pi}{L} \frac{1}{P_1} \right) \left(N \frac{L}{\pi} P_1 \sin \frac{\pi z}{L} \bar{e}_{1t} \right) = V_1 \bar{E}_1 \\ \hat{H} &= \left(\frac{1}{N} I_1^w \right) \left(N \cos \frac{\pi z}{L} \bar{h}_{1t} - \bar{i}_z N k_c^2 \frac{L}{\pi} \psi_1 \sin \frac{\pi z}{L} \right) = I_1 \bar{H}_1 \quad (3-57) \\ N &\triangleq \frac{\pi}{L P_1} \sqrt{\frac{2\tau}{L}}; \end{aligned}$$

Where τ is cavity volume and $P_1 = [k_c^2 + (\pi/L)^2]^{1/2}$. It may be verified that the $\bar{E}_1$ and $\bar{H}_1$ patterns obey Eq. 3-42 with the normalization factor N as defined. Then to convert $\hat{E}_{\tan}(I_1^w)$ of Eq. 3-41 to $\bar{e}_1^{(1)}$ we must

divide the former by V_1 , the amplitude of the *resonant cavity* mode. We can write, then, Eq. 3-41 in the form

$$-\int_{\text{ap}} \left(\frac{\hat{E}_{\text{tan}}}{V_1} \right) \times \left(\frac{\bar{H}_{1t}}{N \cos \pi} \right) \cdot \bar{i}_z dS = -\frac{jI_1^w/V_1}{B/2} \quad (3-58)$$

With the relation between V_1 and V_1^w in Eq. 3-57a and the waveguide mode Eqs. 1-28 and 3-39 we see that

$$\frac{I_1^w}{V_1} = \frac{I_1^w}{V_1^w} \frac{NLP_1}{\pi} = \frac{j\pi}{L\kappa_1 Z_1} \frac{NLP_1}{\pi} = \frac{jNP_1}{\omega\mu} = jN \sqrt{\frac{\epsilon}{\mu}} \quad (3-59)$$

Therefore Eq. 3-58 becomes

$$\left(-\int_{\text{ap}} \bar{e}_1^{(1)} \times \bar{H}_1^{(1)} \cdot \bar{i}_z dS \right) = -\frac{N^2 \sqrt{\epsilon/\mu}}{B/2} = -\frac{2N^2}{377B} \quad (\text{first TE mode; mks units}) \quad (3-60)$$

This integral relates the equivalent circuit susceptance of the first TE waveguide mode to a coupling integral which will turn out to be proportional to the passband width of a (narrow) dispersion curve associated with this mode.

Consider, now, the first TM waveguide mode impinging upon an aperture at a frequency just above the mode cutoff frequency. This mode with the "straight-through" E_z pattern will also have a narrow dispersion curve associated with it. If the aperture has a symmetry such that the propagating TE mode is not excited and if the cutoff modes do not contribute appreciably to the fields at adjacent apertures, then we can evaluate the coupling integral associated with the passband of this mode. Expressions analogous to Eq. 3-57 through 3-59 show that the relevant integral is

$$\left(-\int_{\text{ap}} \bar{e}_1^{(1)} \times \bar{H}_1^{(1)} \cdot \bar{i}_z dS \right) = -\frac{(N^1)^2 \sqrt{\epsilon/\mu}}{B/2} \quad (\text{first TM mode}) \quad (3-61)$$

where $(N^1)^2 = A$, the waveguide cross-sectional area.

Occasionally the sign of Eq. 3-60 may be in doubt. If so, the sign of the integral can be determined by reference to Green's theorem, Eq. 3-43b, with $\bar{B} = \bar{e}_1$, the pattern of the open-circuit resonant cavity mode, and $\bar{A} = \bar{E}_1$, the pattern of the short-circuit mode. Then we obtain the identity

$$(-P_1^2 + P_1^2) \int_V \bar{E}_1 \cdot \bar{e}_1 dV = 2P_1 \left(-\int_{\text{ap}} \bar{e}_1^{(1)} \times \bar{H}_1^{(1)} \cdot \bar{i}_z dS \right) \quad (3-62)$$

The integral over one aperture of the cavity volume V equals the integral over the other aperture because of the assumed mirror symmetry of the

cavity about a transverse bisecting plane. For the TE waveguide mode P_1 is a π phase shift resonant frequency (see Fig. 3-7), whereupon p_1 is the zero phase shift cutoff frequency. The coupling integral has a sign depending on which is higher; the higher frequency can usually be selected from the nature of the aperture susceptance. On the other hand, the first TM waveguide mode with its "straight-through" pattern (see Fig. 3-8) has a cutoff frequency P_1 at zero phase shift per cavity. Its open-circuit mode resonates at a $\varphi = \pi$ phase shift frequency p_1 , which is higher only if the coupling integral is negative.

Problems

1. Show that the fields of Maxwell's equations may be separated into symmetric and antisymmetric portions with respect to the plane of Fig. 3-2 and that the symmetric portion with no reversal of E_{tan} across the hole enables us to satisfy the boundary conditions on the hole.
2. Equations 3-4 and 3-5 imply $(\nabla^2 + k^2)\hat{F} = 4\pi\hat{K}$ only if $\nabla \cdot \hat{F} = -ik\psi$ in the diffracted region. Show that Eqs. 3-6 do imply this condition.
3. Verify Eq. 3-9 by direct evaluation in Eq. 3-8, using Eq. 3-10 as a guide.
4. Verify directly the $E_{tan}/2$ and $H_{tan}/2$ terms in Eq. 3-12.
5. Verify relations 3-15 by direct substitution into Maxwell's and the continuity equations.
6. Verify Eq. 3-17 for patterns $\bar{E}_0$ and $\bar{H}_0$ normalized within a cylindrical cavity.
7. Derive the expressions for A and B of Eq. 3-20 by solving the pair of relations for A and B which arise from Eq. 3-7b.
8. Show that for small $(ka)^2 \ll 1$ the quadrupole term of Eq. 3-34 is negligible for the incident fields of Section 3-5 and 3-6. For what ka does it begin to be important?
9. Evaluate the coupling integral of Eq. 3-55 with reference to the remarks just preceding it.
10. Verify the fact that the mode patterns of Eq. 3-57 obey Eq. 3-42 with N as defined.
11. Check Eq. 3-60 by showing that the expression for the susceptance B of a small round hole as given in Ref. [6, p. 238; Eq. (1b)] yields a coupling integral which agrees with Eq. 3-55 in the limit of a waveguide of infinite cross section.

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Chapter IV

The Single-Passband Behavior of an Infinite Cavity Chain

4-1. Synopsis

In this chapter we formulate the basic equations for the symmetrical periodic cavity chain as we shall use them throughout the remainder of the book. The adjective "symmetrical" means the chain has mirror symmetry with respect to any transverse plane which bisects a cavity. We assume that the frequency of interest lies within or near a passband which is narrow compared to the width of the adjacent stopband(s).¹ Then only the one open-circuit and one short-circuit resonant cavity modes of this passband are strongly excited. This single-passband analysis of the structure may alternatively be called the *single mode-pair* analysis, because both the open- and the short-circuit modes are required to represent the fields.

We first derive the dispersion or ω - φ ($\varphi = \beta_0 L$) relation for a passband in which the fields have even symmetry. This is defined by the E_z component of field having an even number 0, 2, . . . of reversals across a cavity. The field amplitudes of the short-circuit mode are excited by tangential electric field on each hole, as given by one-half the sum of the open-circuit mode expressions for it on either side. The amplitudes of the open- and short-circuit modes are related by the equivalence of both modes for the electric field within a cavity. This enables us to obtain the dispersion relation in which $k^2 = \omega^2 \mu \epsilon$ varies as $1 - \cos \varphi$. A passband parameter M is defined which measures the width of the passband in units of k very simply as an integral of the mode patterns on the coupling hole.

Then we repeat the derivation of the ω - φ relation for a passband in

¹ This restriction will be removed in Chapter 5.

which the fields have odd symmetry: E_z reverses an odd number of times across a cavity. Now k^2 varies as $1 + \cos \varphi$.

We can eliminate the passband parameter from both these dispersion relations and obtain a theorem which says that the shape of a narrow passband is independent of the size or shape of the coupling hole. The hole geometry only determines the open-circuit mode resonant frequency of the passband. Such a theorem enables us to judge on the basis of experiment the relative effect of neighboring passband modes in the field expansions at frequencies in the first passband. The greater this effect the more the dispersion curve for the first passband will differ from a shape specified by $k^2 = (1 \pm \cos \varphi)$.

The sign of the passband parameter M very easily determines the slope of the dispersion curve, and we distinguish four cases, according to the symmetry of the fields and the sign of M .

A given passband of a cavity chain may be characterized by two modes resonating at the cutoff frequencies and defined either within the cavity proper or else within a unit cell centered around the coupling element. A resonant-slot coupled structure exhibits both kinds of modes. We first observe that one passband of such a structure is characterized by two "cavity" modes resonating between electrically and magnetically shorted *slots and centerholes*. Then we point out that a neighboring passband is characterized by two "slot" modes resonating between electrically and magnetically shorted *cavity midplanes*. These two sets of modes are closely related, and we always find the "cavity" modes in the process of cataloging the "slot" modes and vice versa.

We verify the dispersion relations obtained for a single passband by checking the power flow-stored energy-group velocity relation. In this relation we approximate the power flow and stored energy by the modes of the given passband alone.

Then we consider the problem of maximizing both the fractional passband width and the gain per cavity with an electron beam present, but at the single-passband level of approximation. A gain parameter C_B is introduced which is well behaved at all frequencies throughout the passband. The fractional passband width can be increased while keeping C_B fixed by introducing extra cavity-cavity coupling beyond the centerholes. The gain per cavity of the growing wave is nearly proportional at fixed frequency to C_B for frequencies right up to the π cutoff frequency. In Chapter 9 we discuss more of the properties of C_B as related to the Pierce C_P parameter as the latter is naturally defined for the cavity chain. But from the form of C_B we can see that large gain per cavity requires cavities of small volume operated at low frequencies and large beam voltages. About the best cavity chain in this respect is a series of thin cylindrical

cavities operated in the first TM_{010} passband. The cavities cannot be too thin, however, for then the electric field of the resonant cavity mode will not penetrate effectively into the beam region between centerholes. Then the chain behaves as a fin-loaded waveguide with a wider dispersion curve.

We then show that the previously derived "cold" circuit ω - q relation can be obtained by the alternate procedure of expanding the fields of Maxwell's equations with the open-circuit mode of a narrow passband, as excited by tangential magnetic field on each coupling hole. This magnetic field is given by one-half the sum of the *short-circuit* mode expressions for it on either side of the hole. The amplitudes of the two modes are now related by their equivalence for the *magnetic* field within the cavity volume.

Lastly, we derive some equivalent circuits to represent the amplitude equations for the short- and open-circuit modes of the n th passband. We cite examples in which these circuits do not behave according to our intuitive notions of an equivalent circuit. This means that many so-called "equivalent" circuits would not describe the fields and stored energies according to the mode solutions of Maxwell's equations. As an example, the circuit susceptances which represent the effect of apertures on the propagating TE waveguide mode, discussed in Section 3-8, have signs *opposite* to those of the coupling elements in the corresponding equivalent circuit for that TE mode passband.

4-2. Dispersion Relation from the Short-Circuit Mode Expansion

In the preceding chapter we studied the coupling between two cavities shown in Fig. 3-8 and derived the key Eq. 3-54 for the amplitude $V_1^{(1)}$ of the short-circuit mode in cavity (1):

$$(P_1^2 - k^2)V_1^{(1)} = -\frac{P_1}{\tau} \int_{\text{hole}} \frac{1}{2}(V_1^{(1)}\bar{e}_1^{(1)} - V_1^{(2)}\bar{e}_1^{(1)}) \times \bar{H}_1 \cdot \bar{i}_z dS$$

where various expressions for the coupling integral

$$-\int_{\text{hole}} \bar{e}_1^{(1)} \times \bar{H}_1 \cdot \bar{i}_z dS$$

were given in Section 3-10 for various holes and apertures.

The factor $\bar{H}_1$ is the magnetic field pattern of the short-circuit mode which resonates at normalized frequency P_1 (Figs. 3-8 and 4-1b, for example), and $\bar{e}_1^{(1)}$ is the electric field pattern of the open-circuit mode

which resonates when $k = k_\pi$ and $V_1^{(1)} = -V_1^{(2)}$ (Fig. 3-9). The use of just one short-circuit mode for the fields implies that the others are weakly excited; Eq. 3-54 indicates that this is true at frequency k such that

$$\left| \frac{P_1^2 - k^2}{P_2^2 - k^2} \right| \ll 1$$

P_2 being the normalized resonant frequency of the next short-circuit mode.

We now wish to sharpen this description of one passband of a cavity chain so that it will apply to any passband of the chain with quite arbitrary coupling between cavities. Our generalization of Eq. 3-54 will lead to a dispersion relation for the n th passband of a cavity chain which has all the properties mentioned in Section 1-7: symmetry of the ω - φ curve about the $\varphi = 0$ axis [$\omega(\varphi) = \omega(-\varphi)$]; zero slope at $\varphi = 0, \pi$; and group velocity which is consistent with the power flow-stored energy relation of Section 1-4.

A. A Passband Described by Fields of Even Symmetry

Let us assume for definiteness the centerhole-coupled cavity chain shown in Fig. 4-1a along with its first k - φ curve. We are concerned with the behavior of the structure at frequencies within or near the first passband bounded by normalized frequency P_1 of the first short-circuit mode and by the frequency of the open-circuit mode, which we shall call p_1 . We say the fields have even symmetry because the E_z field reverses an even number of times—zero in this case—across the cavity. Since Eq. 3-54 tells us that the short-circuit mode is excited by tangential electric field on a hole as given by the open-circuit mode pattern $\bar{e}_1$, let us define that open-circuit mode by direct analogy with the short-circuit mode. So, just as we defined the short-circuit mode patterns $\bar{E}_1$ and $\bar{H}_1$ by Eqs. 1-21 and 3-42:

$$\nabla \times \bar{E}_1 = P_1 \bar{H}_1 \quad \bar{E}_1 \times \bar{n} = 0 \text{ on walls and holes}$$

$$\nabla \times \bar{H}_1 = P_1 \bar{E}_1 \quad P_1 = \omega_1 \sqrt{\mu\epsilon}$$

$$\int_V \bar{E}_1^2 dV = \int_V \bar{H}_1^2 dV = \tau, \quad \text{cavity volume}$$

so we define the open-circuit mode patterns $\bar{e}_1$ and $\bar{h}_1$ within a cavity by Eq. 1-22:

$$\nabla \times \bar{e}_1 = p_1 \bar{h}_1 \quad \bar{e}_1 \times \bar{n} = 0 \text{ on walls}$$

$$\nabla \times \bar{h}_1 = p_1 \bar{e}_1 \quad \bar{e}_1 \cdot \bar{n} = 0 \text{ on holes}$$

$$p_1 = \omega_1' \sqrt{\mu\epsilon}$$

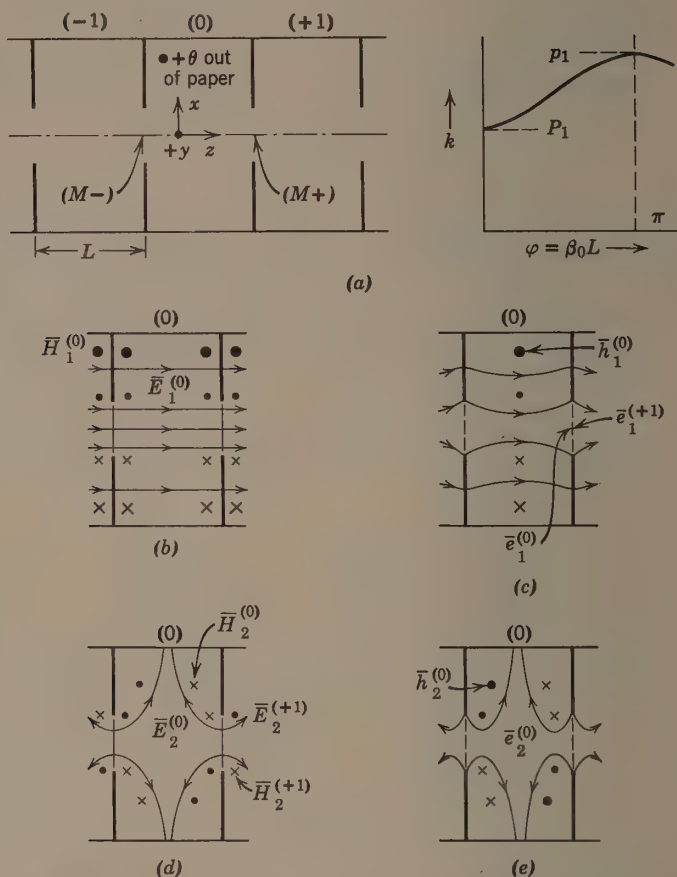


Fig. 4-1. A centerhole-coupled cavity chain and its first four resonant cavity modes. (a) Side view of the physical structure and the first dispersion curve. (b) Patterns of the first short-circuit resonant cavity mode resonating between electrically shorted centerholes at frequency P_1 . Superscript (0) denotes the reference cavity. (c) Patterns of the first open-circuit resonant cavity mode resonating between magnetically shorted centerholes at frequency P_1 . (d) Patterns of the second short-circuit resonant cavity mode. (e) Patterns of the second open-circuit resonant cavity mode. All patterns are repeated from cavity to cavity as shown.

and Green's theorem, Eq. 3-43b, again allows us to normalize both $\vec{e}_1$ and $\vec{h}_1$ as

$$\int_V \vec{e}_1^2 dV = \int_V \vec{h}_1^2 dV = \tau, \quad \text{cavity volume}$$

This $\vec{e}_1$ is the pattern which appears inside the coupling integral defined on the holes and apertures between neighboring cavities and evaluated in Section 3-10.

For the analysis of the infinite cavity chain we may repeat the patterns $\bar{E}_1$ and $\bar{H}_1$ from cavity to cavity with the same orientations, as shown in Fig. 4-1b. But we must satisfy Floquet's theorem, at a given frequency, so we must phase shift the amplitudes V_1 and I_1 from one cavity to the next. Let $V_1^{(0)}$ and $I_1^{(0)}$ within cavity (0) be the reference amplitudes, then $V_1^{(+1)} = V_1^{(0)} \epsilon^{-j\varphi}$ and $I_1^{(+1)} = I_1^{(0)} \epsilon^{-j\varphi}$. Since we have $\epsilon^{j\omega t}$ time dependence, a "wave" of positive φ will have its phase velocity in the $+z$ direction. It is important to note that such a description of the fields implies a discontinuity across each hole; for example, on the left side of hole ($M+$) in Fig. 4-1a we have electric field $V_1^{(0)} \bar{E}_1$, but immediately on the right side we have $V_1^{(0)} \epsilon^{-j\varphi} \bar{E}_1$! We are not certain of how most proponents of continuous field expansions would regard such a representation; we can only say that it not only works but works extremely well, provided we average the field expressions on both sides of the hole. By analogy, we associate amplitudes v_1 and i_1 with the patterns $\bar{e}_1$ and $\bar{h}_1$ and phase shift these amplitudes in the same way.

Now we are ready to reinterpret Eq. 3-54 more significantly. That equation really tells us this: When we represent the fields of Maxwell's equations by $V_1 \bar{E}_1$ and $I_1 \bar{H}_1$ within a cavity and operate on those equations as in Section 3-9 so as to obtain V_1 and I_1 in terms of the tangential electric field on the holes, that tangential electric field on a hole is *one-half the sum of the open-circuit mode expressions for it on either side*. Observe in this connection that the bracket of Eq. 3-54 can be written for $V_1^{(0)}$ of cavity (0) as

$$\frac{1}{2}(v_1^{(0)} \bar{e}_1^{(0)} + v_1^{(+1)} \bar{e}_1^{(+1)})$$

on the right-hand hole of Fig. 4-1a, since pattern $\bar{e}_1$ reverses direction across this hole in Fig. 4-1c! But now we have two amplitudes, $V_1^{(0)}$ and $v_1^{(0)}$, describing the same electric field within cavity (0). It only makes sense to say that they *both represent the same electric field*. When we say that $v_1 \bar{e}_1$ of the open-circuit mode represents (tangential) electric field on a hole, obviously $v_1 \bar{e}_1$ must represent the electric field within the cavity as well and should be equivalent or nearly so to $V_1 \bar{E}_1$ of the short-circuit mode.

Let us write down all these statements concisely to see how they determine the fields within reference cavity (0) of Fig. 4-1. We represent the fields by $V_1 \bar{E}_1$ and $I_1 \bar{H}_1$ [omitting the superscript (0) for brevity] of the short-circuit mode and operate on Maxwell's equations as in Section 3-9, so as to obtain two equations for V_1 and I_1 in terms of $\hat{E}_{\text{tan}}$. The hole excitation is given by

$$\int_{\text{holes}} \hat{E}_{\text{tan}} \times \bar{H}_1 \cdot \bar{n} dS$$

where $\bar{n} = +\bar{i}_z$ on the right-hand hole and $\bar{n} = -\bar{i}_z$ on the left-hand hole.

The $\hat{E}_{\text{tan}}$ on the right-hand hole is $\frac{1}{2}v_1(\bar{e}_1^{(0)} + \epsilon^{-j\varphi}\bar{e}_1^{(+1)})_{\text{on}(M+)}$ and is $\frac{1}{2}v_1(\bar{e}_1^{(0)} + \epsilon^{j\varphi}\bar{e}_1^{(-1)})_{\text{on}(M-)}$ on the left-hand hole. We obtain straightforwardly

$$P_1 V_1 = -\frac{P_1}{\tau} v_1 \int_{M+} \frac{1}{2}(\bar{e}_1^{(0)} + \epsilon^{-j\varphi}\bar{e}_1^{(+1)}) \times \bar{H}_1 \cdot \bar{i}_z dS \quad (4-1)$$

$$- \frac{P_1}{\tau} v_1 \int_{M-} \frac{1}{2}(\bar{e}_1^{(0)} + \epsilon^{+j\varphi}\bar{e}_1^{(-1)}) \times \bar{H}_1 \cdot (-\bar{i}_z) dS - j\omega\mu I_1$$

$$P_1 I_1 = j\omega\epsilon V_1, \quad (4-2)$$

from which we can form the equation for V_1 within cavity (0) as

$$(P_1^2 - k^2)V_1 = -\frac{P_1}{\tau} v_1 \int_{M+} \frac{1}{2}(\bar{e}_1^{(0)} + \epsilon^{-j\varphi}\bar{e}_1^{(+1)}) \times \bar{H}_1 \cdot \bar{i}_z dS$$

$$+ \frac{P_1}{\tau} v_1 \int_{M-} \frac{1}{2}(\bar{e}_1^{(0)} + \epsilon^{+j\varphi}\bar{e}_1^{(-1)}) \times \bar{H}_1 \cdot \bar{i}_z dS \quad (4-3)$$

We have been very careful to attach a superscript in parentheses on each $\bar{e}_1$ to denote on which side of the hole the pattern is measured. As a matter of fact we ought to have written $\bar{H}_1$ as $\bar{H}_1^{(0)}$ because we are evaluating the mode amplitudes within cavity (0). But $\bar{H}_1$ of an even short-circuit mode does not change direction across a hole (Fig. 4-1b), and so the superscript is irrelevant.

In Fig. 4-1c we observe the change of direction of $\bar{e}_1$ across each hole:

$$\bar{e}_1(x, y)_{\text{on}(M-)}^{(-1)} = -\bar{e}_1(x, y)_{\text{on}(M-)}^{(0)} = +\bar{e}_1(x, y)_{\text{on}(M+)}^{(0)}$$

$$= -\bar{e}_1(x, y)_{\text{on}(M+)}^{(+1)} \quad (4-4)$$

at a common transverse point (x, y) . With these relations we can write the right sides of Eqs. 4-1 and 4-3 in terms of a single integral which is the very important *passband parameter* M :

$$M_{11} \triangleq \frac{1}{\tau} \int_{(M-)} \bar{e}_1^{(0)} \times \bar{H}_1^{(0)} \cdot \bar{i}_z dS, \quad \text{units of } k = \omega\sqrt{\mu\epsilon} \quad (4-5)$$

It is defined at the left hole of Fig. 4-1 in accord with the literature. Because we assume cavities with mirror symmetry about a transverse bisecting plane it follows that $\bar{i}_z \times \bar{e}_1^{(0)}$ reverses direction from a point on one hole to the same transverse point on the other hole. Therefore M_{11} is precisely τ^{-1} times the coupling integral as evaluated in Section 3-10 on the right-hand hole of the reference cavity, for passbands of even field symmetry. In terms of M_{11} we can write Eqs. 4-1 and 4-2 as

$$P_1 V_1 = M_{11} v_1 (1 - \cos \varphi) - j\omega\mu I_1 \quad (4-6)$$

$$P_1 I_1 = j\omega\epsilon V_1$$

which combine to give us

$$(P_1^2 - k^2)V_1 = P_1 M_{11}(1 - \cos \varphi)v_1 \quad (4-7)$$

for V_1 and v_1 within cavity (0).

The relation between V_1 and v_1 is obtained by invoking the equivalence of both $V_1 \bar{E}_1$ and $v_1 \bar{e}_1$ for the electric field *within* the cavity:

$$V_1 \bar{E}_1 \approx v_1 \bar{e}_1 \quad (4-8)$$

This equivalence, of course, breaks down on a hole where $\bar{E}_1$ is normal and $\bar{e}_1$ is tangential. But if it is true over most of the volume, we can dot-multiply the equation by either $\bar{E}_1$ or $\bar{e}_1$, integrate over the cavity volume, and obtain

$$\begin{aligned} V_1 &\simeq v_1 \frac{1}{\tau} \int_V \bar{E}_1 \cdot \bar{e}_1 dV \\ v_1 &\simeq V_1 \frac{1}{\tau} \int_V \bar{E}_1 \cdot \bar{e}_1 dV \end{aligned} \quad (4-9)$$

Since both of these relations cannot be exactly true there exists some ambiguity in this single mode-pair description of the fields. The variational treatment of Section 6-7 resolves the ambiguity as

$$V_1 = T_{11}v_1, \quad T_{11} = \frac{1}{\tau} \int_V \bar{E}_1 \cdot \bar{e}_1 dV \quad (4-10)$$

This is entirely reasonable when we recall that our single-passband approximation is valid for a narrow passband for which $P_1 \simeq p_1$, the coupling holes are small, and so $\bar{E}_1 \simeq \bar{e}_1$ within the cavity. Equation 3-54, too, was derived by assuming a small perturbation of a short-circuit cavity mode so that the hole excitation is also proportional to V_1 . The error in taking

$$\frac{1}{\tau} \int_V \bar{E}_1 \cdot \bar{e}_1 dV$$

equal to 1 is a small *fractional* error in M_{11} but M_{11} is still good enough to describe the width of a narrow passband.

Equations 4-7 and 4-10 then furnish us with the determinantal equation for the first passband of the Fig. 4-1 structure:

$$(P_1^2 - k^2)T_{11} = P_1 M_{11}(1 - \cos \varphi) \quad (4-11)$$

which is correct at $\varphi = 0$ where the short-circuit mode resonates at $k = P_1$. We can also check it at $\varphi = \pi$ by invoking Green's theorem Eq. 3-43b. For if we let $\bar{A} = \bar{e}_1$, $\bar{B} = \bar{E}_1$, and use the definition in Eqs. 1-21 and 1-22,

we obtain within cavity (0)

$$\begin{aligned}
 (P_1^2 - p_1^2) \int_V \bar{E}_1 \cdot \bar{e}_1 dV \\
 &= -P_1 \oint_S \bar{e}_1^{(0)} \times \bar{H}_1^{(0)} \cdot \bar{n} dS \\
 &= P_1 \int_{M-} \bar{e}_1^{(0)} \times \bar{H}_1^{(0)} \cdot \bar{i}_z dS - P_1 \int_{M+} \bar{e}_1^{(0)} \times \bar{H}_1^{(0)} \cdot \bar{i}_z dS \\
 &= 2P_1 M_{11} \tau
 \end{aligned} \tag{4-12}$$

where we also used the fact that $\bar{e}_1^{(0)}(x, y)$ at $(M+) \equiv -\bar{e}_1^{(0)}(x, y)$ at $(M-)$. We see that Eq. 4-12 does verify the fact that $k = p_1$ at $\varphi = \pi$ where the open-circuit mode resonates.

Does Eq. 4-11 imply the correct stopband behavior just above $k = p_1$? In this case $|P_1^2 - k^2| > |2P_1 M_{11}|$, and so $|1 - \cos \varphi| > 2$. If we set $j\varphi = j\pi + \alpha$, we obtain

$$(P_1^2 - k^2)T_{11} = P_1 M_{11}(1 + \cosh \alpha), \quad k \geq p_1 \tag{4-13}$$

Detailed calculations on a periodically lumped-loaded transmission line verify the accuracy of Eq. 4-13 (with $p_1 = k_\pi$ in place of P_1 on the right side because $P_1 = 0$ in this line) but only in a narrow frequency range just above the π cutoff frequency. As the frequency rises the modes of the next passband very rapidly become important in the field expressions; only by including them in an extension of this analysis do we obtain an accurate picture of the stopband behavior. The fact that our dispersion relation is accurate for both real and imaginary φ indicates that it should be accurate for *complex* φ which result when an electron beam is introduced into the chain (Section 8-4).

B. A Passband Described by Fields of Odd Symmetry

According to the definition of even symmetry in the preceding subsection we say that fields of odd symmetry have E_z components which suffer an odd number of reversals across a cavity; the predominant modes, naturally, have this same property. Examples are the two modes of Fig. 1-6*d,e* and repeated in Fig. 4-1*d,e*, which describe the second passband of the centerhole-coupled cavity chain provided the cavities are not too short, in which case they describe a higher-frequency passband. The analysis of the preceding subsection is only altered by the fact that the transverse $\bar{e}_2$ pattern of the odd open-circuit mode does not change

direction across a hole, whereas $\bar{H}_2$ does:

$$\begin{aligned}\bar{H}_2(x, y)_{\text{on}(M-)}^{(-1)} &= -\bar{H}_2(x, y)_{\text{on}(M-)}^{(0)} = +\bar{H}_2(x, y)_{\text{on}(M+)}^{(0)} \\ &= -\bar{H}_2(x, y)_{\text{on}(M+)}^{(+1)}\end{aligned}\quad (4-14)$$

This changes Eq. 4-6 for reference cavity (0) to

$$\begin{aligned}P_2 V_2 &= +P_2 M_{22}(1 + \cos \varphi) v_2 - j\omega \mu I_2 \\ P_2 I_2 &= j\omega \varepsilon V_2\end{aligned}\quad (4-15)$$

with the passband parameter defined as before:

$$M_{22} \triangleq \frac{1}{\tau} \int_{M-} \bar{e}_2^{(0)} \times \bar{H}_2^{(0)} \cdot \bar{i}_z dS \quad (4-16)$$

The parameter M_{22} is τ^{-1} times the coupling integrals for TE or TM modes as defined at the $(M+)$ hole in Section 3-10 for passbands in which the fields have the odd symmetry across each cavity. Equations 4-15 combine to give us an equation for V_2 in terms of v_2 :

$$(P_2^2 - k^2) V_2 = P_2 M_{22}(1 + \cos \varphi) v_2 \quad (4-17)$$

The relation between the two mode amplitudes V_2 and v_2 is

$$V_2 = \frac{1}{\tau} \int_V \bar{E}_2 \cdot \bar{e}_2 dV v_2 \triangleq T_{22} v_2$$

and so the determinantal equation for $k(\varphi)$ is

$$(P_2^2 - k^2) T_{22} = P_2 M_{22}(1 + \cos \varphi) \quad (4-18)$$

This equation is correct at $\varphi = \pi$ where the short-circuit mode resonates: $k = P_2$ (see Fig. 4-1*d*), and Green's theorem, Eq. 3-43*b*, verifies its correctness for $k = P_2$ at $\varphi = 0$.

4-3. A Theorem about the Passband Shape

We see that the passband parameter M_{11} or M_{22} very nearly measures the width of a narrow passband. By either Eq. 4-12 for a passband of even symmetry or Eq. 4-17 (with $\varphi = 0$ and $k = p_2$) for one of odd symmetry we have

$$|M_{nn}| \simeq \frac{1}{2} P_n |P_n^2 - p_n^2| \simeq |P_n - p_n|, \quad P_n \simeq p_n \quad (4-19)$$

The exact expression for $|M_{nn}|$ multiplies the right side of Eq. 4-19 by

$$\left| \frac{1}{\tau} \int_V \bar{E}_n \cdot \bar{e}_n dV \right| \triangleq T_{nn}$$

Now let us express M_{11} in terms of P_1 and p_1 by Eq. 4-12 and substitute it into dispersion Eq. 4-11. Then we obtain the passband equation in terms of P_1 and p_1 alone:

$$P_1^2 - k^2 - \frac{1}{2}(P_1^2 - p_1^2)(1 - \cos \varphi) = 0 \quad (4-20)$$

Although Eq. 4-20 was derived with reference to a centerhole-coupled cavity chain it generalizes to a passband of any chain in which the fields have even symmetry. If we express M_{22} in Eq. 4-18 in terms of $P_2^2 - p_2^2$ at $\varphi = 0$, we can write that equation as

$$P_2^2 - k^2 - \frac{1}{2}(P_2^2 - p_2^2)(1 + \cos \varphi) = 0 \quad (4-21)$$

and this equation describes any narrow passband with fields of odd symmetry.

Observe that dispersion relations 4-20 and 4-21 give us $k^2 = \omega^2 \mu \epsilon$ in terms of $\varphi = \beta_0 L$ and the cutoff frequencies alone, *independent of the size or shape of the (small) coupling hole*. We have the following theorem²

Given a passband for a symmetrical periodic structure [one that supports resonant modes of either even or odd symmetry] which is narrow (on a frequency scale) compared to its two adjacent stopbands, then its $\omega^2 - \beta_0$ curve ($\beta_0 = \varphi/L$) is independent of the nature of the coupling and is a $1 \pm \cos \varphi$ function between the cutoff points.

The theorem says that if the structure is operated at a frequency within a narrow passband the fields are almost entirely given by the modes of that passband and the $\omega^2 - \beta_0$ curve "has no choice" but to follow a $1 \pm \cos \varphi$ curve. Since the passband is presumed narrow, this amounts to saying that the $\omega - \beta_0$ curve has the same variation. We have derived this theorem independently of any equivalent circuit representations of the structure!

It is evident that the theorem is valuable as a first approximation to the passband behavior of a periodic cavity chain. The extent to which a dispersion curve does not obey the theorem indicates the relative proximity of other cavity mode resonances to the cutoff frequencies of the passband of interest. For example, in certain resonant-slot coupled structures treated in Section 7-2 the "cavity" passband (defined as one in which the fields tend to phase shift across the slots) grossly violates the theorem because certain resonant slot modes contribute strongly to the fields, even at frequencies within the "cavity" passband. The converse of this statement, however, is not necessarily true; if a passband obeys the theorem well, neighboring modes may contribute rather strongly to the

² See R. M. Bevenssee [1].

fields. This is because the modes of the upper passband tend to push the “sandwiched” $\omega-\beta_0$ curve down, while the modes of the lower passband tend to push it up. The result is a passband that usually very nearly obeys the theorem.

We mentioned earlier that $|M_{nn}|$ is very nearly the passband width of the n th passband in units of k . The sign of M_{nn} gives the slope of the dispersion curve more conveniently than a calculation of relative stored

Table 4-1. Character of the n th Dispersion Curve of a Symmetrical Cavity Chain

E_z Field Symmetry	Sign of M_{nn}	Mode Resonating at $\varphi = 0$	Slope, $\partial\omega/\partial\varphi$, over $0 \leq \varphi \leq \pi$
1. Even	—	Short-circuit	+
2. Even	+	Short-circuit	—
3. Odd	—	Open-circuit	—
4. Odd	+	Open-circuit	+

Examples:

1. First passband of a centerhole-coupled cavity chain.
2. First passband of a cylindrical cavity chain coupled by holes near the outer walls.
3. A passband (it could be the first or second one) defined by a pair of resonant-slot modes (Fig. 4-2).
4. First TE_{10} mode passband of a chain of rectangular cavities coupled through round holes.

energies. For example, we observe that the first passband of the centerhole-coupled chain in Fig. 4-1 is described by a pair of modes of even symmetry. We go to Eq. 4-11, which is appropriate to a passband of this symmetry, and observe that if $M_{11} < 0$ the passband slopes upward from $\varphi = 0$, $k = P_1$; if $M_{11} > 0$, it slopes downward. We find in Eq. 4-5 that $\vec{e}_1^{(0)}$ points inward on the (0) cavity side of hole ($M-$) in Fig. 4-1c and that $\vec{H}_1$ circles in the $+\theta$ direction in Fig. 4-1b. Therefore $M_{11} < 0$ and the first passband slopes upward, Fig. 4-1a. This kind of reasoning is valid even for a passband which is not narrow and with other passbands in close proximity. This is because it is really based upon Green's relation of Eq. 4-12, which is exact.

The four possible choices of field symmetry and sign of M_{nn} for the n th passband are listed in Table 4-1, along with the dispersion curve shape and one example of each choice.

In examples 1 and 2 we can understand the effect of moving a small

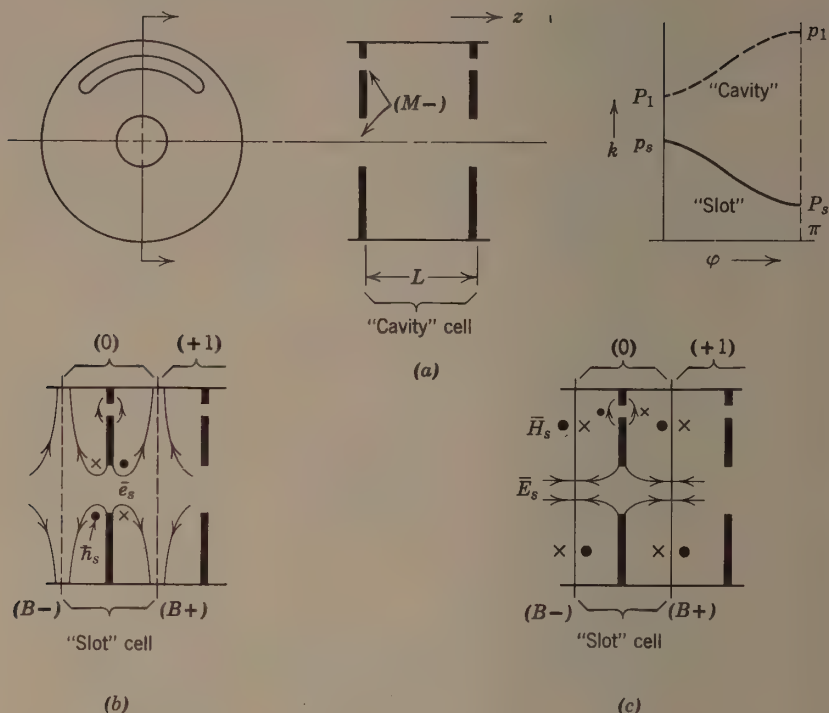


Fig. 4-2. A resonant slot coupled chain and the two modes which resonate at the cutoff frequencies of the "slot" passband. (a) Physical structure and the first two dispersion curves, (b) the first open-circuit mode resonating at frequency p_s between magnetically shorted $(B\pm)$ planes bounding the "slot" cell, (c) the first short-circuit mode resonating at frequency P_s between electrically shorted planes $(B\pm)$.

coupling hole from the center to the edge of a cavity with reference to Eq. 3-55²:

$$M_{11} = \frac{4}{3} P_1 \frac{a^3}{\tau} [2\bar{H}_1^2(0) - \bar{E}_1^2(0)]$$

For the first passband a centerhole will displace $\bar{E}_1$ more than $\bar{H}_1$ so $M_{11} < 0$ and the π resonant frequency rises. The fringing electric field around a magnetically shorted hole creates more stored electric energy, which drives the resonant frequency upward to restore the energy balance. A small hole near the outer wall of a cavity changes the sign of M_{11} to plus and lowers the π resonant frequency with respect to the zero resonance. In this case the introduction of a magnetic short on the hole effectively

² This looks like Bethe's expression but it is actually only half as large because our τ is the full cavity volume, not the volume of a single half-cavity in Fig. 3-8.

raises the stored magnetic energy around the hole and lowers the π -frequency. This latter effect is somewhat difficult to verify in terms of energy, but it is obvious from the sign of M_{11} .

Example 3, illustrated by the resonant-slot coupled structure of Fig. 4-2, is included to show that the phase shift of amplitudes may effectively occur across boundaries ($B\pm$) which bisect the cavities proper rather than across hole or slot surfaces. This "slot" passband could be the first or second passband in the chain, depending upon the slot length. Both the open- and short-circuit resonant slot modes resonate at frequencies such that the slot is very nearly half a free-space wavelength long. We have much more to say about such structures in Sections 7-2 through 7-4. The passband parameter M for such a "slot" passband is defined exactly as before except that the left-hand boundary of the periodic cell is now ($B-$) instead of ($M-$).

$$M_{ss}(\text{"slot" passband}) = \frac{1}{\tau} \int_{B-} \vec{e}_s^{(0)} \times \vec{H}_s^{(0)} \cdot \vec{i}_z dS$$

where (0) denotes the reference "slot" cell. Curiously, none of the ordinary hole-, slot-, and loop-coupled structures supporting axial electric field illustrate case 4 for the first dispersion curve.

Confusion has undoubtedly now arisen concerning the resonant modes defined within a "cavity" cell in Fig. 4-1 and the modes for the "slot" passband defined within the "slot" cell of Fig. 4-2. We wonder which pair of modes to use for which passband, and whether to define both "cavity" and "slot" mode-pairs for a given structure. The answer is: we must be on the lookout for both types of modes, but we will usually find both types in the search for the modes of *either* type. To illustrate, suppose the structure of Fig. 4-2 has the first passband shown, as defined by the two resonant "slot" modes within the "slot" cell, and a second passband defined by the resonant "cavity" modes within the "cavity" cell, like those of Fig. 4-1. But we may not know of these two varieties of resonant modes when we begin searching for resonant "cavity" modes which are defined with respect to electrically and magnetically shorted slots and centerholes. The very first two such modes we find resonating within a "cavity" cell of Fig. 4-2 resemble roughly the patterns of Fig. 4-1c,e, except that half the lines are reversed in Fig. 4-1c so the mode appears as shown in Fig. 4-2c. But *both* these modes resonating within a "cavity" cell are *open-circuit* on the slots and holes; they cannot couple together on the slots and holes to give us a single-passband dispersion relation like Eq. 4-11 or Eq. 4-18 because M would be identically zero! This fact is the clue that these two modes should be redefined for continuity through the "slot" cell and with phase shifts across boundaries ($B\pm$). Sure

enough, when we so redefine them (Fig. 4-2*b,c*) we find that we now have an open-circuit-short-circuit mode-pair which describes the first or "slot" passband. Then if we continue with our search for resonant "cavity" modes we find the next two resemble those of Fig. 4-1*b,c*. These are an open-circuit-short-circuit mode-pair on the slots and holes which do describe the qualitative behavior of the second passband. So we labeled that passband by the adjective "cavity."

Now suppose we had begun searching only for resonant "slot" modes within the "slot" cell of Fig. 4-2, as defined with phase shifts across the $(B\pm)$ boundaries. The very first pair of modes we find do indeed describe the first passband, labeled "slot." But the next pair of modes we find look like those of Fig. 4-1*b,c* except that the patterns of the open-circuit mode to the left of $(M-)$ and right of $(M+)$ in Fig. 4-1*c* are reversed so as to provide for field continuity throughout the "slot" cell. But *both* of these modes resonate within the "slot" cell as defined by electrically shorted $(B\pm)$ and as such cannot describe the behavior of the second passband, according to the single mode-pair analysis. This fact is the clue that these two modes should instead be defined for continuity through the "cavity" cell, with phase shift of amplitudes across the slots and holes. Sure enough, when we so redefine them we can describe the second passband, labeled "cavity."

For the single-passband or single mode-pair analysis of this chapter we must define both the resonant "slot" modes of Fig. 4-2*b,c* to explain that "slot" passband and both the resonant "cavity" modes like those in Fig. 4-1*b,c* to explain that "cavity" passband. In principle, however, we could use either one set of modes or the other to describe the dispersion at any frequency (assuming completeness of both types of modes). As a matter of fact we use both types when we formulate a dispersion relation variationally for the resonant-slot coupled chain (Section 7-5).

Perhaps it would be of interest to show that, as the frequency increases, the density of passbands increases as k^2 and therefore the width of each passband must decrease to zero. This means that, as frequency increases, the $\bar{e}_n$ and $\bar{H}_n$ mode patterns "wriggle" more and more rapidly across the coupling hole and the passband integral M_{nn} (see Eq. 4-5) goes to zero. In order to measure the density of short-circuit mode resonances let us assume the cavity has a uniform cross-sectional area A and write the equation for $\bar{E}_z = \bar{E}_z(x, y) \cos(m\pi z/L)$ of one of those resonances as

$$(\nabla_T^2 + P_n^2 - k_{zm}^2)\bar{E}_z = 0, \quad k_{zm} = \frac{m\pi}{L} \quad (4-22)$$

Here P_n is the normalized resonant frequency of the n th short-circuit cavity mode. This two-dimensional Helmholtz equation for E_z has been

studied for very general boundary conditions, so that if $P_n^2 - k_{zm}^2$ is a very large number we can say that the number of resonances dn between eigenvalue $\sqrt{P_n^2 - k_{zm}^2}$ and $\sqrt{P_n^2 - k_{zm}^2} + d\sqrt{P_n^2 - k_{zm}^2}$ is³

$$dn = \frac{A}{2\pi} \sqrt{P_n^2 - k_{zm}^2} d\sqrt{P_n^2 - k_{zm}^2} \quad (4-23a)$$

Since $d\sqrt{P_n^2 - k_{zm}^2} = (P_n^2 - k_{zm}^2)^{-1/2} P_n dP$ for fixed k_{zm} , the number of resonances between P_n and $P_n + dP$ is

$$dn = \frac{A}{2\pi} P_n dP \quad (4-23b)$$

This same density of resonances corresponds to $k_{zm} = 0, \pi/L, 2\pi/L, \dots$ but diminishes as k_{zm} approaches P_n . But let us use the same density for all k_{zm} up to P_n and calculate the total number of resonances in the dP interval as

$$dN = \frac{A}{2\pi} P_n dP \frac{P_n L}{\pi} = \frac{\tau}{2\pi^2} P_n^2 dP \quad (4-24)$$

The error in dN is a term proportional only to P_n .

In this way we see that the density of short-circuit cavity mode resonances, dN/dP , increases as P^2 as $P \rightarrow \infty$, so that the bandwidth of successive passbands approaches zero as P^{-2} .

4.4. The Power Flow-Stored Energy-Group Velocity Relation

In Section 1-4 we proved that for a periodic lossless structure the relation between the time-average real power flow P_{av} and the total time-average stored energy W_s is

$$P_{av} = \frac{W_s}{L} \frac{\partial \omega}{\partial \beta_0} \quad (1-28)$$

at the frequency at which group velocity $\partial \omega / \partial \beta_0$ is measured. This means the group velocity is not only the velocity of transport of a low-frequency carrier wave envelope, Eq. 1-16, but also the velocity of energy transport.

We now wish to verify our single-passband or single mode-pair relations 4-11 and 4-18 with respect to Eq. 1-14. The time-average power flow may be measured by the averaged tangential electric and magnetic fields on surface ($M-$) of Fig. 4-1, or on ($B-$) of Fig. 4-2, for the n th passband as

$$P_{av} = \frac{1}{2} \operatorname{Re} \int_{M-} \frac{1}{2} v_n (\bar{e}_n^{(0)} + \epsilon^{j\varphi} \bar{e}_n^{(-1)}) \times \frac{1}{2} I_n^* (\bar{H}_n^{(0)} + \epsilon^{j\varphi} \bar{H}_n^{(-1)})^* \cdot \bar{i}_z dS \quad (4-25)$$

³ See P. M. Morse and H. Feshbach [2, p. 761].

If the n th passband has fields (modes) of even symmetry, $\tilde{e}_n^{(-1)}(x, y) = -\tilde{e}_n^{(0)}(x, y)$ on $(M-)$ or $(B-)$; if the fields have odd symmetry, $\tilde{H}_n$ reverses direction across surface $(M-)$ or $(B-)$. We obtain

$$P_{av} = \mp \frac{1}{4} j v_n I_n^* M_{nn} \tau \sin \varphi \quad \begin{array}{l} \text{upper sign, even symmetry;} \\ \text{lower sign, odd symmetry} \end{array} \quad (4-26)$$

Thus the power flow goes to zero at the resonant frequencies where $\varphi = 0, \pi$. We should add that Eq. 4-26 also obtains if we average *either* $\hat{e}_{\tan}$ or $\hat{H}_{\tan}$ in Eq. 4-25, or measure one of them on one side of the hole and the other by the mode expression on the other side.

Our single-passband approximation says $V_n = T_{nn} v_n$ and by Eqs. 4-6 and 4-15 we know $P_n I_n = j \omega \epsilon V_n$, so we may write Eq. 4-26 as

$$P_{av} = \mp \frac{1}{4} V_n^* v_n \frac{\omega \epsilon}{P_n} M_{nn} \tau \sin \varphi \quad (4-27)$$

The time-average electric stored energy is easily computed as

$$W_e = \frac{1}{4} \int_V \epsilon V_n^* \tilde{E}_n \cdot v_n \tilde{e}_n dV = \frac{1}{4} \epsilon V_n^* v_n T_{nn} \tau \quad (4-28)$$

by the definition of T . The time-average magnetic stored energy is equal to this, so the total stored energy is

$$W_s = \frac{1}{2} \epsilon V_n^* v_n T_{nn} \tau \quad (4-29)$$

If we now differentiate Eqs. 4-11 and 4-18 relative to φ we obtain the group velocity as

$$\frac{\partial \omega}{\partial \beta_0} = \frac{\partial k}{\partial \varphi} \frac{L}{\sqrt{\mu \epsilon}} = \mp \frac{L P_n M_{nn} \sin \varphi}{2 k \sqrt{\mu \epsilon} T_{nn}} \quad (4-30)$$

Therefore when we substitute Eqs. 4-27 through 4-30 into Eq. 1-14 we find the latter is indeed true. We realize, however, that we have neglected the contributions of other modes to the fields at a frequency in the n th passband. Nevertheless, we can prove by accounting for all the modes that this formalism verifies Eq. 1-14 exactly (see Appendix A, Section A-1).

4-5. The Maximization of Electronic Gain per Cavity and the Fractional Passband Width

In Section 8-4 we discuss the formal relationship between the resonant cavity mode and traveling wave (Pierce) descriptions of the waves in a cavity chain interacting with an electron beam. Certain necessary, but

not sufficient, conditions which amount to low gain per cavity and near-synchronism of d.c. beam and circuit must be fulfilled for these descriptions to agree. As a matter of fact, close examination of the waves (Chapter 9) indicates that the Pierce analysis which serves so well for helix-type traveling wave tubes is not particularly appropriate for most narrowband cavity-chain amplifiers and oscillators of interest even at frequencies away from cutoff. The cavity chains are better characterized by a new gain parameter C_B (the subscript B distinguishes it from Pierce's C_P) which is roughly equal to the appropriately defined C_P at frequencies away from cutoff and remains well behaved at cutoff. Parameter C_B varies essentially as (see Eq. 9-3)

$$C_B^2 \propto \frac{\bar{e}_z^2 \max}{\omega^2 \tau} \rho_0^{1/2} u_0 \frac{\omega_p}{\omega_q} \quad (4-31)$$

Here ω_p is the plasma frequency, ω_q is the reduced plasma frequency, ρ_0 is the d.c. space charge density, u_0 is the d.c. beam velocity, and relativistic corrections are not included. Parameter C_B is neither an explicit function of the fractional passband width nor one of the "cold" circuit power flow per unit amplitude, the latter proportional to $\sin \varphi^0$ where φ^0 is the "cold" circuit phase shift per cavity.

Physically, gain parameter C_B is very insensitive to the fractional passband width because we can always introduce extra cavity-cavity coupling beyond the centerholes which little affects e_z in the beam region between centerholes.

Equation 4-31 indicates that, for large small-signal gain per cavity, we should design cavities of small volume τ and operate them at low frequencies. We should also operate in the fundamental region $0 \leq \beta_0 L \leq \pi$ of the ω - β_0 curve so that beam velocity u_0 will be large. And, of course, a greater d.c. space charge density ρ_0 will enhance the gain per cavity.

About the best we can do to maximize C_B with respect to cavity dimensions is to design a cylindrical cavity chain for operation in the first TM_{010} passband, at a frequency just above the "straight-through" short-circuit mode cutoff frequency P_1 . Since $P_1 b = 2.405$, b being the outer radius, and $k \simeq P_1$, the cavity cross-sectional area is determined. Now, if we start with rather long cavities, the first passband is narrow and the normalized e_z is essentially $\sqrt{3.71} J_0(P_1 r)$ (no $e^{-j\beta_0 z}$ factor), uniform within a cavity. As we decrease cavity length L , at first $\bar{e}_z$ remains nearly uniform but $\bar{e}_z^2/\tau$ increases because τ decreases. The passband width also increases. When L reaches approximately the centerhole diameter, $\bar{e}_z$ takes on the "bowed" appearance shown in Fig. 4-1c at frequencies within the passband, and $\bar{e}_z$ in the beam region starts to diminish. Then as L decreases still further the chain starts to resemble more and more a fin-loaded waveguide

and should be analyzed as such (Sections 2-6, 6-6). The behavior of the cavity chain composed of cavities with square cross sections is nearly the same.

If we include relativistic corrections, C_B behaves as⁴

$$C_B^2 \propto \frac{\bar{e}_z^2 \max}{\omega^2 \tau} \rho_0^{1/2} u_0 \frac{\omega_p'}{\omega_q'} \left(1 - \frac{u_0^2}{c^2}\right)^{3/4} \quad (4-32)$$

with ω_q'/ω_p' , the ordinary plasma frequency reduction factor, as measured in a coordinate system moving with d.c. beam velocity u_0 . Space charge density ρ_0 is measured in the rest or laboratory frame. But even at a d.c. voltage of 100 kv the last factor in Eq. 4-32 is $(1 - 0.55^2)^{3/4} = 0.76$ and not too serious a reduction.

4-6. Dispersion Relation from the Open-Circuit Mode Expansion

Since we have elevated the open-circuit mode to the status of the short-circuit mode by using them both to represent the electric field within a cavity, we ought to be able to derive the same dispersion relations by expanding the fields of Maxwell's equations with the open-circuit mode. Such proves to be the case, and verifies the consistency of the two mode-expansion techniques.

We now follow the procedure dual to that in Section 3-9. We represent $\hat{E}$ by $v_n \bar{e}_n$ and H by $i_n \bar{h}_n$ at frequencies in the n th passband:

$$\begin{aligned} \nabla \times \hat{e} &= -j\omega\mu(i_n \bar{h}_n) \\ \nabla \times \hat{h} &= j\omega\varepsilon(v_n \bar{e}_n) \end{aligned} \quad (4-33)$$

where the n th open-circuit mode patterns are defined by Eq. 1-22 with subscript 1 changed to n on page 95. We dot-multiply the first equation by $\bar{h}_n$, integrate it over the cavity volume, and integrate the left side by parts as in Eq. 3-45. With

$$\int_{\text{holes}} \hat{E} \times \bar{h}_n \cdot \bar{n} dS \equiv 0$$

because $\bar{h}_n \times \bar{n} \equiv 0$ on the holes, we obtain just

$$p_n v_n = -j\omega\mu i_n \quad (4-34)$$

for the amplitudes within reference cavity (0). We next multiply Eq. 4-33b

⁴ The factor $(1 - u_0^2/c^2)^{3/4}$ appears because of the considerations in Appendix D.

by $\bar{e}_n$, integrate it similarly, and obtain

$$p_n i_n = + \frac{1}{\tau} \int_{\text{holes}} \bar{e}_n^{(0)} \times \hat{H} \cdot \bar{n} dS + j\omega\epsilon v_n \quad (4-35)$$

Notice that $\hat{H}_{\text{tan}}$ on the holes cannot be given by $i\hbar$ which is normal to them. Combining the last two equations, we obtain an equation for open-circuit mode amplitude i_n as excited by $\hat{H}$ on the coupling holes,

$$(p_n^2 - k^2)i_n = p_n \int_{\text{holes}} \bar{e}_n^{(0)} \times \hat{H} \cdot \bar{n} dS \quad (4-36)$$

We now represent $\hat{H}$ on hole ($M-$) of Fig. 4-1, for example, by the short-circuit modes which can represent $\hat{H}_{\text{tan}}$ there:

$$\hat{H}_{\text{tan on } (M-)} = \frac{1}{2} I_n (\epsilon^{j\varphi} \bar{H}_n^{(-1)} + \bar{H}_n^{(0)}) \quad (4-37)$$

with a similar expression for $\hat{H}_{\text{tan}}$ on the ($M+$) hole. All the mode patterns $\bar{e}_n$ and $\bar{H}_n$ on both sides of each hole can be related to those patterns on the $+z$ side of hole ($M-$) by Eqs. 4-4 and 4-14, and so Eq. 4-36 becomes

$$(p_n^2 - k^2)i_n = -p_n M_{nn} (1 \pm \cos \varphi) I_n \quad \begin{array}{l} \text{upper sign, fields} \\ \text{of even symmetry} \end{array} \quad (4-38)$$

Now the two representations $i_n \bar{h}_n$ and $I_n \bar{H}_n$ are equivalent throughout the cavity volume as a whole:

$$i_n \bar{h}_n \approx I_n \bar{H}_n \quad (4-39)$$

If we multiply both sides by $\bar{h}_n$ and integrate over the cavity volume we get

$$i_n \simeq I_n \frac{1}{\tau} \int_V \bar{H}_n \cdot \bar{h}_n dV \quad (4-40a)$$

But we can just as well obtain

$$I_n \simeq i_n \frac{1}{\tau} \int_V \bar{H}_n \cdot \bar{h}_n dV \quad (4-40b)$$

The variational discussion of Section 6-7 resolves this ambiguity as

$$i_n = U_{nn} I_n, \quad U_{nn} \triangleq \frac{1}{\tau} \int_V \bar{H}_n \cdot \bar{h}_n dV \quad (4-41)$$

with $U_{nn} \simeq 1$ for a narrow passband.

With Eq. 4-41 the amplitudes cancel in Eq. 4-38, and we obtain the dispersion relation

$$(p_n^2 - k^2)U_{nn} = -p_n M_{nn} (1 \pm \cos \varphi) \quad (4-42)$$

This is equivalent to Eq. 4-11 for an n th passband with fields of even symmetry (upper sign), since the open-circuit mode resonates at $k = p_n$

where $\varphi = \pi$ and for $k = P_n$, $\varphi = 0$, we obtain just Green's relation 4-12 with the additional relation $P_n U_{nn} = T_{nn} p_n$ derivable from Eq. 3-43a. Similarly we find that Eq. 4-42 for a passband of odd field symmetry (lower sign) is equivalent to Eq. 4-18 for the n th such passband.

Since we obtain essentially the same "cold" circuit dispersion relation by an open-circuit expansion of the fields in Maxwell's equations as by a short-circuit expansion of those fields we may question the utility of the former expansion. We should make an open-circuit mode expansion of the fields if we can obtain a more accurate representation of $\hat{H}_{\tan}$ than of $\hat{e}_{\tan}$ on the coupling surfaces. Such is the case for the cloverleaf structure (Section 7-7), in which we will represent the fields within each "slot" cell by an open-circuit "slot" mode which is excited by the magnetic field of a short-circuit "cavity" mode on the cavity midplanes.

For most single-passband analyses, however, it does not matter which point of view we take even if an electron beam is present, except that we should measure the beam-circuit coupling by $\int \hat{J} \cdot \bar{E}_n dV$ if the operating frequency k is nearer to the short-circuit resonant frequency P_n , and by $\int \hat{J} \cdot \bar{e}_n dV$ if it is nearer to p_n of the open-circuit mode.

4-7. Equivalent Circuits for the n th Passband

In the event that a narrow n th passband is described well by its two resonant cavity modes at frequencies in or near the passband, we can convert the amplitude equations into equivalent circuit form. This is not necessary for a description of the waves in either the "cold" or the "hot" structure; indeed, it proves to be impractical if we wish to include more open- and short-circuit modes in the field expansions. However, the single mode-pair equivalent circuits are very convenient for studying the termination of an amplifier which matches to the growing wave. Therefore we introduce the "cold" circuits here and modify them later to include the presence of an electron beam.

The equivalent circuit for the short-circuit mode amplitudes V_n and I_n of a passband is obtained by converting Eqs. 4-1 and 4-2 into proper form, with the definition of M_{nn} . Let us ignore $T_{nn} \simeq U_{nn} \simeq 1$ for convenience. We define these circuit quantities for reference cavity (0):

$$\begin{aligned} V_{nc} &= \frac{V_n}{P_n}, \quad \text{volts} & L_{nc} &= \frac{\mu}{P_n}, \quad \text{henrys} \\ I_{nc} &= \frac{I_n}{P_n}, \quad \text{amperes} & C_{nc} &= \frac{\epsilon}{P_n}, \quad \text{farads} \end{aligned} \tag{4-43}$$

so that when we write $v_n \simeq V_n = P_n I_n / j\omega\epsilon$ in Eq. 4-1 we can divide these

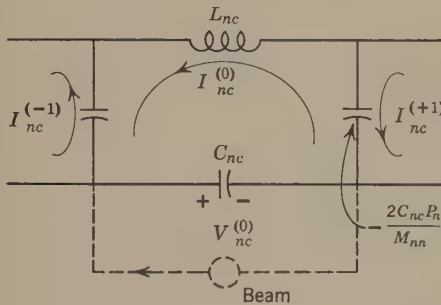
equations by P_n^2 and get, for a passband of even field symmetry,

$$V_{nc}^{(0)} = \frac{I_{nc}^{(0)}}{j\omega C_{nc}} \cdot \frac{M_{nn}}{P_n} - \frac{I_{nc}^{(-1)}}{2j\omega C_{nc}} \cdot \frac{M_{nn}}{P_n} - \frac{I_{nc}^{(+1)}}{2j\omega C_{nc}} \cdot \frac{M_{nn}}{P_n} - j\omega L_{nc} I_{nc}^{(0)} \quad (4-44)$$

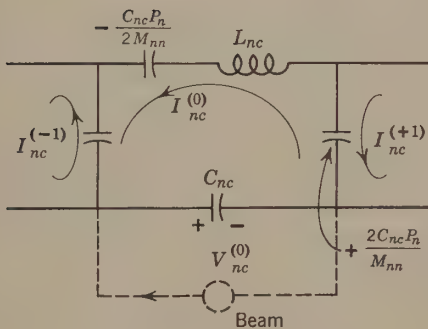
$$I_{nc}^{(0)} = j\omega C_{nc} V_{nc}^{(0)} \quad (\text{passband of even symmetry})$$

If the passband has odd symmetry (Eq. 4-15) the signs in front of the $I_{nc}^{(-1)}$ and $I_{nc}^{(+1)}$ terms change to plus. The circuits for both types of passbands are shown in Fig. 4-3a,b. At $\varphi = 0$ there are no voltage drops across the coupling capacitors of the Fig. 4-3a circuit and $\omega = (L_{nc}C_{nc})^{-1/2} = P_n/\sqrt{\mu\epsilon}$ of the short-circuit mode with even symmetry. The circuit of Fig. 4-3b resonates at $\varphi = \pi$ for the mode of odd symmetry, such that the voltage drops across all the condensers but C_{nc} cancel.

In both circuits of Fig. 4-3 L_{nc} and C_{nc} store the magnetic and electric energy of the cavity proper and the coupling capacitors, $\mp 2(C_{nc}P_n/M_{nn})$,



(a)



(b)

Fig. 4-3. Equivalent circuits from the short-circuit mode representation. (a) Passband of even $\hat{E}_z$ field symmetry, (b) passband of odd symmetry. The effect of a beam interacting with the $\hat{E}_z$ field is represented by the dashed generators.

store energy associated with the coupling hole. The second circuit for the short-circuit mode of odd symmetry has an additional capacitor, $-\frac{1}{2}(C_{nc}P_n/M_{nn})$, which is not combined with C_{nc} in order to illustrate the *reaction of the cavity fields back upon themselves* due to the hole presence. This is a consequence of the fact that the fields on the coupling hole are one-half the sum of the mode fields on either side. No fundamental significance should be attached to the fact that the circuit of Fig. 4-3b shows this reaction while that of Fig. 4-3a does not.

The equivalent circuit for the *open-circuit* mode amplitudes v_n and i_n of the same passband is obtained by converting Eqs. 4-34 through 4-37 into proper form. We define

$$\begin{aligned} v_{nc} &= \frac{v_n}{p_n} & L_{nc} &= \frac{\mu}{p_n} \simeq \frac{\mu}{P_n} \\ i_{nc} &= \frac{i_n}{p_n} & C_{nc} &= \frac{\varepsilon}{p_n} \simeq \frac{\varepsilon}{P_n} \end{aligned} \quad (4-45)$$

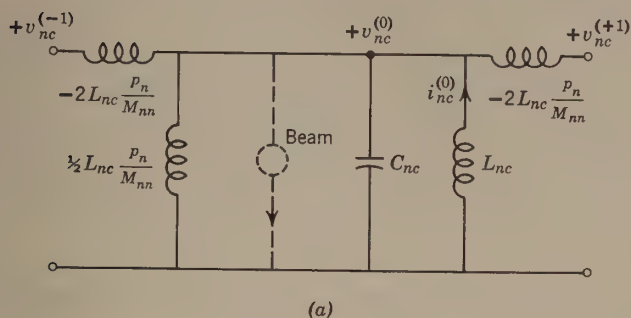
so that when we divide the first two equations by p_n^2 and express $I_n \simeq i_n$ as $p_n v_n / (-j\omega\mu)$ in Eq. 4-35 we get for a passband of *even* field symmetry

$$\begin{aligned} v_{nc}^{(0)} &= -j\omega L_{nc} i_{nc}^{(0)} \\ i_{nc}^{(0)} &= \frac{v_{nc}^{(0)}}{j\omega L_{nc}} \cdot \frac{M_{nn}}{p_n} + \frac{v_{nc}^{(-1)}}{2j\omega L_{nc}} \cdot \frac{M_{nn}}{p_n} + \frac{v_{nc}^{(+1)}}{2j\omega L_{nc}} \cdot \frac{M_{nn}}{p_n} + j\omega C_{nc} v_{nc} \end{aligned} \quad (4-46)$$

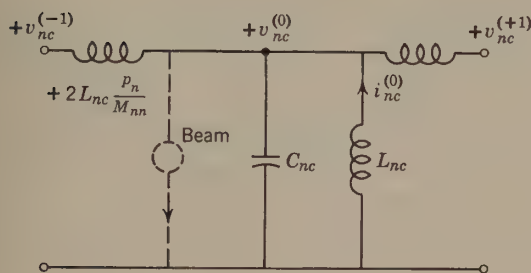
If the passband fields have odd symmetry the signs of the $v_{nc}^{(-1)}$ and $v_{nc}^{(+1)}$ terms change to minus. The circuits appear in Fig. 4-4a,b. We observe in the first circuit that at $\varphi = \pi$ there is zero net current through all the inductors except L_{nc} so this circuit resonates at frequency $(L_{nc}C_{nc})^{-1/2} = p_n/\sqrt{\mu\varepsilon}$ of the even open-circuit mode, as it should. The second circuit resonates at $\varphi = 0$ when no current flows into the adjacent terminals because the same odd open-circuit mode voltage exists across all of them.

In the circuits of Fig. 4-4 L_{nc} and C_{nc} also store the magnetic and electric energy of the cavity proper, while coupling inductors, $\mp 2(L_{nc}P_n/M_{nn})$, store the energy associated with the coupling holes. Inductor $\frac{1}{2}L_{nc}(p_n/M_{nn})$ represents the reaction of the cavity fields back upon themselves in the first but not the second circuit; this difference is not particularly significant.

It is certainly true that more than two equivalent circuits can be constructed to represent the behavior of the n th "cold" circuit passband. The question of which is the *best* equivalent circuit does not become important until we insert a beam into the structure. Then the nature of the growing wave is rather dependent upon the form of the circuit *because of the very particular way in which $\hat{J}$ comes into the second Maxwell equation.*



(a)



(b)

Fig. 4-4. Equivalent circuits from the open-circuit mode representation. (a) Passband of even $\hat{E}_z$ field symmetry, (b) passband of odd symmetry. The effect of a beam interacting with the $\hat{E}_z$ field is represented by the dashed generators.

An obvious example will make this point crystal clear. Suppose the effect of the beam upon the circuit is a current generator I_b in shunt with C_{nc} in Fig. 4-4 (which is the correct effect). Then $v_{nc}^{(\pm 1)} = v_{nc}^{(0)} \epsilon^{\mp \gamma}$ in an infinite line, where complex γ obviously depends upon the strength of the beam current. Now suppose we were to put a current generator $I_b \epsilon^{\gamma/2}$ in shunt with the left coupling inductor $-2(L_{nc} p_n / M_{nn})$ and another with the current $I_b \epsilon^{-\gamma/2}$ in shunt with the other coupling inductor such that the net current out of the $v_{nc}^{(0)}$ terminal is $I_b(\epsilon^{\gamma/2} - \epsilon^{-\gamma/2})$. The generators satisfy the periodicity condition but clearly this excitation is distinctly different from the correct one, especially so near the zero "cold" circuit cutoff frequency. To be certain of the correct circuit excitation we derive the circuit and beam excitations directly from Maxwell's equations in Section 8-4.

Equivalent circuits derived from Maxwell's equations (the solutions to which are based upon variational principles) do not always agree with our intuitive notion of an appropriate circuit. Two such instances clarify this point. For example, Chodorow and Craig [3] say that the positive

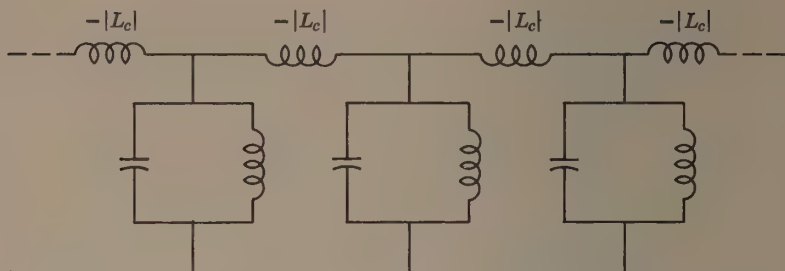


Fig. 4-5. An "equivalent" circuit for the single-passband behavior of a cavity chain (from Chodorow and Craig [3]).

group velocity of the first passband of a centerhole-coupled chain is due to negative mutual coupling in the "equivalent" circuit of Fig. 4-5. In this circuit at $\varphi = \pi$ the mutual inductances in parallel with the cavity inductance lower the total shunt inductance and raise the frequency. But this is not the picture at all in Fig. 4-4a. In our circuit the mutual inductors are positive for this passband since $M_{11} < 0$. If the phase shift changes from $\varphi = 0$ to $\varphi = \pi$, the frequency rises, not because of a negative mutual effect, but because the positive mutual inductors cancel the negative inductance $-\frac{1}{2}(L_{1c}p_1/|M_{11}|)$ of the cavity due to the reaction of its fields back upon themselves, and this effect raises the π cutoff frequency. From another point of view we might expect the centerhole to cause a reduction of capacity for the π mode, and yet in Fig. 4-3a this is the wrong picture. Rather, at $\varphi = \pi$ there are more positive capacitors in the series circuit representing more fringing electric stored energy around the holes and the resonant frequency rises to offset it.

On the other hand Chodorow and Craig say that slots cut in the outer walls of a chain of cylindrical cavities will raise the inductance of the π mode relative to the zero mode so that the π -mode has the lower resonant frequency. This statement does indeed describe the behavior of the circuit of Fig. 4-4a for our example 2 of Table 4-1, with $M_{11} > 0$. If the phase shift changes from zero to π in this circuit the negative mutual inductances cancel $+\frac{1}{2}(L_{1c}p_1/|M_{11}|)$ and lower the π resonant frequency.

We believe this discussion proves that we cannot choose any likely "equivalent" circuit and expect it to describe the fields in a given passband according to the mode solutions of the fundamental Maxwell equations.

Let us now compare Marcuvitz' equivalent circuit element for a given aperture with the coupling element in the circuit of Fig. 4-3. For the first TM mode incident upon the aperture both circuit elements have the same sign; for the first TE mode incident they have opposite signs, a fact which, however, is not of real basic significance. From Ref. [4], p. 218, we find that the aperture for the first TE mode passband of odd symmetry

(see Fig. 3-7 for the symmetry) is represented by a *positive capacitance* if the window is capacitive (naturally). The short-circuit mode of this passband, shown in Fig. 3-7, resonates at $\varphi = \pi$ with the aperture electrically shorted; the zero phase shift resonant frequency must be higher if the circuit element is capacitive. In Table 4-1 this is case 3, for which $M_{11} < 0$. And we see the sign of M_{nn} checks with Eq. 3-60 written as

$$(M_{11}\tau) = -\frac{2N^2}{377B} = -\frac{2N^2}{377\omega C_{ap}}$$

N being a normalizing constant. Now if we glance at our equivalent circuit illustrating shunt coupling between cavities for this passband of odd field symmetry, Fig. 4-3*b*, we find the coupling capacitance, $2(C_{1c}P_1/M_{11})$, is negative! And our derivation of bandwidth parameter M_{11} in terms of C_{ap} is certainly correct.

In fact, this difference of sign will occur between *any* aperture element for the first TE waveguide mode and the coupling element in the circuit of Fig. 4-3*b* for its passband. For the alternative case of an inductive aperture element the $\varphi = 0$ resonant frequency will be lower than the π resonant frequency, case 4 of Table 4-1 applies, M_{11} is positive, and our equivalent circuit coupling element is a positive capacitance; that is, a *negative* inductance. We are not trying to confuse anyone with these statements, for there is no fundamental reason why the signs of these two circuit elements should agree.

Problems

1. Consider the ladder lines of Fig. 2-7*c,d*, the dispersion curves of which appear on Fig. 2-8. The open- and short-circuit modes have which symmetry in Table 4-1? From sketches of the $\vec{e}$ and $\vec{H}$ patterns on a transverse plane midway between two "rungs" of each line verify the sign of M and the slope of the dispersion curve.
2. Derive the single mode-pair dispersion relation for the first passband of a coaxial waveguide loaded periodically by shunt capacitors and show it can be written $\omega = \omega_\pi \sin(\varphi/2)$.
3. Consider a centerhole-coupled cavity chain with nonresonant S-shaped loops effecting additional coupling where the magnetic field is strong. (See Fig. 4-6). The free-space wavelength is assumed to be far greater than

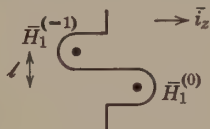


Fig. 4-6

dimension ℓ . Verify from sketches of the $\bar{e}_1$ pattern that the passband width will be increased.

4. Compare the general single mode-pair dispersion Eq. 4-20 with that shown in Fig. 4-27 on p. 70 of *Traveling Wave Tubes* by J. R. Pierce. Let $\omega_L \sim P_1$, $\omega_T \sim p_1$, and $\omega_T = 0.9\omega_L$.
5. Calculate the exact periodic attenuation α vs. k in the first stopband of the lumped-loaded line of Fig. 1-5 with these parameters:

$$\begin{aligned}\omega_2 C_L / Y_0 &= 2, & Y_0 &= 1 \\ L &= 5 \text{ cm}, & C_L &= 106.1 \mu\mu\text{f}\end{aligned}$$

so

$$\omega_2/2\pi = 3.0 \text{ kMc}, \quad \text{the second } \pi \text{ cutoff frequency.}$$

Show that the single mode-pair dispersion Eq. 4-13, written as $p_1^2 - k^2 = -p_1 M_{11} \cdot (1 - \cosh \alpha)$ with $M_{11} < 0$, rapidly becomes inaccurate as k increases above p_1 .

6. Consider the nonsymmetrical periodic chain of cavities shown in Fig. 4-7. Assume a surface M exists such that if it is short-circuited the short-circuit

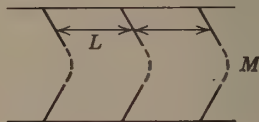


Fig. 4-7

mode resonates at $\varphi = 0$ and if it is open-circuited the open-circuit mode resonates at $\varphi = \pi$. Derive the modified dispersion Eq. 4-11 and show that an inconsistency arises even if all the terms happen to be pure real. Therefore surface M with these properties does not exist.

7. Prove that we can write the time-average real power flow across a coupling surface in at least three equivalent ways, depending on whether $\bar{e}$ or $\bar{H}$ or both are averaged on the surface.
8. Show that the equivalent circuit presented in Fig. 4-3 is not the dual of the circuit in Fig. 4-4 either with or without the current generators representing the beam (refer to E. A. Guillemin [5]).
9. If an electron beam were interacting with a slow wave structure, verify from Maxwell's equations the placement of the equivalent current generators (dashed lines) in Figs. 4-3 and 4-4. How are the equivalent currents evaluated?
10. Suppose alternate cavities of the Fig. 4-1a structure are changed so as to make it doubly periodic but allow it to retain its mirror symmetry about a transverse plane bisecting each cavity. Derive the new single mode-pair dispersion equations by the procedures of Sections 4-2 and 4-6 and show they are consistent. Sketch the two dispersion curves which replace the singly periodic one.

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General Theorems about Symmetric Cavity Chains

5-1. Synopsis

In the preceding chapter we considered the behavior of a cavity chain at frequencies in just one of its narrow passbands. The predominant component of E_z field had either even symmetry characterized by an even number of reversals across the cavity, or odd symmetry. These field symmetries were implied by the mirror symmetry of the structure about a transverse plane bisecting a cavity. The result of our analysis was a theorem stating that the shape of the ω - φ ($\varphi = \beta_0 L$) curve for this passband is, to a first approximation, independent of the nature of the coupling between cavities. In this chapter we make statements about all the dispersion curves and the fields which determine them.

We continue our discussion of only those structures with mirror symmetry about a transverse plane bisecting a cavity. In such structures every open- and short-circuit solenoidal cavity mode which resonates between magnetic or electric shorting planes, respectively, on the coupling surfaces or cavity midplanes has physical significance. For example, the even short-circuit mode with patterns shown in Fig. 4-1*b* represents the true field at an operating frequency corresponding to that mode's resonant frequency. For if the amplitudes phase shift by zero radians across the coupling surfaces, the total field consisting of amplitude times pattern is continuous. This mode, then, represents the true field at a zero phase shift cutoff frequency. This class of symmetrical structures is wide enough to include most hole-, slot-, and loop-coupled cavity chains of practical interest.

We then proceed to generalize the definitions of the solenoidal (curl

but no divergence) short- and open-circuit modes for the field expansions, with respect to periodic scalar permeability and permittivity. Orthogonality relations exist which help us later to evaluate the mode amplitudes. In general, the fields also possess irrotational components (divergence but no curl), so we define sets of irrotational short- and open-circuit modes for the fields. Orthogonality relations also exist for these modes. In addition, every irrotational mode pattern is orthogonal to every solenoidal mode pattern defined with respect to the same boundary conditions.

We expand the fields of Maxwell's equations with all the short-circuit solenoidal and irrotational modes, excited on the coupling surfaces by tangential electric field. By following Schelkunoff's procedure we evaluate the amplitudes of these modes. Irrotational magnetic field exists, but there is no irrotational electric field. Then we follow the dual procedure of expanding the fields of Maxwell's equations with the open-circuit modes, excited by tangential magnetic field on the coupling surfaces. We evaluate the new mode amplitudes and discover that irrotational electric field but no irrotational magnetic field now exists. Thus the two expansion procedures yield different expansions for the same electric and magnetic fields—the short-circuit mode expansions contain no irrotational electric field components but the open-circuit mode expansions do. Schelkunoff gives a simple example to show how both kinds of expansions, one containing no irrotational components but a d.c. component and the other containing an irrotational component, can represent the same magnetic field within a cavity.

From these expressions for the mode amplitudes of the short- or open-circuit mode expressions we show how to derive dispersion relations independent of the amplitudes of the irrotational modes. The expressions for the amplitudes of the irrotational modes merely form a set of auxiliary relations in terms of the relative amplitudes of the solenoidal modes. These relative amplitudes are determined by the value of $k^2(\varphi)$ which satisfies the dispersion relation for the particular ω - φ curve of interest. The irrotational modes do not determine the dispersion characteristics provided we use both the open- and short-circuit mode expansions within the same periodic cell, with phase shift of all amplitudes across the boundaries of that cell. If, however, we expand the fields one way within one unit cell, as excited by the fields of another expansion defined within a disjoint unit cell, and vice versa, the irrotational modes *do* affect the dispersion relation. For example, we may wish to represent the fields in a resonant-slot coupled structure by the "cavity" modes and also by the "slot" modes centered around the slots. In this case the irrotational modes cannot be ignored in principle, but they usually affect the dispersion only slightly.

We then go on to show that the effect of the modes of the second passband in the field expansions for many of these structures is to depress the ω - φ curve of the first passband. Two conditions are sufficient for this effect. The fields in both passbands must tend to phase shift rapidly across the same coupling surfaces and also must tend to have even symmetry in one passband and odd symmetry in the other. In Sections 7-5 and 7-6 we show that the same effect is true in many of the resonant-slot and resonant-loop coupled structures. In order to prove the foregoing statement we derive the 2×2 determinantal equation for $k^2(\varphi)$ using the modes of the first two passbands. The derivation illustrates the procedure to follow with any number of modes in the field expansions. The determinantal equation is the same whether we develop Maxwell's equations in which the fields are expanded in short- or in open-circuit modes.

The form of the dispersion relation for the first two passbands illustrates an interesting conservation relation for the sum of the squares of the frequencies in all the passbands, at any given phase shift φ . It turns out that this sum is equal to the sum of the squares of all the frequencies which lie on single mode-pair or single-passband curves connecting the same cutoff frequencies, for the same phase shift φ .

5-2. Symmetry of the Structure

Figure 5-1 shows a schematic diagram of the general symmetrical cavity chain we wish to treat in this chapter. The chain has mirror symmetry with respect to either transverse plane ($B\pm$) bisecting a cavity. This means the point (x, y, z) goes into the point $(x, y, -z)$ where $z \leq L/2$. Arbitrary symmetrical coupling lies within the dashed pillboxes which enclose coupling surfaces denoted by $(M\pm)$. From elementary group

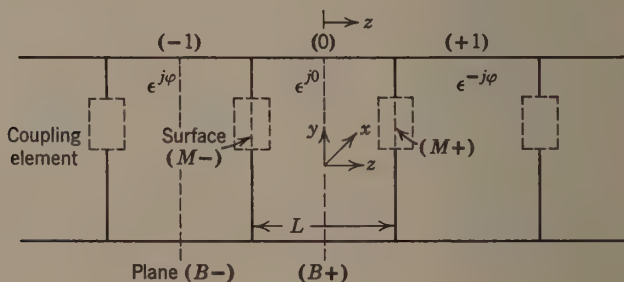


Fig. 5-1. A general symmetrical cavity chain. Cavity (0) is the reference cavity of phase zero and coupled to the neighbors across surfaces $(M\pm)$. Alternatively, a cavity cell may be defined between planes $(B\pm)$.

theory considerations we know that the fields of Maxwell's equations at any given frequency in a cavity can be split into two portions. One portion has the longitudinal symmetry specified by

$$\hat{E}_z(x, y, +z) = +\hat{E}_z(x, y, -z) \quad (5-1a)$$

$$\hat{H}(x, y, z) \times \bar{i}_z = +\hat{H}(x, y, -z) \times \bar{i}_z \quad |z| < \frac{L}{2} \quad (5-1b)$$

$$\hat{H}_z(x, y, z) = -\hat{H}_z(x, y, -z), \quad (5-1c)$$

$$\hat{E}(x, y, z) \times \bar{i}_z = -\hat{E}(x, y, -z) \times \bar{i}_z \quad (5-1d)$$

within the cavity. This portion is said to have *even* symmetry, as defined on page 95, and the fields may be represented by a complete set of even short-circuit modes [each with $\hat{E}_z \neq 0$ and $\hat{E} \times \bar{i}_z \equiv 0$ on the coupling surfaces ($M\pm$), according to definitions on page 95] or, alternatively, by a set of even open-circuit modes [each with $\hat{E}_z = 0$ and $\hat{E} \times \bar{i}_z \neq 0$ on ($M\pm$), as defined on the same page]. The other portion of the fields has the odd symmetry expressed by

$$\hat{E}_z'(x, y, +z) = -\hat{E}_z'(x, y, -z) \quad (5-2)$$

$$\hat{H}'(x, y, z) \times \bar{i}_z = -\hat{H}'(x, y, -z) \times \bar{i}_z, \quad |z| < \frac{L}{2}$$

and so on, and these fields may be constructed with either the odd short-circuit modes or, alternatively, the odd open-circuit modes within each cavity. To represent the complete solenoidal fields within a cavity we can expand the fields of Maxwell's equations with either the short-circuit modes denoted by capital letters as

$$\hat{E}(\bar{r}) = \sum_{\text{even}} V_n \bar{E}_n(\bar{r}) + \sum_{\text{odd}} V_m \bar{E}_m(\bar{r}) \quad (5-3a)$$

and

$$\hat{H}(\bar{r}) = \sum_{\text{even}} I_n \bar{H}_n(\bar{r}) + \sum_{\text{odd}} I_m \bar{H}_m(\bar{r}) \quad (5-3b)$$

where the position vector r lies within reference cavity (0), or, alternatively, with the open-circuit modes denoted by small letters, as

$$\hat{E}(\bar{r}) = \sum_{\text{even}} v_n \bar{e}_n(\bar{r}) + \sum_{\text{odd}} v_m \bar{e}_m(\bar{r}) \quad (5-4a)$$

and

$$\hat{H}(\bar{r}) = \sum_{\text{even}} i_n \bar{h}_n(\bar{r}) + \sum_{\text{odd}} i_m \bar{h}_m(\bar{r}) \quad (5-4b)$$

In these expansions the relative amplitudes $V_n, \dots, i_m$ all depend upon the frequency corresponding to a given value of phase shift φ . The fields in cavities adjacent to reference cavity (0) in Fig. 5-1 are represented by either Eqs. 5-3 or Eqs. 5-4 multiplied by the appropriate factor $\epsilon^{\pm j\varphi}$, as indicated on the figure.

The symmetry means that each open- and short-circuit resonant cavity

mode represents the true field at that cutoff frequency of the chain which is the mode's resonant frequency. For example, suppose we examine the first even open-circuit mode of the centerhole-coupled chain of Fig. 4-1. This mode, Fig. 4-1c, has the symmetry described by Eqs. 5-1c,d. By repeating its patterns $\bar{e}_1$ and $\bar{h}_1$ from cavity to cavity we see that $\bar{e}_1 \times \bar{i}_z$ reverses direction across each hole. Therefore if the mode amplitudes v_1 and i_1 also change sign across each hole, that is, phase shift by π , the total fields $v_1\bar{e}_1$ and $i_1\bar{h}_1$ will be continuous. In addition, the power flow in this mode is zero because only $\bar{e}_1 \times \bar{i}_z$ exists on the (magnetically shorted) holes. Hence this mode represents true field at the first π cutoff frequency of the first passband, where the group velocity is zero. This was the argument presented in Section 1-6 and adopted to cavity chains, to the effect that every ω - β_0 curve of this structure has zero slope and no power flow at every zero and π phase shift resonant frequency.

It also follows that the true field at every cutoff frequency of a passband must have the properties of at least one open- or short-circuit mode. At any cutoff frequency the fields have either even or odd symmetry except in special cases where resonances can exist with both symmetries.¹ If the fields have even character at a zero phase shift cutoff frequency we can insert electric shorting planes on the coupling surfaces without disturbing them, according to the symmetry properties of Eq. 5-1. If the fields have odd character for this phase shift we can insert magnetic shorting planes without disturbing them. It follows that, at a given cutoff frequency, we can always insert the appropriate shorting planes on the coupling surfaces so as to define short- or open-circuit cavity mode(s). If more than one mode exists, we have a degenerate case.

These remarks indicate that we can represent the total solenoidal fields at *any* operating frequency by expansions 5-3 or, alternatively, 5-4, in which each mode has an amplitude partially determined by the proximity of its resonant frequency to the operating frequency. We *assume* completeness of both these expansions although we can only prove it for cavities of constant cross-sectional area. In such cavities each resonant cavity mode can be regarded as composed of a linear combination of TM and TE waveguide modes, each with fields which vary as $\sin(n\pi z/L)$ or $\cos(n\pi z/L)$ across the cavity. These waveguide mode-harmonics do form a complete set for the fields, so that linear combinations of them will also be complete. But despite the cavity shape and precise nature of the coupling within the dashed pillboxes of Fig. 5-1 we expect to obtain good field representations with just the few cavity modes that have resonant

¹ G. B. Walker and G. C. Englefield [1] show that this can happen in a waveguide which supports the TE_{10} mode and which is loaded periodically by dielectric spacers. They speak of a confluence between the first and second passbands at the π phase shift point.

frequencies near the operating frequency. These fields will then yield a dispersion relation that is most accurate in a narrow frequency range which is far from the resonant frequencies of the neglected modes. In line with these statements we obtain, in Section 5-5, 2×2 determinantal equations for the first two passbands of a structure that are most accurate for the first passband and less accurate for the second passband because of the proximity of the modes of the third passband.

If our model did not have mirror symmetry we could still expand the fields with the short- and open-circuit resonant cavity modes. Then, however, each such mode would not represent a true field at a physical cutoff frequency. If we repeated the patterns of these modes from cavity to cavity, the tangential and normal components would be neither continuous nor continuous but for a directional reversal across the coupling surfaces. Thus the mode could not represent true continuous field for either zero or π phase shift of amplitudes across the coupling surfaces. Furthermore, the complete dispersion relation would not contain $\cos \varphi$ and $\sin \varphi$ factors, as in Eqs. 4-20 and 4-21, but would have complex terms representing the asymmetrical couplings of each cavity to its neighbors.

5-3. The General Field Expansions

We now expand the fields of Maxwell's equations with all the short-circuit modes, as excited on the cavity coupling surfaces ($M_{\pm}$) of Fig. 5-1 by the tangential electric field. This latter field may be expanded with the open-circuit modes, as we shall show. Then we follow the dual procedure of expanding the fields with all the open-circuit modes, as excited by tangential magnetic fields on the ($M_{\pm}$) surfaces. However, before we carry out the expansions we redefine the solenoidal short- and open-circuit modes more generally and also introduce the irrotational modes for the sake of completeness.

A. The Solenoidal Mode Characteristics

In Chapter 4 we analyzed a single passband with a pair of solenoidal short- and open-circuit modes defined with respect to free-space permeability μ_0 and permittivity ϵ_0 . To account for the presence of dielectric or possibly magnetic material within the cavities, we redefine the modes with respect to periodic scalar distributions of relative permeability $\mu_r(\vec{r})$ and permittivity $\epsilon_r(\vec{r})$. Our expressions to follow will also be valid if μ_r and ϵ_r are Hermitian dyadics for lossless media, with minor changes in notation.

The n th short-circuit solenoidal mode is therefore redefined within a cavity by the relations

$$\nabla \times \bar{E}_n(\bar{r}) = P_n \mu_r(\bar{r}) \bar{H}_n(\bar{r}) \quad \bar{E}_n \times \bar{n} = 0 \text{ on cavity walls} \quad (5-5a)$$

$$\nabla \times \bar{H}_n(\bar{r}) = P_n \varepsilon_r(\bar{r}) \bar{E}_n(\bar{r}) \quad \text{and coupling surfaces} \quad (5-5b)$$

$$P_n = \omega_n \sqrt{\mu_0 \varepsilon_0} \quad \text{the normalized resonant frequency} \quad (5-5c)$$

Here $\bar{r}$ is merely the position vector. We may prove that $\bar{E}_m$ has weighted orthogonality with respect to $\bar{E}_n$ if $P_m \neq P_n$ as follows. We multiply Eq. 5-5a by μ_r^{-1} , take the curl, and substitute Eq. 5-5b. After dot-multiplying by $\bar{E}_m$ and integrating over the cavity volume τ we obtain

$$\int_V \bar{E}_m \cdot \nabla \times \mu_r^{-1} \nabla \times \bar{E}_n dV = P_n^2 \int_V \bar{E}_m \cdot \varepsilon_r \bar{E}_n dV \quad (5-6)$$

By the divergence theorem we can convert the left side so this expression reads

$$\int_V \mu_r^{-1} \nabla \times \bar{E}_n \cdot \nabla \times \bar{E}_m dV = P_n^2 \int_V \bar{E}_m \cdot \varepsilon_r \bar{E}_n dV \quad (5-7)$$

because $\oint \mu_r^{-1} \bar{E}_m \times \nabla \times \bar{E}_n \cdot \bar{n} dS = 0$ on the cavity walls and coupling surfaces. Now, by subtracting Eq. 5-7 with n and m interchanged we find that

$$\int_V \bar{E}_m \cdot \varepsilon_r \bar{E}_n dV = 0 \quad \text{if } P_n \neq P_m \quad (5-8)$$

Similarly, we may show that

$$\int_V \bar{H}_m \cdot \mu_r \bar{H}_n dV = 0$$

at the same time. Moreover, we may dot-multiply Eq. 5-5a by $\bar{H}_n$ and Eq. 5-5b by $\bar{E}_n$, subtract the equations, and integrate the result over the cavity volume to show that

$$\int_V \varepsilon_r \bar{E}_n^2 dV = \int_V \mu_r \bar{H}_n^2 dV$$

We may then normalize both $\bar{E}_n$ and $\bar{H}_n$ so these latter integrals are both equal to the cavity volume.

Summing up these relations we have the orthonormalities

$$\int_V \bar{E}_m \cdot \varepsilon_r \bar{E}_n dV = \int_V \bar{H}_m \cdot \mu_r \bar{H}_n dV = \begin{cases} 0 & \text{if } P_n \neq P_m \\ \tau & \text{cavity volume, if } P_n = P_m \end{cases} \quad (5-9)$$

which are useful when evaluating the mode amplitudes.

² If several short-circuit modes resonate at the same frequency P_n , we can extract linear combinations of them which are orthogonal among themselves as well as with respect to all other short-circuit modes. The Schmidt process is an orderly procedure for doing this.

The n th open-circuit solenoidal mode is defined more generally as

$$\begin{aligned}\nabla \times \bar{e}_n(\bar{r}) &= p_n \mu_r(\bar{r}) \bar{h}_n(\bar{r}) & \bar{e}_n \times \bar{n} &= 0 \text{ on cavity walls} \\ \nabla \times \bar{h}_n(\bar{r}) &= p_n \epsilon_r(\bar{r}) \bar{e}_n(\bar{r}) & \bar{e}_n \cdot \bar{n} &= 0 \text{ on coupling surfaces} \\ p_n &= \omega_n' \sqrt{\mu_0 \epsilon_0}\end{aligned} \quad (5-10)$$

and we can also normalize the $\bar{e}_n$ and $\bar{h}_n$ patterns so as to have the orthonormalities

$$\int_V \bar{e}_m \cdot \epsilon_r \bar{e}_n dV = \int_V \bar{h}_m \cdot \mu_r \bar{h}_n dV = \begin{cases} 0 & \text{if } p_n \neq p_m \\ \tau & \text{cavity volume, if } p_n = p_m \end{cases} \quad (5-11)$$

Those d.c. short- and open-circuit modes for $P_0 = 0$ and $p_0 = 0$ are regarded as degenerate *solenoidal* modes.

B. Irrotational Mode Characteristics

In general a field may be broken into two portions: a solenoidal portion with curl but no divergence, represented by the modes of Eqs. 5-5 and, alternatively, the modes of Eqs. 5-10, and an irrotational portion with divergence but no curl. Now it is true that the total electric and magnetic fields within an empty cavity have no divergence. Nevertheless, one or the other of them may have an expansion containing irrotational modes, each having divergence but whose sum total has none within the cavity. We prove this statement by construction. We define a set of irrotational modes each of which is orthogonal to all the solenoidal modes within a cavity, evaluate their amplitudes by Schelkunoff's procedure as on page 85 and discover that certain of the modes have nonzero amplitudes.

The irrotational cavity mode patterns and amplitudes are labeled with the superscript i , with capital letters for the short-circuit modes and small letters for the open-circuit modes as before. Let us define the n th short-circuit mode pattern $\bar{E}_n^i$ for the electric field as

$$\bar{E}_n^i(\bar{r}) = -\nabla \Phi_n(\bar{r}) \quad \text{within a cavity} \quad (5-12a)$$

so that its curl will be identically zero. Scalar function Φ_n is taken to satisfy the modified Helmholtz equation

$$\nabla \cdot (\epsilon_r \nabla \Phi_n) + Q_n^2 \Phi_n = 0, \quad Q_n \neq 0 \quad (5-12b)$$

such that $\bar{E}_n^i$ is normal to the walls and coupling surfaces,

$$\nabla \Phi_n \times \bar{n} = 0 \quad \text{on walls and coupling surfaces} \quad (5-12c)$$

The n th irrotational short-circuit mode for the magnetic field is defined analogously as

$$\vec{H}_n^i(\vec{r}) = -\nabla\Psi_n^i(\vec{r}) \quad (5-13a)$$

$$\nabla \cdot (\mu_r \nabla \Psi_n^i) + R_n^2 \Psi_n^i = 0, \quad R_n \neq 0 \quad (5-13b)$$

such that $\vec{H}_n^i$ is tangential to the cavity walls and coupling surfaces,

$$\nabla \Psi_n^i \cdot \vec{n} = 0 \quad \text{on walls and coupling surfaces} \quad (5-13c)$$

Modes with $Q_0 = 0$ or $R_0 = 0$ are d.c. modes that are grouped with the solenoidal modes.

The n th irrotational open-circuit mode for the electric field is defined analogously to Eqs. 5-12, with respect to a scalar function ϕ_n : $\vec{E}_n^i = -\nabla\phi_n$. The scalar function ϕ_n satisfies the boundary conditions of $\vec{E}_n^i \times \vec{n} = 0$ on the cavity walls and $\vec{E}_n^i \cdot \vec{n} = 0$ on the (magnetically shorted) coupling surfaces. The n th irrotational open-circuit mode for the magnetic field is defined by an equation for a scalar function ψ_n analogous to Eq. 5-13, such that $\vec{H}_n^i \cdot \vec{n} = -\nabla\psi_n \cdot \vec{n} = 0$ on the cavity walls and $\nabla\psi_n \times \vec{n} = 0$ on the coupling surfaces. There are both even and odd varieties of all these irrotational modes, according to whether $\vec{E}_{nz}^i$ and $\vec{H}_n^i \times \vec{i}_z$, or $\vec{E}_{nz}^i$ and $\vec{H}_n^i \times \vec{i}_z$ suffer an even or an odd number of reversals across each cavity.

We can also develop orthogonality relations for these irrotational modes that are useful when evaluating their amplitudes. Suppose we consider the short-circuit $\vec{E}_n^i$ and $\vec{E}_m^i$ patterns, defined by Eqs. 5-12. Then we can write this identity within a cavity:

$$\begin{aligned} \int_V \nabla \cdot (\Phi_m \epsilon_r \nabla \Phi_n) dV &= \int_V \Phi_m \nabla \cdot (\epsilon_r \nabla \Phi_n) dV + \int_V \nabla \Phi_m \cdot \epsilon_r \nabla \Phi_n dV \\ &= \oint_S \Phi_m \epsilon_r \nabla \Phi_n \cdot \vec{n} dS = 0, \quad \Phi_{m,n} = 0 \text{ on } S \end{aligned} \quad (5-14)$$

If we interchange m and n in this equation and subtract it from Eq. 5-14 as it stands, we obtain, with the definition of Φ in Eq. 5-12,

$$(-Q_n^2 + Q_m^2) \int_V \Phi_m \Phi_n dV = 0 \quad (5-15)$$

Thus, if $Q_m \neq Q_n$, then Φ_m is orthogonal to Φ_n , and this implies in Eq. 5-14 that $\vec{E}_m^i$ can be adjusted so that

$$\int_V \vec{E}_m^i \cdot \epsilon_r \vec{E}_n^i dV = \begin{cases} 0 & \text{if } Q_m \neq Q_n \\ \tau & \text{cavity volume, if } Q_n = Q_m \end{cases} \quad (5-16)$$

where we have normalized $\vec{E}_n^i$ to the cavity volume for convenience. By

a similar procedure we may obtain the weighted orthonormality of the $\bar{H}_n^i$ patterns as

$$\int_V \bar{H}_m^i \cdot \mu_r \bar{H}_n^i dV = \begin{cases} 0 & \text{if } R_m \neq R_n \\ \tau & \text{cavity volume, if } R_n = R_m \end{cases} \quad (5-17)$$

We may also verify analogous relations for the open-circuit mode patterns $\bar{e}_n^i$ and $\bar{h}_n^i$ as well.

Lastly we note that every short-circuit or open-circuit irrotational mode pattern is orthogonal to every short-circuit or open-circuit solenoidal mode pattern, respectively, of the same type. For example, consider the patterns $\bar{E}_n$ and $\bar{E}_m^i$. From the identity

$$\begin{aligned} \int_V \nabla \cdot (\nabla \Phi_m \times \bar{H}_n) dV &= - \int_V \nabla \Phi_m \cdot \nabla \times \bar{H}_n dV \\ &= P_n \int_V \bar{E}_m^i \cdot \epsilon_r \bar{E}_n dV = \oint_S \nabla \Phi_m \times \bar{H}_n \cdot \bar{n} dS = 0, \end{aligned} \quad (5-18)$$

$$\nabla \Phi_m \times \bar{n} = 0 \text{ on } S$$

we see that $\int_V \bar{E}_n \cdot \epsilon_r \bar{E}_m^i dV \equiv 0$ for all m and n .

C. Fields of Maxwell's Equations Expanded in Short-Circuit Modes

To generalize the single-passband solution of Maxwell's equations in Chapter 4 let us now expand the fields, but not their curls, with a presumably complete set of short-circuit solenoidal and irrotational modes. We seek to determine their amplitudes V_n , V_n^i , I_n , and I_n^i so as to satisfy the equations

$$\nabla \times \hat{E} = -j\omega\mu_0\mu_r \left(\sum_n I_n \bar{H}_n + \sum_m I_m^i \bar{H}_m^i \right) \quad (5-19a)$$

$$\nabla \times \hat{H} = j\omega\epsilon_0\epsilon_r \left(\sum_n V_n \bar{E}_n + \sum_m V_m^i \bar{E}_m^i \right) \quad (5-19b)$$

within reference cavity (0) in the periodic chain of Fig. 5-1. We again follow Schelkunoff's procedure (page 85) and dot-multiply the first of these equations by solenoidal $\bar{H}_n$ and the second by solenoidal $\bar{E}_n$ so as to obtain expressions for V_n and I_n . When we integrate the equations over the cavity (0) volume and integrate the left sides by parts, as in Eq. 3-45, we obtain

$$\begin{aligned} P_n V_n &= -\frac{1}{\tau} \int_{M^\pm} \hat{e} \times \bar{H}_n^{(0)} \cdot \bar{n} dS - j\omega\mu_0 I_n \\ P_n I_n &= j\omega\epsilon_0 V_n \end{aligned} \quad (5-20)$$

Here $\hat{e}$ (tangential) on the coupling surfaces is unspecified as yet; it will be represented by open-circuit modes. Superscript (0) reminds us that the field pattern is measured on the (0) cavity side of a coupling surface ($M\pm$). In deriving Eqs. 5-20 we have employed the volume orthogonalities of the $\bar{E}_n$ and $\bar{H}_n$ patterns with all the other solenoidal and irrotational mode patterns.

To obtain expressions for I_m^i and V_m^i in Eqs. 5-19 we first dot-multiply Eq. 5-19a by $\bar{H}_m^i$, integrate over reference cavity (0), and integrate the left side by parts. The $\bar{H}_m^i$ pattern is orthogonal with respect to the other irrotational patterns and all the solenoidal patterns and so we obtain just

$$+j\omega\mu_0 I_m^i = -\frac{1}{\tau} \int_{M^\pm} \hat{e} \times \bar{H}_m^{i(0)} \cdot \bar{n} dS \quad (5-21)$$

This proves that irrotational field components do contribute to the magnetic field when we solve Maxwell's equations this way. However, when we dot-multiply Eq. 5-19b by $\bar{E}_m^i$, integrate throughout the cavity volume, and employ the orthogonality relations, we find that

$$V_m^i \equiv 0 \quad \text{for all } m, \quad (5-22)$$

that is, the short-circuit electric field generated within each cavity is purely solenoidal. We use this fact in Section 5-4 in order to justify the neglect of the irrotational components in the equivalent open-circuit mode expansion for this electric field.

D. Fields of Maxwell's Equations Expanded in Open-Circuit Modes

We can alternatively extend the work of Section 4-6 and expand the fields of Maxwell's equations in a presumably complete set of solenoidal and irrotational open-circuit modes, as

$$\nabla \times \hat{e} = -j\omega\mu_0\mu_r \left(\sum_n i_n \bar{h}_n + \sum_m i_m^i \bar{h}_m^i \right) \quad (5-23a)$$

$$\nabla \times \hat{h} = j\omega\epsilon_0\epsilon_r \left(\sum_n v_n \bar{e}_n + \sum_m v_m^i \bar{e}_m^i \right) \quad (5-23b)$$

within reference cavity (0). To isolate the solenoidal amplitudes v_n and i_n we dot-multiply the first of these equations by $\bar{h}_n$ and the second by $\bar{e}_n$, integrate over the cavity volume, integrate the left sides by parts again, and utilize the orthogonalities of these patterns with all the other solenoidal and irrotational patterns. Since $\bar{h}_n \times \bar{n} = 0$ but $\bar{e}_n \times \bar{n} \neq 0$ on

the $(M\pm)$ surfaces in Fig. 5-1 we obtain just

$$\begin{aligned} p_n v_n &= -j\omega\mu_0 i_n \\ p_n i_n &= +\frac{1}{\tau} \int_{M^\pm} \bar{e}_n^{(0)} \times \hat{H} \cdot \bar{n} dS + j\omega\epsilon_0 v_n \end{aligned} \quad (5-24)$$

The field $\hat{H}$, which will be expanded with the short-circuit modes, is not yet specified on $(M\pm)$, and $\bar{e}_n^{(0)}$ lies on the (0) cavity side of these surfaces.

To obtain the irrotational v_m^i amplitude we dot-multiply Eq. 5-23b by $\bar{e}_m^i$ and proceed similarly. Since

$$\int_{M^\pm} \hat{H} \times \bar{e}_m^i \cdot \bar{n} dS \neq 0$$

because $\bar{e}_m^i \times \bar{n} \neq 0$ on $(M\pm)$ we obtain

$$-j\omega\epsilon_0 v_m^i = \frac{1}{\tau} \int_{M^\pm} \bar{e}_m^{i(0)} \times \hat{H} \cdot \bar{n} dS \quad (5-25)$$

This shows that irrotational electric field exists when we satisfy Maxwell's equations within the cavity with the open-circuit modes. However, when we dot-multiply Eq. 5-23a by $\bar{h}_m^i$ and evaluate the i_m^i amplitude we find that, because $\hat{e} \times \bar{h}_m^{i(0)} \cdot \bar{n} dS \equiv 0$ on the coupling surfaces,

$$i_m^i \equiv 0 \quad \text{for all } m \quad (5-26)$$

This fact will help justify the complete absence of irrotational magnetic field components when we obtain the dispersion relation from Eqs. 5-24 in the latter portion of Section 5-5.

The short-circuit mode expansions have a different structure from the open-circuit mode expansions. We see by Eqs. 5-21 and 5-22 that only irrotational magnetic field exists in the short-circuit field representation but that only irrotational electric field exists in the open-circuit mode representation, Eqs. 5-25 and 5-26. Both mode expansions in principle represent the same fields. Nevertheless, when we expand the fields with the short-circuit modes, the tangential electric field on the coupling surfaces excites only the irrotational magnetic field components, whereas if we expand the fields with the open-circuit modes the tangential magnetic field on the coupling surfaces excites only the irrotational electric field components. These statements imply the nonuniqueness of the representation for a field within a bounded region of space, such as a cavity. Schelkunoff [2] gives a simple example of this nonuniqueness for the magnetic field within a shorted section of rectangular waveguide. We now discuss his example.

Schelkunoff considers the rectangular cavity of length ℓ shown in Fig. 5-2, with an incident TE_{10} waveguide field at $z = 0$. Because of the

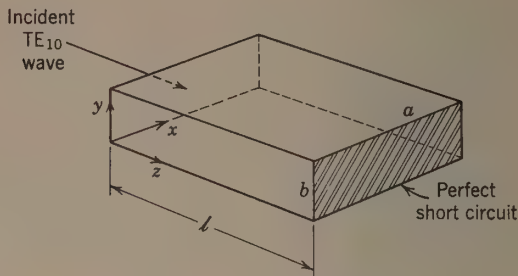


Fig. 5-2. A waveguide cavity.

symmetry the fields in the range $0 \leq z \leq \ell$ remain TE_{10} and therefore vary as $\sin(\pi x/a)$ or $\cos(\pi x/a)$. One expansion for the fields $\hat{E}_y$, $\hat{H}_x$, and $\hat{H}_z$ within the cavity is the solenoidal

$$\begin{aligned}\hat{E}_y(x, z) &= \sum_{n=1}^{\infty} A_n \sin \frac{\pi x}{a} \sin \frac{n\pi z}{\ell} \\ \hat{H}_x(x, z) &= \sum_{n=1}^{\infty} \frac{n\pi A_n}{j\omega_n \mu \ell} \sin \frac{\pi x}{a} \cos \frac{n\pi z}{\ell} + A_0 \sin \frac{\pi x}{a} \cosh \frac{\pi(\ell - z)}{a} \\ \hat{H}_z(x, z) &= \sum_{n=1}^{\infty} -\frac{\pi A_n}{j\omega_n \mu a} \cos \frac{\pi x}{a} \sin \frac{n\pi z}{\ell} + A_0 \cos \frac{\pi x}{a} \sinh \frac{\pi(\ell - z)}{a} \\ \omega_n^2 \mu \epsilon &= \left(\frac{\pi}{a}\right)^2 + \left(\frac{n\pi}{\ell}\right)^2\end{aligned}\quad (5-27)$$

The A_0 mode is the d.c. mode of zero curl and divergence, which we regard as a degenerate solenoidal mode. The n th mode of amplitude A_n resonates at frequency ω_n within the cavity shorted at $z = 0$ by an electric shorting plane.

Schelkunoff chooses these amplitudes so as to make the current of each mode unity at the $z = 0$ "terminals" and proceeds to obtain the correct input admittance Y at any frequency ω from his formula [3]

$$Y(j\omega) = \sum_n \frac{j\omega}{2\mathcal{E}_n(\omega_n^2 - \omega^2)} \quad (5-28)$$

where $\mathcal{E}_n$ is the energy stored in the n th mode as determined by the A_n amplitude.

He then alternatively satisfies Maxwell's equations within the cavity with the new expansions

$$\begin{aligned}\hat{E}_y &= \sin \frac{\pi x}{a} \sum_{n=1}^{\infty} A_n' \sin \frac{n\pi z}{\ell}, & 0 < z \leq \ell, \\ \hat{H}_x &= \sin \frac{\pi x}{a} \sum_{n=0}^{\infty} B_n \cos \frac{n\pi z}{\ell}, & 0 \leq z \leq \ell\end{aligned}\quad (5-29)$$

in which the $\hat{E}_y$ expansion is identically zero right at $z = 0$ and the $\hat{H}_x$ expansion includes the $n = 0$ irrotational mode

$$\hat{H}_z = 0, \quad \hat{H}_x = \sin \frac{\pi x}{a} B_0, \quad \nabla \cdot \vec{H}_0 \neq 0 \quad (5-30)$$

By substituting these expansions directly into Maxwell's equations for $\hat{E}_y$ and $\hat{H}_x$ alone,

$$\frac{\partial \hat{E}_y}{\partial z} = j\omega\mu\hat{H}_x, \quad \frac{\partial \hat{H}_x}{\partial z} = \left(j\omega\epsilon + \frac{\pi^2}{j\omega\mu a^2}\right)\hat{E}_y \quad (5-31a,b)$$

he evaluates the amplitude B_0 in terms of electric field E_y (at $z = 0$) = E_0 by integrating the first equation from $z = 0$ to ℓ . The result is

$$j\omega\mu\ell B_0 = -E_0 \quad (5-32)$$

He then multiplies both sides of Eq. 5-31a by $\cos(n\pi z/\ell)$, integrates from $z = 0$ to ℓ , integrates the left side by parts, and obtains the relation

$$-E_0 + \frac{n\pi}{2} A_n' = \frac{1}{2}j\omega\mu\ell B_n, \quad n = 1, 2, \dots \quad (5-33)$$

If we now substitute relations 5-29 and 5-30 directly into the Maxwell Eq. 5-31a and eliminate A_n' with Eq. 5-33, we find that there is a discrepancy term of value

$$\begin{aligned} \sin \frac{\pi x}{a} \sum_{n=1}^{\infty} \frac{2E_0}{\ell} \cos \frac{n\pi z}{\ell} - j\omega\mu \sin \frac{\pi x}{a} B_0 \cos \frac{n\pi z}{\ell} \\ = \sin \left(\frac{\pi x}{a} \right) \frac{E_0}{\ell} \left(1 + 2 \sum_{n=1}^{\infty} \cos \frac{n\pi z}{\ell} \right) \end{aligned} \quad (5-34)$$

The quantity within the last parentheses is merely an expansion for an impulse (delta) function at $z = 0$ and of strength ℓ . Hence we see that the amplitude relations of 5-32, 5-33 do check Maxwell's Eq. 5-31a directly *within* the cavity over the open interval $0 < z \leq \ell$.

Schelkunoff then multiplies Eq. 5-31b by $\sin(n\pi z/\ell)$ and operates on it to obtain the second amplitude relation

$$-\frac{n\pi}{\ell} B_n = \left(j\omega\epsilon + \frac{\pi^2}{j\omega\mu a^2}\right) A_n' \quad (5-35)$$

which verifies Maxwell's Eq. 5-31b directly. Then Eqs. 5-32, 5-33, and 5-35 determine all the amplitudes for expansions 5-29 and lead to the same input admittance as previously derived.

Here is a simple example in which the magnetic field may be expanded either in a set of solenoidal modes by Eq. 5-27 or, alternatively, by a set

of solenoidal modes and one irrotational mode by Eq. 5-29. The two expansions are equivalent *within* the cavity volume, which is all we ever require. If an expansion for a field component, such as either of the preceding ones for $\hat{H}_x$, is continuous from a point within the cavity to a point *on* the coupling surface, we can, of course, use the volume expansion to represent that field component on the coupling surface. In this way our short-circuit mode expansion for magnetic field within the cavity yields the tangential magnetic field *on* the coupling surfaces, and the open-circuit mode expansion yields the tangential electric field on those surfaces.

We shall now discuss the effect of the irrotational modes in expansions 5-19 and 5-23 upon the dispersion characteristics of the cavity chain.

5-4. Effect of the Irrotational Modes upon the Dispersion Characteristics

We now demonstrate how the short-circuit mode amplitude relations of Section 5-3C may be solved to yield a complete dispersion relation unaffected by the irrotational modes. Then we show how the dual relations of Section 5-3D may be solved to yield a dispersion relation unaffected by those irrotational modes. The irrotational modes are not important when we reexpand the exciting $\hat{e}$ field of Eq. 5-20a with the open-circuit modes which also phase shift their amplitudes across the $(M\pm)$ coupling surfaces, or reexpand the exciting $\hat{H}$ field of Eq. 5-24b by the short-circuit modes with the same phase shift properties.

To begin the discussion, consider the short-circuit mode expansions for the field, Eqs. 5-19 through 5-22. All the irrotational and solenoidal modes are considered to be defined within the cavity proper, between coupling surfaces $(M\pm)$ in Fig. 5-1. Alternatively these coupling surfaces could be taken as planes $(B\pm)$. This means all their amplitudes will phase shift across those surfaces. The V_n , I_n , and I_m^i ($V_m^i \equiv 0$) amplitudes all depend upon $\hat{e} \times \bar{n}$ on the coupling surfaces. Let us alternatively expand the electric field $\hat{e}$ within each cavity by a set of open-circuit modes defined by Eq. 5-10 so as to have nonvanishing tangential electric field on $(M\pm)$. We presume that such a set of modes is complete for the electric field *within* the cavity and, by continuity, for the tangential electric field *on* $(M\pm)$. For cavities of uniform cross section the expansion will be complete since each open-circuit mode is just a linear combination of waveguide TM and TE mode-harmonics which are known to form a complete set. But since the electric field has no divergence in this chain we can employ just the *solenoidal* open-circuit modes. Schelkunoff's

example in the preceding section illustrated in general the existence of either a mode expansion composed of solenoidal modes alone or an expansion composed of both solenoidal and irrotational modes for the same field. Here we choose the solenoidal expansion and write electric field $\hat{e}$ as

$$\hat{e}(\vec{r}) = \sum_n v_n \bar{e}_n(\vec{r}), \quad \text{solenoidal within a cavity} \quad (5-36)$$

in which we do not yet know the v_n 's. They will be related back to the short-circuit V_n amplitudes in the process of deriving the dispersion relation.

As supporting evidence for the fact that we may expand $\hat{e}$ in this way we observe that no irrotational electric field has been generated in the short-circuit mode expansion: $V_m^i \equiv 0$ in Eq. 5-22. Now, if irrotational $\sum_m v_m^i \bar{e}_m^i$ field were present in Eq. 5-36, we could proceed to evaluate the amplitudes of the short-circuit modes for electric field by equating the short-circuit mode expansion on the right side of Eq. 5-19b to $\sum_n v_n \bar{e}_n + \sum_m v_m^i \bar{e}_m^i$ within the cavity volume. Suppose we do this and evaluate the amplitudes V_m^i to see if they are still zero. We dot-multiply both sides of this volume equivalence by $\epsilon_r \bar{E}_m^i$, integrate over the cavity volume, observe the orthogonality of $\bar{E}_m^i$ with all the other short-circuit modes, as well as with all $\bar{e}_n$:

$$\begin{aligned} \int_V \bar{E}_m^i \cdot \epsilon_r \bar{e}_n dV &\propto \int_V \nabla \Phi_m \cdot \nabla \times \bar{h}_n dV = \int_V \nabla \cdot (\Phi_m \nabla \times \bar{h}_n) dV \\ &\propto \oint_S \Phi_m \bar{e}_n \cdot d\vec{S} \equiv 0, \\ \Phi_m &= 0 \text{ on cavity walls and coupling surfaces} \end{aligned} \quad (5-37)$$

(see Eqs. 5-10, 5-12), so that we would obtain

$$V_m^i = \sum_k v_k^i \frac{1}{\tau} \int_V \bar{e}_k^i \cdot \epsilon_r \bar{E}_m^i dV \neq 0 \quad \text{in general}$$

Each V_m^i could conceivably still be zero for nonzero v_k^i but the fact that every one is identically zero in the generated short-circuit electric field expansion is good reason to take the open-circuit expansion for electric field as purely solenoidal. Teichmann and Wigner [4] have arrived at the same conclusion.

With *solenoidal* open-circuit mode expansion 5-36, it is easy to see why the irrotational short-circuit modes in Eqs. 5-19 do not affect the dispersion. All the solenoidal V_n and I_n amplitudes are related to solenoidal $\hat{e}$ in Eq. 5-20, and the v_n amplitudes of $\hat{e}$ can be related to the V_n 's by equating

the two expansions for electric field within the cavity volume:

$$\sum_n V_n \bar{E}_n = \sum_m v_m \bar{e}_m \quad (5-38)$$

We can dot-multiply both sides of this equation by $\epsilon_r \bar{E}_n$, integrate it over the cavity volume, use orthonormality Eq. 5-9, and obtain

$$V_n = \sum_m T_{nm} v_m, \quad T_{nm} = \frac{1}{\tau} \int_V \bar{E}_n \cdot \epsilon_r \bar{e}_m dV \quad (5-39)$$

The phase shift enters the problem in our expression for tangential $\hat{e}$ on a coupling surface ($M\pm$), which is³

$$\hat{e}_{\text{on } (M\pm)} = \frac{1}{2} \left(\sum_n v_n \bar{e}_n^{(0)} + v_n e^{\mp j\varphi} \bar{e}_n^{(\pm 1)} \right) \quad (5-40)$$

in Fig. 5-1. We used this averaging technique successfully in Chapter 4 and use it again in Section 5-5 in order to derive the dispersion relation for the first two passbands of a structure. Observe that we do have enough relations now to obtain a set of homogeneous equations for the v_m amplitudes alone, the determinant of which gives us a dispersion relation for $k^2(\varphi)$ without invoking Eq. 5-21 for the irrotational I_m^i amplitudes. This irrotational magnetic field is simply an auxiliary magnetic field within the cavities.

Now let us outline the dual reasoning for the expansion of the fields of Maxwell's equations in open-circuit modes, Section 5-3D. Now it is the tangential magnetic field on the ($M\pm$) coupling surfaces of Fig. 5-1 which determines the v_n , i_n , and v_n^i open-circuit mode amplitudes in Eqs. 5-24 and 5-25. We regard that magnetic field as expanded *within* the cavity by the alternative short-circuit mode representation

$$\hat{H} = \sum I_n \bar{H}_n \quad (5-41)$$

in terms of solenoidal modes with initially unknown amplitudes. In general, irrotational magnetic field cannot be present in Eq. 5-41 or else it would imply the presence of irrotational open-circuit mode components of magnetic field within the cavity by the volume equivalence of the two expansions. These irrotational components are identically zero by Eq. 5-26.

By the continuity of tangential component of each $\bar{H}_n$ pattern at the coupling holes, Eq. 5-41 will also give us the tangential magnetic field *on*

³ In Section 6-7 we introduce a variational expression for $k^2(\varphi)$ in terms of electric field alone, in which expression 5-40 will be implicit. The variational expression will also imply Eqs. 5-39 in the process of deriving the determinantal equation for the dispersion from the relations among the v_m amplitudes alone.

an $(M\pm)$ coupling surface provided we properly average the expansions on both sides of the surface.⁴

$$\hat{H}_{\text{on } (M\pm)} = \frac{1}{2} \left(\sum_n I_n \bar{H}_n^{(0)} + I_n e^{\mp j\varphi} \bar{H}_n^{(\pm 1)} \right) \quad (5-42)$$

in Fig. 5-1. Thus we obtain one set of relations for the v_n 's and i_n 's in terms of the I_n 's, the frequency, and the phase shift φ in Eq. 5-24, and we can relate all the i_n 's to all the I_n 's through the volume equivalence of both solenoidal expansions for magnetic field within a cavity:

$$\sum_n i_n \bar{h}_n = \sum_m I_m \bar{H}_m \quad (5-43)$$

If we dot-multiply both sides of this relation by $\mu_r \bar{h}_n$, integrate over the cavity, and use orthonormality relation 5-11, we obtain

$$i_n = \sum_m U_{mn} I_m, \quad U_{mn} = \frac{1}{\tau} \int_V \bar{H}_m \cdot \mu_r \bar{h}_n dV \quad (5-44)$$

With this relation we can form a set of homogeneous equations for the I_n amplitudes alone, the determinant of which yields the dispersion relation for $k^2(\varphi)$. Equation 5-25 simply gives us the amplitudes of the auxiliary irrotational electric field within the cavity, in terms of the solenoidal $\hat{H} = \sum I_n \bar{H}_n$ field, the relative amplitudes of which are known once $k^2(\varphi)$ in a given passband is known.

As a result of the foregoing considerations we have the following useful theorem [5]:

Closed cavity modes with divergence but no curl [irrotational modes] are not necessary for obtaining the frequency-phase shift characteristics of a periodic structure.

The theorem means that we may neglect the irrotational modes (but not the d.c. modes) within each cavity in the process of deriving the dispersion characteristics from the properties of the specially defined open- and short-circuit solenoidal modes. All these solenoidal modes must be defined *either* within a cavity bounded by surfaces $(M\pm)$ in Fig. 5-1 so as to have phase shift of amplitudes across these surfaces, *or* within a cell bounded by planes $(B\pm)$ and with phase shifts across *those* surfaces.

It so happens there is a modified procedure for solving Maxwell's equations in periodic structures which is particularly convenient for certain resonant-slot and resonant-loop coupled structures in which the

⁴ Equation 5-42 is implied by a variational expression in Section 6-7 for $k^2(\varphi)$ in terms of magnetic field alone. The expression will also imply Eq. 5-44 in the process of deriving from it the dispersion relation from the equations for the I_n amplitudes alone.

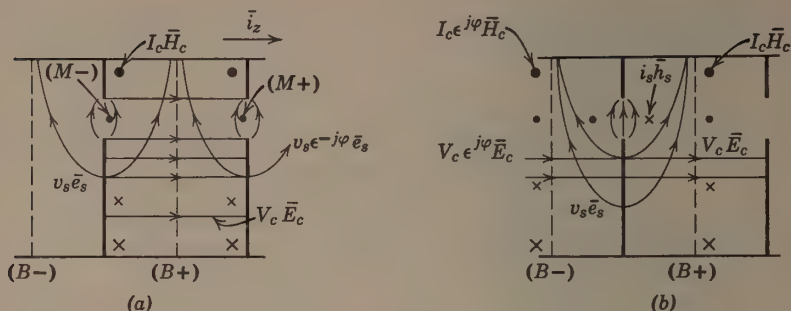


Fig. 5-3. Field representations in a resonant-slot coupled structure. (a) "cavity" mode (subscript c) excited on the slots $(M^\pm)$ by $\vec{E}$ of the "slot" mode (subscript s), (b) "slot" mode excited on the planes $(B^\pm)$ by $\vec{H}$ of the "cavity" mode.

fields tend to phase shift across the $(M^\pm)$ coupling surfaces in one passband and across the $(B^\pm)$ coupling surfaces in a neighboring passband. An example of such a structure was the resonant-slot coupled chain discussed on page 105. In this modified procedure we employ two sets of modes, one set of "cavity" modes defined between the $(M^\pm)$ coupling surfaces of Fig. 5-1 with phase shift of amplitudes across these surfaces, and another set of "coupling" modes defined between planes $(B^\pm)$ so as to have phase shift of amplitudes across those surfaces. In Fig. 5-3 we show the first such short-circuit "cavity" mode as well as the first open-circuit "coupling" or "slot" mode for the aforementioned resonant-slot coupled chain. The V_c and I_c "cavity" mode amplitudes phase shift across the $(M^\pm)$ surfaces and the patterns $\vec{E}_c$ and $\vec{H}_c$ repeat from cavity to cavity with the orientations shown. The v_s and i_s "slot" mode amplitudes phase shift across the $(B^\pm)$ surfaces and the $\vec{E}_s$ and $\vec{H}_s$ patterns repeat from "slot" cell to "slot" cell. Let us now describe how we could proceed to obtain the dispersion relation in terms of both these mode sets.

Within the cavity between the two slots we can expand the fields of Maxwell's equations with the short-circuit "cavity" modes, as excited by tangential electric field $\vec{E}$ on the slots according to Eq. 5-20. This $\vec{E}$ field we choose to expand with the open-circuit "slot" modes, which converge rapidly to the true slot field at frequencies such that the slot is about half a free-space wavelength long. In order to evaluate the "slot" mode amplitudes we can expand the fields of Maxwell's equations with these "slot" modes within each "slot" cell, excited by tangential $\vec{H}$ field on $(B^\pm)$ according to Eq. 5-24. We choose to expand this $\vec{H}$ -field with the more rapidly convergent "cavity" modes. To summarize: We excite the "cavity" modes with the "slot" modes on $(M^\pm)$ and the "slot" modes with the "cavity" modes on $(B^\pm)$. There are as many amplitude equations

as there are mode amplitudes, and so we are assured of obtaining a dispersion relation for $k^2(\varphi)$.

The point we now wish to make is that both the irrotational "cavity" and "slot" modes now contribute to the dispersion relation. Irrotational magnetic field is present in the short-circuit "cavity" mode expansion, Eq. 5-21, and it helps excite the "slot" modes on surfaces ($B_{\pm}$). In addition, irrotational electric field is present in the open-circuit "slot" mode expansion, Eq. 5-25, and it helps excite the "cavity" modes on ($M_{\pm}$).

Table 5-1. The Resonant Modes of a Symmetrical Cavity Chain

"Cavity" Modes Defined within a Periodic Cell Which is the Cavity Proper				"Coupling" Modes Defined within a Periodic Cell Centered around the Coupling Element			
Solenoidal		Irrotational		Solenoidal		Irrotational	
Even	Odd	Even	Odd	Even	Odd	Even	Odd
S O	S O	S O	S O	S O	S O	S O	S O

S and O stand for short-circuit and open-circuit modes.

Although we cannot neglect in principle the irrotational modes when we expand the fields in two disjoint cells of the structure in this way, it often happens that the irrotational field affects the dispersion curves only slightly. In Section 7-2 we find this is true for this class of resonant-slot coupled structures useful for high-power microwave beam amplifiers. In many of these resonant-slot and resonant-loop coupled structures we can simply ignore the irrotational modes, derive the form of the dispersion relation in terms of a few appropriately chosen solenoidal modes, and then adjust several parameters in that relation so as to obtain a good "fit" to the measured dispersion curve(s). This procedure is usually satisfactory for the analysis of such cavity chains interacting with electron beams, since the irrotational space charge field is better represented by the beam excitations than by the slowly convergent irrotational cavity modes.

We have just introduced the "coupling" modes, which have all the properties of the "cavity" modes except that they are defined within a different unit cell of the structure. Let us categorize all these different varieties of modes: "coupling" modes, irrotational modes, open-circuit modes, and so on, in Table 5-1. Reading downward in the table, any combination of adjectives such as "cavity, irrotational, even, open-circuit" applies to at least one mode of the structure.

5-5. Corrections to the First Dispersion Curve due to the Modes of the Second Passband

In this section we develop the dispersion relation for the first passband of a symmetrical cavity chain, Fig. 5-1, in terms of the solenoidal open- and short-circuit modes of the first two passbands. The development will illustrate the general procedure to follow for deriving the dispersion relation for more passbands in terms of more resonant cavity modes. We demonstrate that, for a great many structures, the modes of the second passband have the effect of depressing the $\omega-\varphi$ ($\varphi = \beta_0 L$) curve of the first passband, so that ω for a given φ tends to be less than specified by the single mode-pair or single-passband formula of Eq. 4-20 or 4-21. We consider structures that behave like the centerhole-coupled chain of Fig. 4-1; in Sections 7-5 and 7-6 we explain why many resonant-slot and resonant-loop coupled structures have this same property.

We treat here only those cavity chains in which the fields in *both* the first two passbands tend to phase shift abruptly across the coupling surfaces ($M\pm$) in Fig. 5-1. Our treatment will also describe chains in which the fields in *both* passbands tend to phase shift abruptly across the planes ($B\pm$) since these structures only differ in the definition of the unit cell. In Section 4-3 we pointed out how to decide whether the fields in a given passband tend to phase shift across the coupling surfaces or the cavity midplanes. The criterion was this: If *both* cavity modes which resonate at the two cutoff frequencies of the passband are either short-circuit or open-circuit on the coupling surfaces, say, then the fields tend to phase shift rapidly across the cavity midplanes instead, and vice versa. In such a case the patterns of the two modes should be defined for continuity across the coupling surfaces and with phase shift of amplitudes across the midplanes, whereupon they will form an open-circuit-short-circuit mode-pair on the midplanes. This mode-pair will then describe in first approximation the $\omega-\varphi$ curve of this passband according to the analysis of Chapter 4.

In view of these remarks we now wish to treat cavity chains for which *each* of the first two passbands is characterized by one open-circuit and one short-circuit mode resonating between coupling surfaces ($M\pm$) [or ($B\pm$)] in Fig. 5-1, with phase shift of amplitudes across those surfaces.

Having selected the type of symmetrical chain for discussion, let us solve Maxwell's equations with the short-circuit modes as in Section 5-3C. We have two such solenoidal modes, of amplitudes V_1 , I_1 and V_2 , I_2 in this approximation. From Eq. 5-20 we obtain the expression for

$V_{1,2}$ of either mode as

$$(P_n^2 - k^2)V_n = -\frac{P_n}{\tau} \int_{M\pm} \hat{e} \times \bar{H}_n^{(0)} \cdot \bar{n} dS, \quad n = 1, 2 \quad (5-45)$$

where P_n is the cutoff frequency of the mode, $k^2 = \omega^2 \mu \epsilon$, and the (0) superscript measures a field on the (0) cavity side of coupling surface ($M\pm$) in Fig. 5-1. Tangential $\hat{e}$ field on the coupling surfaces is given by Eq. 5-40,

$$\hat{e}_{\text{on}(M\pm)} = \frac{1}{2} \left(\sum_{m=1}^2 v_m \bar{e}_m^{(0)} + v_m \epsilon^{\mp j\varphi} \bar{e}_m^{(\pm 1)} \right), \quad m = 1, 2 \quad (5-40')$$

where the fields phase shift by $\epsilon^{\mp j\varphi}$ from cavity (0) to cavity (± 1). If the field of the n th mode ($n = 1$ or 2) is even, then $\bar{e}_n$ reverses direction across a coupling hole; the $\bar{e}_1$ field of Fig. 4-1c is a good example. If the field has odd character $\bar{e}_n$ does not reverse direction, as in Fig. 4-1e. Therefore we can relate the $\bar{e}_n$ patterns on both sides of coupling surfaces ($M\pm$) by the relations

$$\begin{aligned} \mp \bar{e}_n(x, y)^{(-1)}|_{\text{on}(M-)} &= \bar{e}_n(x, y)^{(0)}|_{\text{on}(M-)} = \mp \bar{e}_n(x, y)^{(0)}|_{\text{on}(M+)} \\ &= \bar{e}_n(x, y)^{(+1)}|_{\text{on}(M+)} \quad \begin{array}{l} \text{upper sign, even mode,} \\ \text{lower sign, odd mode} \end{array} \end{aligned} \quad (5-46)$$

The upper sign of a double sign prefixing a mode pattern will always be taken for an even mode, and the lower sign for an odd mode. With 5-46, Eq. 5-40' can be written as

$$\begin{aligned} \hat{e}_{\text{on}(M\pm)} &= \frac{1}{2} \sum_{m=1}^2 v_m (\mp \epsilon^{j\varphi} + 1) \bar{e}_m^{(0)}|_{\text{on}(M-)}, \\ &= \frac{1}{2} \sum_{m=1}^2 v_m (1 \mp \epsilon^{-j\varphi}) \bar{e}_m^{(0)}|_{\text{on}(M+)} \end{aligned} \quad (5-47)$$

Next we substitute Eqs. 5-47 into Eq. 5-45 and relate the integral over the ($M+$) coupling surface on the (0) cavity side to that over ($M-$) by means of the relation

$$\int_{M+} \bar{e}_m^{(0)} \times \bar{H}_n^{(0)} \cdot \bar{n} dS \equiv \int_{M-} (\mp \bar{e}_m^{(0)}) \times (\pm \bar{H}_n^{(0)}) \cdot \bar{i}_z dS \quad (5-48)$$

in which we recognize the fact that only $\bar{H}_n$ reverses direction from one coupling surface to the other if the mode is odd, Eq. 5-2. In this way, we can write Eq. 5-45 so as to represent both possible cases as

$$\begin{aligned} (P_n^2 - k^2)V_n &= -\frac{P_n}{\tau} \frac{1}{2} \sum_{m=1}^2 v_m (\mp \epsilon^{j\varphi} + 1) \int_{M-} \bar{e}_m^{(0)} \times \bar{H}_n^{(0)} \cdot (-\bar{i}_z) dS \\ &\quad - \frac{P_n}{\tau} \frac{1}{2} \sum_{m=1}^2 v_m (1 \mp \epsilon^{-j\varphi}) \int_{M-} (\mp \bar{e}_m^{(0)}) \times (\pm \bar{H}_n^{(0)}) \cdot \bar{i}_z dS, \quad n = 1, 2 \end{aligned} \quad (5-49)$$

We must add that the sign prefixing $\epsilon^{j\varphi}$ and $\epsilon^{-j\varphi}$ is chosen in accord with the symmetry of $\bar{e}_m$, the upper sign for an even $\bar{e}_m$ pattern.

Equation 5-49 represents two equations for $V_{1,2}$ in terms of $v_{1,2}$, the phase shift φ , and frequency k . The relations we need in order to eliminate $V_{1,2}$ from the problem are obtained from the volume equivalence of electric field expansions *within* the cavity,

$$\sum_{n=1}^2 V_n \bar{E}_n = \sum_{m=1}^2 v_m \bar{e}_m \quad (5-38')$$

whereupon

$$V_n = \sum_{m=1}^2 T_{nm} v_m, \quad T_{nm} = \frac{1}{\tau} \int_V \bar{E}_n \cdot \epsilon_r \bar{e}_m dV \quad (5-39')$$

according to the variational treatment of Section 6-7.

Now let us particularize these equations to more specific structures. Suppose the fields and the modes in *both* the first two passbands are either even (upper signs) or odd. An example is a centerhole-coupled structure of narrow cavities such that the short-circuit electric field E_z pattern varies as $J_0(P_1 r)$ and $J_0(P_2 r)$ in the respective passbands. Then we can obtain two homogeneous equations for v_1 and v_2 alone from Eq. 5-49, with appropriate sign choices, and the preceding Eq. 5-39'. If we set the determinant of the matrix of coefficients equal to zero, we obtain the determinantal equation for $k^2(\varphi)$ in the form

$$\begin{vmatrix} (P_1^2 - k^2)T_{11} - P_1 M_{11}(1 \mp \cos \varphi) & (P_1^2 - k^2)T_{12} - P_1 M_{12}(1 \mp \cos \varphi) \\ (P_2^2 - k^2)T_{21} - P_2 M_{21}(1 \mp \cos \varphi) & (P_2^2 - k^2)T_{22} - P_2 M_{22}(1 \mp \cos \varphi) \end{vmatrix} = 0$$

with

$$\begin{aligned} T_{ij} &= \frac{1}{\tau} \int_V \bar{E}_i \cdot \epsilon_r \bar{e}_j dV \\ M_{ij} &= \frac{1}{\tau} \int_{M-} \bar{e}_j^{(0)} \times \bar{H}_i^{(0)} \cdot \bar{i}_z dS \end{aligned} \quad (5-50)$$

We see that the 11 or 22 term alone set equal to zero gives the single-passband dispersion relation according to Chapter 4. The terms M_{11} and M_{22} are the passband parameters for the first and second passbands; M_{12} and M_{21} represent crosscoupling between the modes of the two passbands on the coupling surfaces.

Unfortunately we cannot make any general statements about the effect of the crosscoupling upon the shape of the first passband because we do not know the relative magnitudes of $(P_1^2 - k^2)T_{12}$ and $(P_2^2 - k^2)T_{21}$ relative to $P_1 M_{12}$ and $P_2 M_{21}$, respectively.

Now consider those cavity chains for which the first passband has fields of even (or odd) symmetry, while the second passband has fields of odd (or even) symmetry. Then the appropriately chosen signs in Eq. 5-49 and Eq. 5-39' yield for the dispersion

$$\begin{vmatrix} (P_1^2 - k^2)T_{11} - P_1M_{11}(1 \mp \cos \varphi) & \pm j \sin \varphi P_1M_{12} \\ \mp j \sin \varphi P_2M_{21} & (P_2^2 - k^2)T_{22} - P_2M_{22}(1 \mp \cos \varphi) \end{vmatrix} = 0 \quad (5-51a)$$

which, upon expansion, is

$$[(P_1^2 - k^2)T_{11} - P_1M_{11}(1 \mp \cos \varphi)][(P_2^2 - k^2)T_{22} - P_2M_{22}(1 \mp \cos \varphi)] - P_1P_2M_{12}M_{21} \sin^2 \varphi = 0 \quad (5-51b)$$

The upper signs of both j terms are selected if the field in the first passband has even symmetry and that in the second passband has odd symmetry. We have made use of the fact that $T_{12} = T_{21} \equiv 0$ because the volume coupling of an even and an odd mode pattern is zero. This dispersion relation predicts that $k^2(\varphi)$ in the first passband will almost always be lower due to the $\sin^2 \varphi$ term. To see why, we first observe that $M_{12}M_{21}$ is almost always positive. This follows from the fact that the whole $\bar{H}_2$ field pattern flows very much like the $\bar{H}_1$ pattern on a coupling surface, since the resonant frequencies of both modes are relatively low and fairly close together. Then the $\bar{E}_2$ and $\bar{E}_1$ lines of flow are much alike. Since the $\bar{e}_2$ and $\bar{e}_1$ lines are perturbations of the $\bar{E}_2$ and $\bar{E}_1$ lines obtained by covering the coupling surfaces with magnetic shorts, the $\bar{e}_2$ and $\bar{e}_1$ lines also flow much the same way. Therefore

$$\int_{M-} \bar{e}_2^{(0)} \times \bar{H}_1^{(0)} \cdot \bar{i}_z dS$$

will usually have the same sign as

$$\int_{M-} \bar{e}_1^{(0)} \times \bar{H}_2^{(0)} \cdot \bar{i}_z dS$$

and $M_{12}M_{21}$ will be positive. To satisfy Eq. 5-51b, $k^2(\varphi)$ will then decrease in the first passband so as to make the first bracketed expression positive, the second bracketed expression already being positive.

Our conclusion is this: In a cavity chain with the first passband characterized by an even (or odd) short-circuit and open-circuit mode-pair with phase shift of amplitudes across surfaces ($M\pm$) or ($B\pm$) in Fig. 5-1 and with the second passband characterized by an odd (or even) mode-pair with these same properties, the first passband will almost always be depressed. In reaching this conclusion the modes of the third passband were neglected since they are weakly excited at frequencies in the first

passband. The conclusion rests essentially upon the similarity of the $\bar{e}_1$ and $\bar{e}_2$ patterns and of the $\bar{H}_1$ and $\bar{H}_2$ patterns on the coupling surfaces.

We have reached this conclusion from dispersion relations derived by expanding the fields of Maxwell's equations with the short-circuit modes, as excited by tangential electric field on the coupling surfaces. We can alternatively follow the procedure of Section 5-3D and expand the fields with the open-circuit modes, but our final dispersion relations will be identical to Eqs. 5-50 and 5-51. The remainder of this section is devoted to a demonstration of this fact in order to show the consistency between these two expansion techniques.

From Eqs. 5-23 and 5-24a we have, upon elimination of v_n ,

$$(p_n^2 - k^2)i_n = + \frac{p_n}{\tau} \int_{M\pm} \bar{e}_n^{(0)} \times \hat{H} \cdot \bar{n} dS, \quad n = 1, 2 \quad (5-52)$$

where the (0) superscript reminds us that the field pattern $\bar{e}_n$ is measured on the (0) cavity side of surface ($M\pm$) in Fig. 5-1. For magnetic field $\hat{H}$ we use the averaged expression

$$\hat{H}_{\text{on}(M\pm)} = \frac{1}{2} \sum_{m=1}^2 (I_m \bar{H}_m^{(0)} + I_m \epsilon^{\mp j\varphi} \bar{H}_m^{(\pm 1)}) \quad (5-42')$$

Because the tangential $\bar{H}_n$ pattern does not reverse direction from ($M-$) to ($M+$) if the mode is even (upper signs) but does if the mode is odd (lower signs), we have

$$\begin{aligned} \pm \bar{H}_m^{(-1)}(x, y)|_{\text{on}(M-)} &= \bar{H}_m(x, y)^{(0)}|_{\text{on}(M-)} = \pm \bar{H}_m(x, y)^{(0)}|_{\text{on}(M+)} \\ &= \bar{H}_m(x, y)^{(+1)}|_{\text{on}(M+)} \end{aligned} \quad (5-53)$$

whereupon Eq. 5-42' reads

$$\begin{aligned} \hat{H}_{\text{on}(M\pm)} &= \frac{1}{2} \sum_{m=1}^2 I_m (\pm \epsilon^{j\varphi} + 1) \bar{H}_m^{(0)}|_{\text{on}(M-)} \\ &= \frac{1}{2} \sum_{m=1}^2 I_m (1 \pm \epsilon^{-j\varphi}) \bar{H}_m^{(0)}|_{\text{on}(M+)} \end{aligned} \quad (5-54)$$

Next, we substitute Eq. 5-54 into Eq. 5-52 and relate the integral on the ($M+$) coupling surface, (0) cavity side, to that on the ($M-$) surface, (0) cavity side, by Eq. 5-48. Then Eq. 5-52 reads

$$\begin{aligned} (p_n^2 - k^2)i_n &= \frac{p_n}{\tau} \frac{1}{2} \sum_{m=1}^2 I_m (\pm \epsilon^{j\varphi} + 1) \int_{M-} \bar{e}_n^{(0)} \times \bar{H}_m^{(0)} \cdot (-\bar{i}_z) dS \\ &\quad + \frac{p_n}{\tau} \frac{1}{2} \sum_{m=1}^2 I_m (1 \pm \epsilon^{-j\varphi}) \int_{M-} (\mp \bar{e}_n^{(0)}) \times (\pm \bar{H}_m^{(0)}) \cdot \bar{i}_z dS, \\ &\quad n = 1, 2 \end{aligned} \quad (5-55)$$

We must add that the sign prefixing $\epsilon^{j\varphi}$ and $\epsilon^{-j\varphi}$ is chosen according to the symmetry of the field (upper sign, even symmetry).

The remaining two relations between $i_{1,2}$ and $I_{1,2}$ are furnished by the volume equivalence of the two (solenoidal) mode expansions for magnetic field within the cavity,

$$\sum_{n=1}^2 i_n \bar{h}_n = \sum_{m=1}^2 I_m \bar{H}_m \quad (5-43')$$

whereupon

$$i_n = \sum_{m=1}^2 U_{mn} I_m, \quad U_{mn} = \frac{1}{\tau} \int_V \bar{H}_m \cdot \mu_r \bar{h}_n dV \quad (5-44')$$

Now let the first and second passbands both be described by even modes (upper signs) or by odd modes (lower signs). The preceding equations furnish two homogeneous equations for $I_{1,2}$ and the determinant of their matrix of coefficients yields the dispersion relation

$$\begin{vmatrix} (p_1^2 - k^2)U_{11} + p_1 M_{11}(1 \pm \cos \varphi) & (p_1^2 - k^2)U_{21} + p_1 M_{21}(1 \pm \cos \varphi) \\ (p_2^2 - k^2)U_{12} + p_2 M_{12}(1 \pm \cos \varphi) & (p_2^2 - k^2)U_{22} + p_2 M_{22}(1 \pm \cos \varphi) \end{vmatrix} = 0$$

with

$$U_{ij} = \frac{1}{\tau} \int_V \bar{H}_i \cdot \mu_r \bar{h}_j dV \quad (5-56)$$

$$M_{ij} = \frac{1}{\tau} \int_{M-} \bar{e}_j^{(0)} \times \bar{H}_i^{(0)} \cdot \bar{i}_z dS$$

We can show the complete equivalence of this relation with Eq. 5-50 as follows. In order to transform the 11 and 22 terms we need two relations, which may be obtained from the following divergence identity inside reference cavity (0):

$$\begin{aligned} \int_V \nabla \cdot (\bar{A} \times \alpha^{-1} \nabla \times \bar{B}) dV &= \int_V (\alpha^{-1} \nabla \times \bar{B} \cdot \nabla \times \bar{A} - \bar{A} \cdot \nabla \times \alpha^{-1} \nabla \times \bar{B}) dV \\ &= \oint_S \bar{A}^{(0)} \times \alpha^{-1} \nabla \times \bar{B}^{(0)} \cdot \bar{n} dS \end{aligned} \quad (5-57)$$

Vectors $\bar{A}$ and $\bar{B}$ and scalar function α are presumed to have continuous derivatives in the cavity. By subtracting the same relation with $\bar{A}$ and $\bar{B}$ interchanged we obtain

$$\begin{aligned} \int_V (\bar{B} \cdot \nabla \times \alpha^{-1} \nabla \times \bar{A} - \bar{A} \cdot \nabla \times \alpha^{-1} \nabla \times \bar{B}) dV \\ = \oint_S (\bar{A}^{(0)} \times \alpha^{-1} \nabla \times \bar{B}^{(0)} - \bar{B}^{(0)} \times \alpha^{-1} \nabla \times \bar{A}^{(0)}) \cdot \bar{n} dS \end{aligned} \quad (5-58)$$

This is a generalized Green's relation analogous to Eq. 3-43b. Now to transform the 11 term of Eq. 5-56 let $\bar{B} = \bar{H}_1$, $\alpha = \varepsilon_r$, and $\bar{A} = \bar{h}_1$ in Eq. 5-58. By the definitions of the mode patterns in Eq. 5-5 and 5-10 we obtain

$$\begin{aligned} \int_V (\bar{H}_1 \cdot P_1^2 \mu_r \bar{h}_1 - \bar{h}_1 \cdot P_1^2 \mu_r \bar{H}_1) dV \\ = \oint_S (\bar{h}_1^{(0)} \times P_1 \bar{E}_1^{(0)} - \bar{H}_1^{(0)} \times p_1 \bar{e}_1^{(0)}) \cdot \bar{n} dS \\ = p_1 \int_{M\pm} \bar{e}_1^{(0)} \times \bar{H}_1^{(0)} \cdot \bar{n} dS \end{aligned} \quad (5-59a)$$

since $\bar{E}_1^{(0)} \times \bar{n} \equiv 0$ on walls and $(M\pm)$. This relation reduces to

$$(P_1^2 - P_1^2)U_{11} = -2p_1 M_{11} \quad (5-59b)$$

provided *both* the open-circuit and short-circuit modes are either even or odd by the definitions of U_{11} and M_{11} given earlier. With Eq. 5-59b the 11 term of Eq. 5-56 can be rewritten as

$$(P_1^2 - k^2)U_{11} - p_1 M_{11}(1 \mp \cos \varphi) \quad (5-60)$$

The other relation we need is obtained by substituting $\bar{B} = \bar{H}_1$, $\alpha = \varepsilon_r$, and $\bar{A} = \bar{h}_1$ into Eq. 5-57:

$$\int_V (P_1 \bar{E}_1 \cdot p_1 \varepsilon_r \bar{e}_1 - \bar{h}_1 \cdot P_1^2 \mu_r \bar{H}_1) dV = \oint_S \bar{h}_1^{(0)} \times P_1 \bar{E}_1^{(0)} \cdot \bar{n} dS = 0 \quad (5-61a)$$

with $\bar{E}_1^{(0)} \times \bar{n} = 0$ on walls and $(M\pm)$. This is just

$$p_1 T_{11} = P_1 U_{11} \quad (5-61b)$$

With this, Eq. 5-60 for the 11 term becomes

$$\frac{P_1}{P_1} [(P_1^2 - k^2)T_{11} - P_1 M_{11}(1 \mp \cos \varphi)] \quad (5-62)$$

The 22 term of Eq. 5-56 can be similarly transformed into

$$\frac{P_2}{P_2} [(P_2^2 - k^2)T_{22} - P_2 M_{22}(1 \mp \cos \varphi)] \quad (5-63)$$

To transform the 12 term we write Eq. 5-58 with $\bar{B} = \bar{H}_2$, $\alpha = \varepsilon_r$, and $\bar{A} = \bar{h}_1$ so as to develop the relation

$$(P_1^2 - P_2^2)U_{21} = -2p_1 M_{21} \quad (5-64)$$

provided *both* the first open-circuit and the second short-circuit modes are either even or odd. We also find

$$p_1 T_{21} = P_2 U_{21} \quad (5-65)$$

With these expressions the 12 term of Eq. 5-56 can be written as

$$\frac{P_1}{P_2} [(P_2^2 - k^2)T_{21} - P_2 M_{21}(1 \mp \cos \varphi)] \quad (5-66)$$

Finally, the 21 term can be transformed into

$$\frac{P_2}{P_1} [(P_1^2 - k^2)T_{12} - P_1 M_{12}(1 \mp \cos \varphi)] \quad (5-67)$$

With relations 5-62 through 5-67 our determinantal Eq. 5-56 reads

$$0 = \begin{vmatrix} P_1 & 0 \\ 0 & P_2 \end{vmatrix} \times \begin{bmatrix} (P_1^2 - k^2)T_{11} - P_1 M_{11}(1 \mp \cos \varphi) & (P_2^2 - k^2)T_{21} - P_2 M_{21}(1 \mp \cos \varphi) \\ (P_1^2 - k^2)T_{12} - P_1 M_{12}(1 \mp \cos \varphi) & (P_2^2 - k^2)T_{22} - P_2 M_{22}(1 \mp \cos \varphi) \end{bmatrix} \times \begin{vmatrix} P_1^{-1} & 0 \\ 0 & P_2^{-1} \end{vmatrix} \quad (5-68)$$

which is the same as the previously derived Eq. 5-50 in spite of the transposition of the 12 and 21 terms in the bracketed portion.

Now, if the first passband fields are characterized instead by even (or odd) modes and the second passband fields by odd (or even) modes, we can modify this procedure to show that the determinantal equation obtained from Eqs. 5-55 and 5-44',

$$\begin{vmatrix} (p_1^2 - k^2)U_{11} + p_1 M_{11}(1 \pm \cos \varphi) & \mp j \sin \varphi p_1 M_{21} \\ \pm j \sin \varphi p_2 M_{12} & (p_2^2 - k^2)U_{22} + p_2 M_{22}(1 \pm \cos \varphi) \end{vmatrix} = 0 \quad (5-69)$$

can be converted into Eq. 5-51 (the upper sign for the j terms is taken if the first passband is "even").

These last few paragraphs have shown the equivalence of the 2×2 determinantal equations for $k^2(\varphi)$ in the first or second passband, as obtained by expanding the fields of Maxwell's equations with either the short- or the open-circuit modes. The equivalence can be generalized⁵

⁵ R. M. Bevensee [5]. In this reference extra minus signs were introduced into the definitions of the even-odd coupling elements such as M_{12} so that the extended version of Eq. 5-68 in matrix form differs from the extended version of Eq. 5-50 by a trivial conjugate sign.

for all the passbands and we have the following statement.

In principle, the dispersion characteristics of a symmetrical cavity chain can be obtained by employing the solenoidal short- and open-circuit modes defined within a particular cell. It does not matter which set of modes is used to expand the fields of Maxwell's equations.

5-6. The Frequency-Squared Sum Rule

We can consider the complete dispersion relation for all the passbands as written in a particular determinantal form, with the various elements of the determinant expressed by the parameters of the solenoidal short- and open-circuit modes. In this form the diagonal elements alternately represent passbands of even and odd field symmetry. Then the upper left-hand corner of the complete determinant for $k^2(\varphi)$ looks like Eq. 5-51 or the equivalent Eq. 5-69. If the first two open- and short-circuit modes of a structure are both even or both odd the infinite determinant cannot be approximated by the upper left-hand 2×2 portion at frequencies in the first or second passband; Eq. 5-50 or Eq. 5-56 should be used instead. But the form of Eq. 5-51 exemplifies a conservation property of all the passbands, so let us examine it.

If we solve quadratic Eq. 5-51 for the two values of $k^2(\varphi)$ in the first two passbands at a given φ , we obtain

$$k_{1,2}^2(\varphi) = \frac{1}{2} \left[k_{10}^2(\varphi) + k_{20}^2(\varphi) \pm \sqrt{(k_{10}^2 - k_{20}^2)^2 + \frac{4P_1P_2M_{12}M_{21}}{T_{11}T_{22}} \sin^2 \varphi} \right] \quad (5-70)$$

where $k_{10}^2(\varphi) = P_1^2 - P_1M_{11}(1 \mp \cos \varphi)/T_{11}$ is the single mode-pair estimate of k^2 in the first passband, according to Eq. 4-11 or Eq. 4-18, and $k_{20}^2(\varphi) = P_2^2 - P_2M_{22}(1 \mp \cos \varphi)/T_{22}$ is the single-passband estimate of k^2 in the second passband. This relation tells us that, for structures described by Eq. 5-51,

$$k_1^2(\varphi) + k_2^2(\varphi) = k_{10}^2(\varphi) + k_{20}^2(\varphi) \quad (5-71)$$

that is, the sum of the squares of the two frequencies in the actual passbands as calculated here is equal to the sum of the squares of the frequencies on the first two single mode-pair curves between the same cutoff frequencies.

A general statement [5] can be proved from the complete dispersion relation for all the passbands:

At a given phase shift per cavity, φ , the sum of the squares of the frequencies of all the passbands equals the sum of the $k^2(\varphi)$ for all the single-passband curves, each of which follows a simple $1 \pm \cos \varphi$ variation between the actual cutoff points. This rule is independent of the nature of the coupling in the symmetrical periodic cavity chain.

Since the rule is independent of the nature of the symmetrical coupling it would apply to the tape ladder lines discussed in Section 2-5, for example. Unfortunately we do not have a complementary statement to the effect that the deviation of the n th dispersion curve from its single mode-pair form goes to zero in a prescribed fashion as n goes to infinity.

Problems

1. Show the mode relations of Eqs. 5-5 through 5-18 are valid if μ and ϵ are Hermitian dyadics (such that the transpose conjugate of μ equals μ and similarly for ϵ), with slight modifications of notation.
2. Show that Eq. 5-20 also obtains if we solve Maxwell's Eq. 5-19 for the solenoidal mode amplitudes without integrating by parts but with magnetic surface current $\bar{n}_i \times \hat{e}$ on the holes, where $\bar{n}_i$ is the unit normal inward.
3. Suppose three solenoidal modes exist with the same resonant frequency. Construct linear combinations of them which will be orthogonal among themselves, with each combination normalized to the cavity volume. Develop an argument to show that all the coefficients of the mode patterns in each combination may be considered real if the patterns are real vectors. Do the $\bar{H}$ patterns have the same relations among themselves as the $\bar{E}$ patterns, given $P\bar{H}_i = \nabla \times \bar{E}_i$?
4. Suppose the fields of the Fig. 5-1 chain are expanded in short-circuit modes within reference cavity (0), as driven by tangential $\hat{e}$ field on the $(M\pm)$ surfaces, and that field is given by an open-circuit mode expansion between planes $(B\pm)$, driven by tangential *magnetic* field on the $(B\pm)$ surfaces. How do the irrotational modes determine the amplitudes of each mode? The relevance of the irrotational modes in this particular description of the chain puts the theorem on page 137 in proper perspective.
5. Equations 5-29 and 5-30 describe a short-circuit mode expansion for the rectangular cavity considered by Schelkunoff. Derive the input impedance by expanding the fields with *open*-circuit modes, driven by tangential magnetic field on the $z = 0$ surface.
6. Why would Eqs. 5-50 and 5-51 be inappropriate dispersion equations for the first passband if the second one were characterized by the "slot" modes of Fig. 5-3b, defined as open- or short-circuit on the midplanes $(B\pm)$? Would the *complete* dispersion determinant in terms of all the "cavity" modes be correct (in principle) in spite of the field behavior in the second passband?

7. Use Eq. A-9 in Appendix A to obtain an expression for the power flow across a coupling surface in terms of the resonant cavity modes of both the first and second passbands. Equations 5-50 or 5-51a infer the relative amplitudes of the short-circuit modes.
8. Prove the general equivalence of the dispersion relations obtained from the complete short- and open-circuit mode expansions. If each relation is written in matrix form it can be converted into the other by means of the generalized relations 5-59b and 5-61b also in matrix form. (The answer is in Ref. [5].)
9. Consider the sum of the squares of the short-circuit mode resonant frequencies as compared to the sum of the squares of the open-circuit mode resonant frequencies. From the behavior of the first three or four dispersion curves in the line of Prob. 5, Chapter 4, show that these two sums are apparently not equal.
10. Consider again the doubly periodic structure described in Prob. 10 of Chapter 4. Obtain the determinantal equation for $k^2(\varphi)$, φ being the periodic phase shift across any two cavities, by expanding the fields of Maxwell's equations with the first two short-circuit modes of each cell "driven" by the corresponding open-circuit modes. Also derive the dispersion relation by following the procedure of Section 4-6 for the doubly periodic structure. Transform either of these determinantal equations into the other with Eqs. 5-57 through 5-69 as a guide.

References

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Chapter VI

Variational Analyses of “Cold” Slow Wave Structures

6-1. Synopsis

In this chapter we discuss the applications of specific variational expressions to the calculation of the dispersion characteristics of various helices and cavity chains, including the ridged waveguide. The variational technique is valuable in slow wave structure analysis because with it we can form expressions for the frequency associated with a given propagation constant or phase shift per period which have the stationary property. This means small errors in the field expressions create much smaller errors in the calculated frequency, so that with relatively crude trial fields we can often obtain an accurate dispersion relation. We must take care to employ realistic functions which are capable of representing the fields well.

We first present some examples of the variational technique in order to acquaint the reader with its philosophy and the notation. Then we develop a very general variational expression for $j\omega$ which serves for a wide variety of helices, including the usual tape and wire helices and the ring and contrawound helices described by C. K. Birdsall. After verifying the fact that this variational expression yields the correct sheath helix dispersion relation, we use it to derive the dispersion relation for the wave on a narrow tape helix with primary n th mode ($n \neq 0$) in the range $0 \leq \beta_n a \leq 0.5 \cot \psi$. The fields are approximated by just the n th sheath helix modes with variational amplitudes, and certain field continuities are explicitly written on the cylindrical surface between inner and outer helix regions. The resultant dispersion relation describes well the slow wave nature of the $n = 1$ mode fields in the higher range of $\beta_1 a$ and

their fast wave nature as $\beta_1 a$ approaches zero near the first forbidden zone.

We then apply this same expression to a derivation of the dispersion of the tape helix wave with predominant $n = 0$ mode component in this same range $0 \leq \beta_0 a \leq 0.5 \cot \psi$. By specifying only the explicit continuity of electric field parallel to the tape on the helix surface we obtain a well-behaved determinantal equation for $\omega(\beta_0)$. This relation has the asymptotic form of the dispersion relation for the $n = 0$ sheath helix wave as ka approaches both zero and infinity and is well behaved in the intermediate range. The trial fields have the asymptotic forms of the sheath helix fields provided we reverse the signs of the external trial fields. This sign peculiarity means the electric field at low frequencies, $2ka \ll \cot \psi$, actually behaves as though the $n = 0$ mode component of electric field parallel to the tape were *discontinuous* across the helix surface; that is, zero on that surface.

We calculated the ω - β_0 curve and impedance of a helix measured by Watkins and Siegman using these $n = 0$ mode variational expressions for the fields. The calculated dispersion curve was quite accurate, but the impedance was erroneous by a factor of 2 or more.

Next, we apply the variational expression for ω to a calculation of the dispersion in a particular ring-plane helix consisting of a chain of rings alternating with gaps and supported at opposite points by two planes. There are two dispersion curves of interest, both of which are asymptotic to the velocity of light line at low frequencies. One of these waves has even symmetry of E_z about the two planes and the other wave has odd symmetry. We represent the fields of each wave by appropriate sheath helix modes which propagate only as $e^{-j\beta_0 z}$; that is, we neglect the other axial harmonics. The derived dispersion relation for the wave of odd symmetry is very accurate over the range $0 \leq \beta_0 p \leq \pi$; the relation for the other wave is accurate over a smaller range.

Although the calculated impedance $e_z^2/2\beta_0^2 P_{av}$ is not particularly accurate, we estimate the impedance of the wave of odd symmetry to be equal to or greater than that for the wave of even symmetry. However, we can reduce the effect of the former wave in beam amplifiers by cutting notches in the planes between the rings.

The contrawound and multifilar helices may be treated analogously with trial fields appropriate to the symmetry. The dispersion diagrams are qualitatively those of the single-tape helix.

Next we apply our variational expression to ridged waveguides. Our technique for a waveguide with very thin ridges is quite similar to that of Slater, who divided the waveguide into two regions, expanded the fields with appropriate modes in each region, and then matched them on the

common boundary. Our analysis is not as elegant or accurate as the Chu-Schwinger method, but it avoids the use of cumbersome Green's functions for expanding the fields on the common boundary. It has the advantage of ready extension to waveguides with finite-thickness teeth or ridges. We illustrate this variational analysis by calculating the first passband of the fin-loaded waveguide studied by Slater and by Chu and Hansen. Our dispersion relation is in the form of a 3×3 determinant and it checks well with the measured curve.

To conclude this chapter we quote several variational expressions for $k^2 = \omega^2 \mu \epsilon$ appropriate for a periodic cavity chain. We show how we may derive from them the same dispersion relations as were obtained in Chapters 4 and 5 from the mode expansion point of view.

After analyzing all these structures we have reached certain conclusions about the advantages and disadvantages of variational techniques, and they are summarized in Section 6-8.

6-2. Examples of the Variational Technique¹

A. Minimization of Mean-Square Error by Fourier Expansion

Suppose we wish to represent a scalar function $f(\theta)$ over the interval $(0, 2\pi)$ by a finite set of sines and cosines so as to minimize the mean-square error δ ,

$$\delta = \int_0^{2\pi} \left[f(\theta) - \sum_{n=0}^N (a_n \cos n\theta + b_n \sin n\theta) \right]^2 d\theta \quad (6-1)$$

We now determine a_n and b_n variationally so as to make δ an extremum. Then we show that the extremum is actually a minimum.

When we expand the integrand of Eq. 6-1 and invoke the orthogonalities

$$\int_0^{2\pi} \cos m\theta \cos n\theta d\theta = \int_0^{2\pi} \sin m\theta \sin n\theta d\theta = \pi$$

¹ H. Goldstein [1, pp. 30-38] explains concisely how the calculus of variations enables us to describe the "path" followed by any "system" knowing an integral between the endpoints of this path is an extremum. He shows how Hamilton's principle leads to Lagrange's equations of motion when the integral of the Lagrangian function between the endpoints in time is made an extremum. In our work "system" will refer to fields and perhaps electron beam excitations and "path" will refer to the spatial rather than the temporal distribution of these excitations.

for $n = m \neq 0$, and

$$\int_0^{2\pi} \cos m\theta \sin n\theta d\theta = 0$$

we obtain

$$\begin{aligned} \varepsilon = \int_0^{2\pi} f^2(\theta) d\theta - 2 \sum_{n=1}^N \int_0^{2\pi} f(\theta)(a_n \cos n\theta + b_n \sin n\theta) d\theta \\ + 2\pi a_0^2 + \pi \sum_n (a_n^2 + b_n^2) \quad (6-2) \end{aligned}$$

We make ε an extremum by setting its variation $\delta\varepsilon$ equal to zero, first with respect to a variation in a_0 , δa_0 , with the other amplitudes held fixed:

$$\frac{\delta\varepsilon}{\delta a_0} = 0 = -2 \int_0^{2\pi} f(\theta) d\theta + 4\pi a_0 \quad (6-3)$$

and also with respect to variations δa_n and δb_n , $n \neq 0$:

$$\begin{aligned} \frac{\delta\varepsilon}{\delta a_n} = 0 = -2 \int_0^{2\pi} f(\theta) \cos n\theta d\theta + 2\pi a_n \\ \frac{\delta\varepsilon}{\delta b_n} = 0 = -2 \int_0^{2\pi} f(\theta) \sin n\theta d\theta + 2\pi b_n \end{aligned} \quad (6-4)$$

We see that the coefficients are just the Fourier coefficients

$$\begin{aligned} a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta, \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta \\ b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta \end{aligned} \quad (6-5)$$

To prove that these coefficients make ε a minimum we let one of them, say a_m , vary by δa_m . Then from Eq. 6-1

$$\begin{aligned} \varepsilon + \delta\varepsilon = \int_0^{2\pi} \left[f(\theta) - \sum_n (a_n \cos n\theta + b_n \sin n\theta) - \delta a_m \cos m\theta \right]^2 d\theta \\ = \varepsilon - 2\delta a_m \int_0^{2\pi} [f(\theta) \cos m\theta - a_m \cos^2 m\theta] d\theta + (\delta a_m)^2 \pi (1 + \delta_{m0}) \end{aligned} \quad (6-6)$$

The symbol δ_{m0} is 1 only if $m = 0$; otherwise it is zero. The integral is identically zero by definition of a_m in Eq. 6-5, and so δa_m has increased ε . It follows that ε is also a minimum with respect to changes in every b_m as well.

B. Solutions of the Scalar Wave Equation

Consider the scalar wave equation for a function $\psi(x, y, z)$ within a closed volume V :

$$\begin{aligned} \nabla^2 \psi + k^2 \psi &= 0 & \text{within } V \\ \psi \text{ or } \nabla \psi \cdot \vec{n} &= 0 & \text{on } S \text{ bounding } V \end{aligned} \quad (6-7)$$

This equation will generally have an infinite number of solutions denoted by ψ_0 of "eigenvalue" k_0^2 , ψ_1 of eigenvalue k_1^2 , ... The solutions ψ_0 , ψ_1 , and so on, are called the "eigenfunctions" of the equation. Suppose we wish to form an expression which will give us a very good value for the lowest eigenvalue k_0^2 from only a fairly good estimate of eigenfunction ψ_0 . We form an integral expression for k^2 which tends to "smooth" errors in ψ . We multiply Eq. 6-7 by ψ , integrate it over volume V , and isolate k^2 to obtain

$$k^2 = - \frac{\int_V \psi \nabla^2 \psi dV}{\int_V \psi^2 dV} \quad (6-8)$$

We now prove that Eq. 6-8 is variational in the sense that a small linear error $\delta\psi$ in ψ from the correct value of ψ_0 creates only a quadratic error in k^2 , $\delta(k^2)$, which is on the order of

$$\frac{\delta(k^2)}{k^2} \approx - \frac{\int_V (\delta\psi)^2 dV}{\int_V \psi_0^2 dV} \quad (6-9)$$

Thus the fractional error in k^2 will be much smaller than the fractional error in ψ .

To obtain Eq. 6-9 we let $\psi = \psi_0 + \delta\psi$ and $k^2 = k_0^2 + \delta(k^2)$ in Eq. 6-8 and move the denominator to the left side:

$$\begin{aligned} [k_0^2 + \delta(k^2)] \int_V [\psi_0^2 + 2\psi_0 \delta\psi + (\delta\psi)^2] dV \\ = - \int_V [\psi_0 \nabla^2 \psi_0 + \psi_0 \nabla^2 (\delta\psi) + \delta\psi \nabla^2 \psi_0 + \delta\psi \nabla^2 (\delta\psi)] dV \end{aligned} \quad (6-10)$$

We shall consider $\delta\psi$ to be a small continuous functional change of ψ_0 throughout the volume, which may therefore be differentiated just like ψ_0 . Since ψ_0 satisfies the wave equation $\nabla^2 \psi_0 = -k_0^2 \psi_0$ by definition we may cancel the unvaried terms in Eq. 6-10. One of the $k_0^2 \psi_0 \delta\psi$ terms on

the left cancels with $-\nabla^2\psi_0 \delta\psi$ on the right. We obtain

$$\begin{aligned} k_0^2 \int_V [\psi_0 \delta\psi + (\delta\psi)^2] dV + \delta(k^2) \int_V [\psi_0^2 + 2\psi_0 \delta\psi + (\delta\psi)^2] dV \\ = - \int_V [\psi_0 \nabla^2(\delta\psi) + \delta\psi \nabla^2(\psi_0)] dV \quad (6-11) \end{aligned}$$

We can now eliminate the linear $\delta\psi$ terms by inverting the first term on the right side of Eq. 6-11. We use the identity

$$\int_V \nabla \cdot (\alpha \nabla \beta) dV = \int_V (\alpha \nabla^2 \beta + \nabla \alpha \cdot \nabla \beta) dV = \oint_S \alpha \nabla \beta \cdot \bar{n} dS \quad (6-12)$$

With α and β interchanged we obtain another identity which, when subtracted from Eq. 6-12, yields

$$\int_V (\alpha \nabla^2 \beta - \beta \nabla^2 \alpha) dV = \oint_S (\alpha \nabla \beta - \beta \nabla \alpha) \cdot \bar{n} dS \quad (6-13)$$

If we let $\alpha = \psi_0$ and $\beta = \delta\psi$ in this latter equation we obtain

$$\int_V [\psi_0 \nabla^2(\delta\psi) - \delta\psi \nabla^2 \psi_0] dV = \oint_S [\psi_0 \nabla(\delta\psi) - \delta\psi \nabla \psi_0] \cdot \bar{n} dS \quad (6-14)$$

which we can use to reexpress $\int_V \psi_0 \nabla^2(\delta\psi) dV$ in Eq. 6-11.

But what of the surface integral in Eq. 6-14? If ψ_0 obeys the boundary condition $\psi_0 \equiv 0$ on S , its gradient is generally not zero on S and the second surface term in Eq. 6-14 is not necessarily zero. The variation $\delta\psi$ has, as yet, been unspecified on S . Similarly, if ψ_0 satisfies the boundary condition $\nabla \psi_0 \cdot \bar{n} \equiv 0$ on S , the first surface integral is not necessarily zero. We ignore mixed boundary conditions for ψ_0 . In order to simplify the variational argument we now *specify* $\delta\psi$ on S so that, whichever boundary condition ψ_0 satisfies, the surface integral in Eq. 6-14 is identically zero.²

With variation $\delta\psi$ so specified on S we can then write

$$- \int_V \psi_0 \nabla^2(\delta\psi) dV$$

in Eq. 6-11 as

$$- \int_V \delta\psi \nabla^2 \psi_0 dV$$

which is

$$+ k_0^2 \int_V \delta\psi \psi_0 dV$$

² Nonzero surface integrals demand appropriate surface integrals in the variational expressions. We shall encounter them presently in connection with helices and cavity chains.

from Eq. 6-7. Then this term cancels the first term on the left side of Eq. 6-11 and we obtain for $\delta(k^2)$

$$\delta(k^2) \int_V [\psi_0^2 + 2\psi_0 \delta\psi + (\delta\psi)^2] dV = -k_0^2 \int_V (\delta\psi)^2 dV - \int_V \delta\psi \nabla^2(\delta\psi) dV \quad (6-15)$$

If we have $\nabla^2(\delta\psi) \approx -k_0^2 \delta\psi$ and also $\delta\psi \ll \psi_0$ we can verify Eq. 6-9 and show that the relative error in k^2 is much smaller than the relative error in ψ .

This somewhat ponderous proof of the variational nature of Eq. 6-8 shows that k^2 is stationary at k_0^2 for small deviations of ψ about ψ_0 . We can formulate the variational property alternatively as follows: The condition of stationary k^2 will *constrain* ψ to satisfy the wave equation for eigenvalue k^2 with neglect of the quadratic $(\delta\psi)^2$ terms. As we improve upon the function ψ the variational procedure will determine a value of k^2 closer to k_0^2 and a ψ closer to ψ_0 throughout volume V . The next example will make this clear. To verify that Eq. 6-8 has this variational property we set a variation of k^2 , $\delta(k^2)$, equal to zero with respect to any variation in the parameters of the *trial function* ψ :

$$\delta(k^2) = 0 = -\delta \frac{\int_V \psi \nabla^2 \psi dV}{\int_V \psi^2 dV} \quad (6-16)$$

By means of the convenient formula

$$\delta \left[\frac{\text{numerator}}{\text{denominator}} \right] = (\text{denom})\delta(\text{num}) - (\text{num})\delta(\text{denom}) = 0 \quad (6-17)$$

or

$$\delta(\text{num}) - \left[\frac{\text{num}}{\text{denom}} \right] \delta(\text{denom}) = 0$$

we can write Eq. 6-16, with neglect of the quadratic $\delta\psi \nabla^2(\delta\psi)$ term, as

$$0 = - \int_V [\psi \nabla^2(\delta\psi) + \delta\psi \nabla^2 \psi] dV - k^2 \int_V 2\psi \delta\psi dV \quad (6-18)$$

When we invert the first integral by means of Eq. 6-14 and again assume that $\delta\psi$ is defined to make the surface integral vanish completely, we obtain

$$0 = - \int_V 2\delta\psi \nabla^2 \psi dV - k^2 \int_V 2\psi \delta\psi dV \quad (6-19)$$

or

$$\int_V (\nabla^2 \psi + k^2 \psi) \delta\psi dV = 0 \quad \text{within } V$$

This last equation, true for *arbitrary continuous* variation $\delta\psi$ within V , shows that ψ must satisfy the wave equation—at least on an integral basis. As the number of variational parameters in ψ approaches infinity ψ tends to satisfy the wave equation at every point within V . This completes the proof of the variational nature of Eq. 6-8 by the alternative formulation.

If we have made a good choice in the form of the trial function for ψ so that it can represent ψ_0 reasonably well within V , Eq. 6-18 will determine a number of equations for the variational parameters in ψ which, when solved, will yield a value of k^2 quite close to k_0^2 . As we insert more parameters into ψ and vary them, the number of equations increases and one of the k^2 eigenvalues approaches k_0^2 and ψ more nearly satisfies the wave equation at every point within V . Unfortunately we cannot give general rules for this rate of convergence of k to k_0 .

We can, however, show that k^2 converges to k_0^2 from above, where $k_0^2 > 0$. To show that $k_0^2 > 0$ we need only identify α and β in Eq. 6-12 with ψ_0 so that

$$k_0^2 = \frac{+\int_V (\nabla \psi_0)^2 dV}{\int_V \psi_0^2 dV} > 0$$

In order to demonstrate the upper-bound nature of Eq. 6-8 for k_0^2 we introduce the set of eigenfunctions defined as

$$\begin{aligned} \nabla^2 \psi_n + k_n^2 \psi_n &= 0 & \text{in } V \\ \psi_n \text{ or } \nabla \psi_n \cdot \vec{n} &= 0 & \text{on } S \text{ bounding } V \end{aligned} \quad (6-20)$$

The first of these is ψ_0 , which belongs to the k_0^2 being sought. We do not have to know these functions explicitly but only that they exist. Suppose, now, we have determined a trial function ψ with some error in it:

$$\psi = \psi_0 + \sum_{n=1}^{\infty} a_n \psi_n \quad (6-21)$$

in which we need not know the value of each a_n explicitly. Then we could estimate k_0^2 by Eq. 6-8 as

$$k^2(\text{estimate of } k_0^2) = - \frac{\int_V \left(\psi_0 + \sum_{n=1}^{\infty} a_n \psi_n \right) \nabla^2 \left(\psi_0 + \sum_n a_n \psi_n \right) dV}{\int_V \left(\psi_0 + \sum_n a_n \psi_n \right)^2 dV} \quad (6-22)$$

To simplify this we can invoke the orthogonality property of the ψ_n 's

based on Eq. 6-13, with $\alpha = \psi_m$ and $\beta = \psi_n^3$

$$\int_V \psi_n \psi_m dV = \begin{cases} 0 & \text{if } k_n \neq k_m, \\ \neq 0 & \text{if } k_n = k_m \end{cases} \quad (6-23)$$

With Eq. 6-23 we can rearrange Eq. 6-22 to read

$$\begin{aligned} (k_0^2 + \Delta k^2) \int_V (\psi_0^2 + \sum_{n=1}^{\infty} a_n^2 \psi_n^2) dV &= - \int_V (\psi_0 \nabla^2 \psi_0 + \sum_{n=1}^{\infty} a_n^2 \psi_n \nabla^2 \psi_n) dV \\ &= + \int_V (k_0^2 \psi_0^2 + \sum_{n=1}^{\infty} k_n^2 a_n^2 \psi_n^2) dV \end{aligned} \quad (6-24)$$

The zero-subscript terms cancel and give us

$$k_0^2 \sum_{n=1}^{\infty} a_n^2 \int_V \psi_n^2 dV + \Delta k^2 (+ \text{number}) = \sum_{n=1}^{\infty} a_n^2 k_n^2 \int_V \psi_n^2 dV \quad (6-25)$$

and this rearranges into

$$\Delta k^2 (+ \text{number}) = \sum_{n=1}^{\infty} a_n^2 (k_n^2 - k_0^2) \int_V \psi_n^2 dV > 0 \quad (6-25')$$

because $k_0 < k_1, k_2, \dots$, by definition.

In this way we have shown that the variational calculation of the first eigenvalue k_0^2 by Eq. 6-8 will always be high.

C. The First Vibration of a Circular Membrane

As a specific and convincing example of the value of the variational method we consider a vibrating circular membrane of radius a and vertical displacement $\psi(r)$, clamped to zero at $r = a$. We wish to determine the eigenvalue k_0^2 of the first circularly symmetric mode with zero slope of ψ at $r = 0$.⁴ Equation 6-8 can be written after integration by parts as

$$(ka)^2 = \frac{\int_0^1 \left[\frac{d\psi(x)}{dx} \right]^2 x dx}{\int_0^1 \psi(x)^2 x dx}, \quad x = \frac{r}{a}, \quad (6-26)$$

and the exact value of $(k_0 a)^2 = 2.4048^2 = 5.78316$, corresponding to the first root of Bessel function $J_0(ka)$.

³ If several of the ψ_n 's have the same $k_n \neq k_0$ we form linear combinations of them which are mutually orthogonal and relabel k_n as $k_{n1}, k_{n2}, \dots$

⁴ See Ref. [2, p. 1116].

First we try $\psi = A(1 - x^2)$, which has the correct zero slope at $x = 0$ and the correct value of zero at $x = 1$. When we substitute this into Eq. 6-26, the A^2 coefficients cancel and there are no parameters to vary. A direct evaluation yields $(ka)^2 = 6.00$, greater than the correct 5.78+.

Next we try $\psi = A(1 - x^2) + B(1 - x^2)^2$. When we substitute this into Eq. 6-26 and set $\partial(k^2)/\partial A$ and $\partial(k^2)/\partial B$ separately equal to zero [which amounts to setting variation $\delta(k^2) = 0$ with respect to both δA and δB], we obtain two homogeneous equations for A and B . The resultant 2×2 determinantal equation for $(ka)^2$ is

$$\begin{vmatrix} \frac{1}{6}(ka)^2 - 1 & \frac{1}{8}(ka)^2 - \frac{3}{8} \\ \frac{1}{8}(ka)^2 - \frac{3}{8} & \frac{1}{10}(ka)^2 - \frac{3}{8} \end{vmatrix} = 0 \quad (6-27)$$

The smaller root of this determinant is $(ka)^2 = 5.7837$, amazingly close to the correct 5.78316 and slightly high, as we proved it must be in the preceding subsection.

This value of $ka \cong k_0 a$ determines B in terms of A from either of the homogeneous equations. A plot of $\psi(x)$ is almost identical to the correct $\psi_0(x)$ provided they are both normalized to the same value at $x = 0$.

D. Variational Expression for $j\omega$ of the Periodically Loaded Transmission Line

In this subsection we introduce variational expressions in the forms which conveniently generalize to helix slow wave structures and cavity chains. Unfortunately the mathematical formulation of these expressions tends to look more involved than the ideas behind the technique. We want to reassure the reader that once he becomes familiar with the notation he will be able to understand the variational method as applied to all sorts of slow wave structures.

Instead of the one wave equation we now have the two transmission line equations

$$\frac{dV(z)}{dz} = -j\omega L(z)I(z) \quad (6-28a)$$

$$\frac{dI(z)}{dz} = -j\omega C(z)V(z) \quad (6-28b)$$

in which $C(z)$ and $L(z)$ represent the arbitrary periodic distributions of capacitance and inductance per unit length. One or both of these may be regarded as Dirac delta functions on lines with periodic lumped loading (Fig. 6-1, for example). Let us try to formulate a variational expression

for $j\omega$ by extension of the procedure leading to Eq. 6-8: we multiply Eq. 6-28a by $I(z)$ and Eq. 6-28b by $V(z)$, integrate them both over the length L of a periodic cell, and either add or subtract the two equations—which ever will prove to be correct. We obtain

$$-j\omega = \frac{\int_L (V'I \pm VI') dz}{\int_L (LI^2 \pm CV^2) dz}, \quad ' = \frac{\partial}{\partial z} \quad (6-29)$$

in which we take both upper signs of $\pm$ or both lower ones. If this

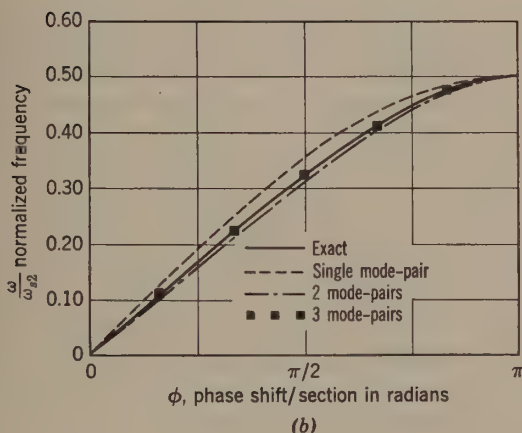
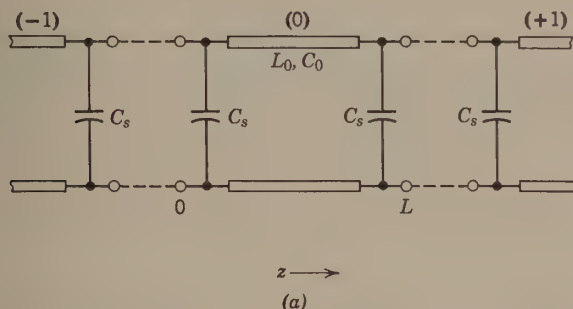


Fig. 6-1. The first passband of a periodically lumped-loaded transmission line (from Goblick and Bevensee [3]). (a) A unit cell consists of a section of transmission line with inductance L_0 and capacitance C_0 per unit length and shunted on both sides by capacitors C_s , as shown; (b) exact and calculated dispersion curves for the first passband. ω_{02} is the resonant frequency at which voltage and current vary as shown in Fig. 1-5d.

expression is variational we ought to find that, by forcing a variation $\delta(j\omega)$ equal to zero with respect to variation $\delta I(z)$, say, we can constrain one of the transmission line Eqs. 6-28 to be satisfied, with neglect of the $(\delta I)^2$ terms. Here $\delta I(z)$ is a small continuous function within the cell.

When we set $\delta(j\omega) = 0$ and use the rule 6-17 we obtain

$$\int_L [V' \delta I \pm V(\delta I)'] dz + j\omega 2 \int_L LI \delta I dz = 0 \quad (6-30)$$

which looks somewhat like

$$\int_L (V' + j\omega LI) \delta I dz = 0 \quad (6-31)$$

If we could obtain this latter form for arbitrary δI , $V(z)$ and $I(z)$ would satisfy the first transmission line equation within the cell of length L and we would have completed half the proof of the variational property. To convert Eq. 6-30 into the form of Eq. 6-31 we must invert the second integral as

$$\int_L V(\delta I)' dz = - \int_L V' \delta I dz + V(z) \delta I(z) \Big|_0^L \quad (6-32)$$

Now if we *define* $\delta I(0) = \delta I(L) = 0$ at the edge of the cell where V is not generally zero, the last term of Eq. 6-32 is zero. Then Eq. 6-32 with Eq. 6-30 shows that Eq. 6-31 is indeed satisfied, provided we choose the *lower* sign of $\pm$.

Having chosen this sign we must now verify the remainder of the variational property, namely that $\delta(j\omega) = 0$ with respect to arbitrary $\delta V(z)$ will constrain Eq. 6-28b to be satisfied. By the preceding reasoning we find that such is the case, provided we *define* variations $\delta V(0) = \delta V(L) = 0$, since $I(0)$ and $I(L)$ are not generally zero there.

In order to make Eq. 6-29 (with the lower sign of $\pm$) variational for $j\omega$ we have had to place rather severe restrictions on the variations of the trial functions for $V(z)$ and $I(z)$ at the cell boundaries. We can remove such restrictions and still avoid the bothersome last term in Eq. 6-32 by utilizing the periodic nature of the fields. If this last term were $V^* \delta I$, where the asterisk denotes an ordinary complex conjugate, it would be identically zero provided all trial functions phase shift by $\epsilon^{-j\varphi}$ from one cell boundary to the next one. Obviously, then, we will vary the trial functions so as to maintain this $\epsilon^{-j\varphi}$ phase shift. Thus, $V^*(z)$ phase shifts by $\epsilon^{+j\varphi}$, and $\delta I(z)$ by $\epsilon^{-j\varphi}$, and the two boundary terms of $V^* \delta I$ cancel each other.

In order to have unrestricted (but periodic) variations of trial fields $V(z)$ and $I(z)$ on the cell boundaries, let us form a new variational expression for $j\omega$ by writing Eq. 6-28 for one set of trial functions, denoted by

a subscript plus, which phase shift by $\epsilon^{-j\varphi}$ across the periodic cell of length L :

$$\frac{dV_+(z)}{dz} = -j\omega L(z)I_+(z) \quad (6-33a)$$

$$\frac{dI_+(z)}{dz} = -j\omega C(z)V_+(z) \quad (6-33b)$$

Here φ is regarded as known and we want a variational expression for the ω values associated with it. We also define, for mathematical reasons, an independent pair of trial functions $V_-(z)$ and $I_-(z)$, denoted by a subscript minus, which also phase shift by $\epsilon^{-j\varphi}$ across the cell. These functions are just like V_+ and I_+ except that they have independent amplitudes and perhaps other variational parameters. We assume continuity of $V_+(z), \dots, I_-(z)$ across the cell boundaries, although there may be rapid changes of phase across these boundaries. To form the new variational expression for ω we multiply Eq. 6-33a by I_-^* and Eq. 6-33b by V_-^* , integrate them both over the cell length L , add them (since subtraction, it turns out, does not work), and then isolate $j\omega$. We obtain

$$-j\omega = \frac{\int_L (V_+'I_-^* + I_+'V_-^*) dz}{\int_L (LI_+I_-^* + CV_+V_-^*) dz}, \quad ' = \frac{\partial}{\partial z} \quad (6-34)$$

To prove that Eq. 6-34 is indeed variational we set $\delta(j\omega) = 0$ first with respect to $\delta I_-(z)$. By the rule of Eq. 6-17 we obtain

$$\int_L V_+' \delta I_-^* dz + j\omega \int_L L(z)I_+ \delta I_-^* dz = \int_L (V_+' + j\omega LI_+) \delta I_-^* dz = 0 \quad (6-35)$$

so that, with δI_- arbitrary within the cell, the first transmission line equation 6-33a is constrained on an integral basis. If we set $\delta(j\omega) = 0$ with respect to $\delta V_-(z)$ we also constrain Eq. 6-33b.

Equation 6-34 must also be variational for the fields $V_-(z)$ and $I_-(z)$; to see if it is we set $\delta(j\omega) = 0$ for a variation δI_+ and obtain

$$\int_L (\delta I_+)' V_-^* dz + j\omega \int_L LI_-^* \delta I_+ dz = 0 \quad (6-36)$$

By Eq. 6-32 we integrate the left side by parts and obtain

$$\int_L -V_-'^* \delta I_+ dz + V_-^* \delta I_+|_0^L + j\omega \int_0^L LI_-^* \delta I_+ dz = 0 \quad (6-37)$$

However, V_-^* phase shifts by $\epsilon^{+j\varphi}$ and δI_+ by $\epsilon^{-j\varphi}$ from $z = 0$ to $z = L$ so the two boundary contributions cancel and Eq. 6-37 becomes just

$$\int_L (V_-'^* - j\omega LI_-^*) \delta I_+ dz = 0 \quad (6-38)$$

For arbitrary $\delta I_+(z)$ within the cell $V_-(z)$ and $I_-(z)$ do indeed satisfy Eq. 6-33a on an integral basis. By similar reasoning we can show that if $\delta(j\omega) = 0$ relative to all $\delta V_+(z)$ then V_- and I_- must also obey Eq. 6-33b.

Therefore Eq. 6-34 is variational for both the plus and minus trial function fields in the sense specified. We may also verify the alternative statement of the variational property: that if the fields $V_+, \dots, I_-$ are varied linearly about their correct values at $\omega = \omega_0$ (this is not necessarily the lowest frequency for a given phase shift φ), then the variation of ω , $\delta\omega$, will be quadratic in $(\delta V_+)^2, \dots, (\delta I_-)^2$ and therefore much smaller.

E. Variational Expression for ω^2 of the Periodically Loaded Transmission Line

It is not necessary to write a variational expression for ω in terms of both voltage and current in any periodic cell. We can work in terms of voltage alone, say, by dividing Eq. 6-28a by $L(z)$, differentiating it relative to z , and introducing Eq. 6-28b, to obtain

$$\frac{d}{dz} \left[L(z)^{-1} \frac{dV(z)}{dz} \right] = -\omega^2 C(z) V(z) \quad (6-39)$$

To form a variational expression for ω^2 we regard trial voltage $V_+(z)$ as satisfying Eq. 6-39, multiply both sides by independent trial voltage $V_-(z)^*$, integrate over the cell length L , and isolate ω^2 :

$$\omega^2 = - \int_L V_-^* \frac{d}{dz} \left[\frac{1}{L(z)} \frac{dV_+}{dz} \right] dz \bigg/ \int_L C(z) V_+ V_-^* dz \quad (6-40)$$

This equation is variational because if we set $\delta(\omega^2) = 0$ relative to $\delta V_-^*(z)$ we constrain $V_+(z)$ to satisfy Eq. 6-39 within the cell. And if we set $\delta(\omega^2) = 0$ relative to δV_+ and integrate the numerator by parts in order to put it into proper form, the boundary terms cancel as before and V_- is constrained to obey the same equation 6-39.

In practice, we would vary only the parameters of one set of trial functions in Eqs. 6-34 and 6-40, usually those of the minus set. The statement that $\delta(j\omega)$ or $\delta(\omega^2)$ is zero with respect to these variations leads to a set of homogeneous equations for the trial field amplitudes and other variational parameters (if there are any) of the plus field set. Solution of

these equations gives the allowed values of $j\omega$ or ω^2 that go with a given phase shift φ . Each value of $j\omega$ or ω^2 determines the relative amplitudes of the plus trial field set of functions which compose a wave at this frequency and for this phase shift φ . The fields in some of these waves are more accurate than in others; for example, if we employ the first N short-circuit resonant modes of the cell in Fig. 4-1 to represent V_- and I_- (see Section 5-3C) and the first N open-circuit modes to represent V_+ and I_+ , then the lowest-frequency wave will be the most accurate and the N th wave with larger components of neglected $N + 1, N + 2, \dots$ modes will be the least accurate.

The thoughtful reader will have noticed our implicit assumption that, as the number of variational parameters in the trial functions goes to infinity, the values of $j\omega$ or ω^2 determined in the variational method will all converge to the correct ones. This is certainly reasonable if we use more and more open- and short-circuit resonant modes of a structure with periodic abrupt discontinuities, or an ever-increasing number of waveguide modes for helix-type structures. The procedure followed is known as the Rayleigh-Ritz technique [2, p. 1116], which converges to the correct series of eigenvalues of $j\omega$ or ω^2 for a given phase shift φ provided the two sets of resonant cell modes and the set of waveguide modes are complete for the fields within a cell. For a cell of the periodically loaded transmission line, both the open-circuit and the short-circuit mode sets as defined in Section 1-6 are complete for voltage and current. But we can prove completeness of the set of resonant cavity modes and the waveguide modes only for particularly simple cavity and waveguide configurations.

In general, we can say nothing about the rate of convergence of a calculated value of $j\omega$ or ω^2 to the true value. However, we can show that the convergence is nonuniform, in general, by referring to the periodically lumped-loaded transmission line shown in Fig. 6-1. The first passband has been calculated [3] by employing the open-circuit and short-circuit resonant cell modes shown in Fig. 1-5 as trial functions either for $V_{\pm}$ and $I_{\pm}$ in Eq. 6-34 or for $V_{\pm}$ in Eq. 6-40. The minus fields are expanded in short-circuit modes and the plus fields in open-circuit modes.⁵ The single mode-pair estimate of this passband, as calculated with the modes of

⁵ Formally it does not matter whether the plus trial fields are expanded with the open- or the short-circuit modes. However, the zero-frequency short-circuit mode of Fig. 1-5b has no voltage pattern so we want to avoid a dispersion relation involving $T_{11} \sim \int E_1 C(z) e_1 dz$, E_1 being identically zero. With the plus fields expanded in open-circuit modes the relevant volume integral is $U_{11} \sim \int H_1 h_1 dz$, which is finite. For the same reason Eq. 6-40 is not useful but the analogous variational expression in terms of currents is.

Fig. 1-5*b,c* which resonate at the cutoff frequencies of that passband, is high. The two mode-pair estimate based on all the modes of Fig. 1-5 for the first two passbands is low. And the three mode-pair estimate is essentially the correct one. We conclude by analogy that our variational procedure will not always determine upper or lower bounds for $\omega(\varphi)$ in the first or any other passband of a symmetrical cavity chain.

6-3. A Variational Analysis for Wire and Tape Helices

A. The Variational Expression for $j\omega$

When we look back at the conventional analyses of tape helices reviewed in Chapter 2 we notice that they depend upon an assumed electric current distribution on the tape. However, the solution to Maxwell's equations for the helix is a pure field problem which should not demand any a priori assumptions about this distribution. We should be able to represent the fields of a wave by a suitable linear combination of traveling modes and compute their relative amplitudes and their common propagation constant β_0 *without specifying any current on the tape*. After the field is determined the current is specified by $\vec{n} \times \vec{H}$ on the tape or ring surfaces of the helix, but it is really of no interest whatsoever. Further evidence of the insensitivity of the field solutions to the actual current distribution is furnished by Sensiper's work. He has found that the characteristics of the $n = 0$ mode employed in traveling wave tube work little depend upon whether the current is assumed to be of constant magnitude across the tape or whether it is assumed to go to infinity as the inverse square root of the distance from the edge.

Accordingly, let us formulate a variational expression for $j\omega$ from Maxwell's equations

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \quad (6-41a)$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E} \quad (6-41b)$$

for the wire or tape helix shown in Fig. 6-2, without mentioning the current on the tape. Instead, we introduce a surface integral on the surface $r = a$ which will tend to constrain continuity of the fields on the open surface and $\vec{E}_{\text{tan}}$ to be zero on the metal surface. We shall develop the expression in such a way as to be valid for a wire helix of arbitrary cross section wound on a cylindrical surface of arbitrary cross section, such as an ellipse or rectangle. We assume for simplicity that the helix is not surrounded by an outer conductor; if one is present, we could

The braces around $j\omega$ will remind us that it is to be kept constant when we vary the parameters (amplitudes) in the trial fields $\hat{E}_-$, $\hat{H}_+$, and so on.

It is most convenient to define one set of the plus and minus trial fields throughout the internal portion of the slab, denoted by A_i for $r \leq a$, and another set of trial fields for the external or A_o region. Each set of fields is taken to be a linear combination of sheath helix modes as defined by Eqs. 2-1 and 2-2 within the internal region and as defined by Eqs. 2-3 and 2-4 within the external region. Each of these modes satisfies Maxwell's equations identically within either region A_i or A_o and so a linear combination of them will leave each bracketed expression identically zero in Eq. 6-42. However, we must retain these integrals for the proof of the variational properties, with the understanding they will be zero when we take variations of $\hat{E}_-^*$ and $\hat{H}_-^*$ within the slab.

In order to distinguish the trial fields within these two separate regions we denote the ones inside the A_i or internal region by lower-case letters: $\hat{e}_+$, $\hat{e}_-$, $\hat{h}_+$, and $\hat{h}_-$. Those within the external or A_o region are identified by capital letters: $\hat{E}_+$, $\hat{E}_-$, $\hat{H}_+$, and $\hat{H}_-$. At this stage of development, Eq. 6-42 looks as if it will tend to constrain $\hat{e}_+$ and $\hat{h}_+$ to satisfy Maxwell's equations within the A_i region and $\hat{E}_+$ and $\hat{H}_+$ to satisfy them within the A_o region, but we must also constrain continuity of the tangential field components across the boundary surface D at $r = a$ in Fig. 6-2:

$$\begin{aligned} (\hat{e}_+ - \hat{E}_+) \times \bar{i}_r &= 0 \\ (\hat{e}_- - \hat{E}_-) \times \bar{i}_r &= 0 \\ (\hat{h}_+ - \hat{H}_+) \times \bar{i}_r &= 0 \\ (\hat{h}_- - \hat{H}_-) \times \bar{i}_r &= 0 \end{aligned} \quad \text{on open surface } D \text{ at } r = a \quad (6-43)$$

We recall from previous discussion that the proper variational expression must imply the same properties of the plus fields as of the minus fields because they have equal physical significance. The remaining condition we must constrain is that of zero tangential electric field on the helix wire or tape:

$$\begin{aligned} \hat{e}_- \times \bar{n}' &= \hat{e}_+ \times \bar{n}' = 0 && \text{on inner wire surface,} \\ \hat{E}_- \times \bar{n}' &= \hat{E}_+ \times \bar{n}' = 0 && \text{on outer wire surface} \end{aligned} \quad (6-44)$$

Unit vector $\bar{n}'$ points into the wire. For the special case of a thin flat tape $\bar{n}' = \bar{i}_r$ on the inner tape surface and $\bar{n}' = -\bar{i}_r$ on the outer surface.

To modify Eq. 6-42 we must include integrals over the tape of the form $\int \hat{e}_+ \times \hat{h}_-^* \cdot \bar{n}' dS$ so that if $\delta\{j\omega\} = 0$ relative to $\delta\hat{h}_-^*$ on the tape surface $\hat{e}_+ \times \bar{n}'$ will, in principle, be zero there. We also need integrals of the form $\int \hat{e}_-^* \times (\hat{h}_+ - \hat{H}_+) \cdot \bar{i}_r dC$ on the discontinuity surface D so that

$(\hat{h}_+ - \hat{H}_+) \times \bar{i}_r$ will tend to be continuous for the same reason. A trial-and-error procedure shows that the appropriately modified Eq. 6-42 is

$$\begin{aligned}
 0 = & \int_{A_i} \hat{e}_-^* \cdot [\nabla \times \hat{h}_+ - \{j\omega\} \epsilon \hat{e}_+] dS - \int_{A_i} \hat{h}_-^* \cdot [\nabla \times \hat{e}_+ + \{j\omega\} \mu \hat{h}_+] dS \\
 & + \int_{A_o} \hat{E}_-^* \cdot [\nabla \times \hat{H}_+ - \{j\omega\} \epsilon \hat{E}_+] dS - \int_{A_o} \hat{H}_-^* \cdot [\nabla \times \hat{E}_+ \\
 & + \{j\omega\} \mu \hat{H}_+] dS + \int_{W_i} \hat{e}_+ \times \hat{h}_-^* \cdot \bar{n}' dC + \int_{W_o} \hat{E}_+ \times \hat{H}_-^* \cdot \bar{n}' dC \\
 & + \frac{1}{2} \int_D (\hat{e}_-^* + \hat{E}_-^*) \times (\hat{h}_+ - \hat{H}_+) \cdot \bar{i}_r dC \\
 & + \frac{1}{2} \int_D (\hat{e}_+ - \hat{E}_+) \times (\hat{h}_-^* + \hat{H}_-^*) \cdot \bar{i}_r dC \quad (6-45)
 \end{aligned}$$

We understand that these integrals extend over a cross-sectional slab one unit thick, in which $W_{i,o}$ refers to the inner or outer surface of the wire and D to the open surface at $r = a$ on Fig. 6-2.

We can easily prove that Eq. 6-45 is variational with respect to variations in the minus trial fields. For example, if we set $\delta\{j\omega\} = 0$ with respect to an arbitrary variation $\delta\hat{e}_-^*$ within region A_i we constrain the relation

$$0 = \int_{A_i} \delta\hat{e}_-^* \cdot (\nabla \times \hat{h}_+ - j\omega\epsilon\hat{e}_+) dS + \frac{1}{2} \int_D \delta\hat{e}_-^* \cdot (\hat{h}_+ - \hat{H}_+) \times \bar{i}_r dC \quad (6-46)$$

If this is true for arbitrary $\delta\hat{e}_-^*$ within region A_i and on surface D , the plus fields must satisfy Maxwell's Eq. 6-41b and have the continuity of Eq. 6-43c. If we now set $\delta\{j\omega\} = 0$ for arbitrary $\delta\hat{h}_-^*$ within region A_i and on D and the wire surfaces we constrain the plus fields to obey Eq. 6-41a, 6-43a, and 6-44. Similar remarks apply for variations $\delta\hat{E}_-$ and $\delta\hat{H}_-$ taken within the external A_o region and on the external wire surface.

To prove the variational properties of Eq. 6-45 with respect to variations in the plus trial fields we must integrate the $\nabla \times \delta\hat{h}_+$, $\nabla \times \delta\hat{e}_+$, $\nabla \times \delta\hat{H}_+$, and $\nabla \times \delta\hat{E}_+$ terms by parts, whereupon the $\frac{1}{2}$ factors appended to the $\int_D$ integrals guarantee the variational properties. For example, we set $\delta\{j\omega\} = 0$ for $\delta\hat{h}_+$. We first have

$$\begin{aligned}
 0 = & \int_{A_i} \hat{e}_-^* \cdot \nabla \times \delta\hat{h}_+ dS - \int_{A_i} \hat{h}_-^* \cdot j\omega\mu \delta\hat{h}_+ dS \\
 & + \frac{1}{2} \int_D (\hat{e}_-^* + \hat{E}_-^*) \times \delta\hat{h}_+ \cdot \bar{i}_r dC \quad (6-47)
 \end{aligned}$$

We integrate the first term by parts throughout the slab one unit thick up to the wire and discontinuity surface D :

$$\begin{aligned} \int_{A_i} \hat{e}_-^* \cdot \nabla \times \delta \hat{h}_+ dS &= \int_{A_i} \delta \hat{h}_+ \cdot \nabla \times \hat{e}_-^* dS + \oint_{\text{wire}, D} \delta \hat{h}_+ \times \hat{e}_-^* \cdot \bar{i}_r dC \\ &+ \frac{\partial}{\partial z} \int_{A_i} \delta \hat{h}_+ \times \hat{e}_-^* \cdot \bar{i}_z dS \quad (6-48) \end{aligned}$$

The very last integral is zero by our specification that the plus and minus trial fields both propagate as $e^{-j\beta_0 z}$ as they rotate together along the wire. The contour integral can be split into two portions. Then Eq. 6-47 can be converted into

$$\begin{aligned} 0 &= \int_{A_i} \delta \hat{h}_+ \cdot (\nabla \times \hat{e}_-^* - j\omega\mu\hat{h}_-^*) dS + \int_{W-} \delta \hat{h}_+ \cdot \hat{e}_-^* \times \bar{i}_r dC \\ &+ \int_D \delta \hat{h}_+ \times \hat{e}_-^* \cdot \bar{i}_r dC + \frac{1}{2} \int_D (\hat{e}_-^* + \hat{E}_-^*) \times \delta \hat{h}_+ \cdot \bar{i}_r dC \quad (6-49a) \end{aligned}$$

By exchanging the order of cross-multiplication in the first integral over D we can combine it with the second to obtain

$$\frac{1}{2} \int_D (-\hat{e}_-^* + \hat{E}_-^*) \times \delta \hat{h}_+ \cdot \bar{i}_r dC = \frac{1}{2} \int_D (\hat{e}_-^* - \hat{E}_-^*) \times \bar{i}_r \cdot \delta \hat{h}_+ dC \quad (6-49b)$$

Therefore, if Eq. 6-49 is true for arbitrary $\delta \hat{h}_+$ within region A_i and on the wire and surface D , the $\hat{e}_-$ and $\hat{h}_-$ fields are constrained to satisfy Eqs. 6-41a, 6-44a, and 6-43b. We can proceed similarly to prove with respect to variations $\delta \hat{e}_+$, $\delta \hat{H}_+$, and $\delta \hat{E}_+$ that the minus fields tend to satisfy Eq. 6-41b and the other boundary conditions on the wire and open surface D .

We have now demonstrated the variational nature of Eq. 6-45 and will proceed to use it to derive the dispersion characteristics of various helices.

B. Verification on the Sheath Helix

We first verify Eq. 6-45 appropriately modified to provide for the special sheath boundary conditions at $r = a$,

$$\begin{aligned} \hat{e}_\perp - \hat{E}_\perp &= 0 \\ \hat{e}_\parallel - \hat{E}_\parallel &= 0 \\ \hat{h}_\parallel - \hat{H}_\parallel &= 0 \end{aligned} \quad (2-5)$$

Figure 2-1 identifies the perpendicular and parallel field components on

the sheath surface. We may verify that, in order to constrain these boundary conditions, we must replace the last four integrals in Eq. 6-45 by

$$2\pi a[\hat{e}_{\parallel+} \times \hat{h}_{\perp-}^* - \hat{E}_{\parallel+} \times \hat{H}_{\perp-}^* + \frac{1}{2}(\hat{e}_{\perp-}^* + \hat{E}_{\perp-}^*) \times (\hat{h}_{\parallel+} - \hat{H}_{\parallel+}) \\ + \frac{1}{2}(\hat{e}_{\perp+} - \hat{E}_{\perp+}) \times (\hat{h}_{\parallel-}^* + \hat{H}_{\parallel-}^*)] \cdot \bar{i}_r \quad (6-50)$$

In the sheath helix the two n th modes representing the TM and TE portions of the field do not couple to any of the others (see Section 2-2), so we can represent $\hat{e}_+$ and $\hat{h}_+$ throughout the internal region A_i by the fields of the n th mode of Eqs. 2-1 and 2-2 with amplitudes $(B_1)_n$ and $(B_3)_n$. We represent fields $\hat{e}_-$ and $\hat{h}_-$ the same way but with different (independent) amplitudes—call them $(C_1)_n$ and $(C_3)_n$. We represent the external $\hat{E}_+$ and $\hat{H}_+$ fields by the n th mode of Eqs. 2-3 and 2-4 involving amplitudes $(B_2)_n$ and $(B_4)_n$. $\hat{E}_-$ and $\hat{H}_-$ are similarly represented, but with amplitudes $(C_2)_n$ and $(C_4)_n$. These modes satisfy Maxwell's equations identically within their respective regions, so the area integrals of Eq. 6-45 (as modified by the terms 6-50) vanish.

We then take variations $\delta(C_1)_n^*, \dots, \delta(C_4)_n^*$ in Eq. 6-50 alone set equal to zero, with ω regarded as fixed. We obtain four homogeneous equations, each in terms of $(B_1)_n, \dots, (B_4)_n$. It would be unnecessarily tedious to obtain the simple sheath helix dispersion relation from the determinant of this 4×4 matrix of coefficients. Instead we can verify that Sensiper's Eq. 2-11 for the B 's together with the correct dispersion Eq. 2-6 *do indeed* satisfy each of our equations in terms of the B 's.

Variational Eq. 6-45 has just passed the first test of a useful method for describing dispersion of general helices.

C. Tape Helix Dispersion for the Waves with Predominant $n \neq 0$ Mode

Refer to the tape helix ω - β diagram of Fig. 2-4, page 45, part of which appears in Fig. 6-3. We wish to calculate the shape of the ω - β_n curve of the n th mode near the first forbidden region, in the range $0 \leq \beta_n a \leq 0.5 \cot \psi$. Figure 6-3 shows that, as $\beta_{+1}a$, for example, approaches zero in this range, the $ka - \beta_{+1}a$ curve rises, reaches a plateau corresponding to a resonance of the structure, and then approaches zero along the $\beta a = ka$ line as an asymptote.

According to the discussion of Section 2-3 we are really investigating the behavior of a wave with two $n = 0$ mode components (representing the TM and TE portions of the field) of propagation constant β_0 , two $n = +1$ mode components of propagation constant $\beta_{+1} = \beta_0 + 2\pi/p$,

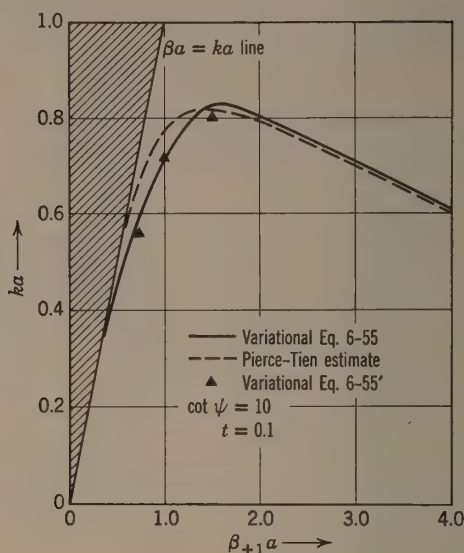


Fig. 6-3. Dispersion of the tape helix wave with predominant $n = +1$ mode component. $\cot \psi = 10$ and $t = 0.1$.

and so on, all of which are coupled together on the tape and discontinuity surface D . Now it so happens that the predominant fields in such a wave are given by that mode of radial propagation constant nearest to zero. This is reasonable from the fact that the fields decay less rapidly away from the helix wire if the radial propagation constant is smaller. Hence the wave we are investigating has predominant n th mode, such that β_n is closest to k and radial propagation $\gamma_n = (\beta_n^2 - k^2)^{1/2}$ is smallest. Accordingly, let us represent the internal trial fields for the variational Eq. 6-45 by the n th mode of Eqs. 2-1 and 2-2:

$$\left. \begin{aligned}
 \hat{e}_{z+} &= (B_3)_n I_n(\gamma_n r) \\
 \hat{e}_{r+} &= (B_3)_n (j\beta_n/\gamma_n) I_n' - (B_1)_n (n\omega\mu/\gamma_n^2 r) I_n \\
 \hat{e}_{\theta+} &= -(B_3)_n (n\beta_n/\gamma_n^2 r) I_n - (B_1)_n (j\omega\mu/\gamma_n) I_n' \\
 \hat{h}_{z+} &= (B_1)_n I_n \\
 \hat{h}_{r+} &= (B_3)_n (n\omega\varepsilon/\gamma_n^2 r) I_n + (B_1)_n (j\beta_n/\gamma_n) I_n' \\
 \hat{h}_{\theta+} &= (B_3)_n (j\omega\varepsilon/\gamma_n) I_n' - (B_1)_n (n\beta_n/\gamma_n^2 r) I_n
 \end{aligned} \right\} e^{jn\theta - j\beta_n z} \quad (6-51)$$

($r \leq a$)

Each mode satisfies Maxwell's equations exactly. The prime refers to differentiation with respect to argument $\gamma_n r$ and the variational amplitudes are $(B_1)_n$ and $(B_3)_n$, with β_n to be determined as a function of ω or

k . The minus trial fields are represented the same way but with independent amplitudes $(C_1)_n$ and $(C_3)_n$.

Similarly we represent the plus trial fields over the external region of Fig. 6-2 with the n th mode of Eqs. 2-3 and 2-4:

$$(r \geq a) \left. \begin{aligned} E_{z+} &= (B_4)_n K_n(\gamma_n r) \\ E_{r+} &= (B_4)_n (j\beta_n/\gamma_n) K_n' - (B_2)_n (n\omega\mu/\gamma_n^2 r) K_n \\ E_{\theta+} &= -(B_4)_n (n\beta_n/\gamma_n^2 r) K_n - (B_2)_n (j\omega\mu/\gamma_n) K_n' \\ H_{z+} &= (B_2)_n K_n \\ H_{r+} &= (B_4)_n (n\omega\varepsilon/\gamma_n^2 r) K_n + (B_2)_n (j\beta_n/\gamma_n) K_n' \\ H_{\theta+} &= (B_4)_n (j\omega\varepsilon/\gamma_n) K_n' - (B_2)_n (n\beta_n/\gamma_n^2 r) K_n \end{aligned} \right\} e^{jn\theta - j\beta_n z} \quad (6-52)$$

The minus trial fields are the same but for new amplitudes $(C_2)_n$ and $(C_4)_n$.

We now consider for definiteness a helix with $\cot \psi = 2\pi a/p = 10$ and specify a tape width w such that the ratio of the cross-sectional tape width $w \csc \psi$, to the open cross section is 0.1. This means in Fig. 6-2 that our tape width parameter t is

$$t \triangleq \frac{w \csc \psi}{2\pi a - w \csc \psi} = 0.1 \quad (6-53)$$

This corresponds to Sensiper's x parameter of value

$$x = \frac{\pi w}{p} = \frac{\pi w \cot \psi}{2\pi a} \cong \pi \left(\frac{w \csc \psi}{2\pi a} \right) = 0.286$$

If we now substitute these trial fields into Eq. 6-45 with the area integrals identically zero, we have an expression for $\{j\omega\}$ in terms of four amplitudes of the plus field and four amplitudes of the minus field. Rather than vary all four of the C amplitudes let us reduce the number of amplitudes to two in each trial field by specifying these explicit continuities at $r = a$:

$$\hat{e}_\perp(\gamma_n a) - \hat{E}_\perp(\gamma_n a) = 0 \quad (6-54a)$$

$$h_\parallel(\gamma_n a) - \hat{H}_\parallel(\gamma_n a) = 0 \quad (6-54b)$$

The first continuity is true for the complete field even on the tape where $\hat{e}_\perp = \hat{E}_\perp = 0$. The second continuity is the assumption of very small currents flowing across the tape, which works so well in the conventional analyses. We need not write this latter continuity if we choose, but we *must* write either Eq. 6-54a or a similar continuity for $\hat{e}_\parallel$ explicitly if we are to obtain field expressions which converge rapidly to the true ones. It is easy to see why this is so. If we examine the trial field expressions of Eqs. 6-51, 6-52, and the variational Eq. 6-45 we find that nowhere do they introduce the parameter $\cot \psi$ explicitly, since we need not relate

β_n to $\beta_0 = \beta_n - n \cot \psi/a$ in this n th-mode representation of the fields. However, Eq. 6-54a will introduce $\cot \psi$ in the process of establishing one of the important continuities across the discontinuity surface D at $r = a$.

When we express the trial field amplitudes $(B_3)_n$ and $(B_4)_n$ in terms of $(B_1)_n$ and $(B_2)_n$ and also $(C_3)_n$ and $(C_4)_n$ in terms of $(C_1)_n$ and $(C_2)_n$ by Eq. 6-54 we are left with only the amplitudes $(B_1)_n$, $(B_2)_n$, $(C_1)_n$, and $(C_2)_n$ in variational Eq. 6-45. We set $\delta\{j\omega\} = 0$ with respect to $\delta(C_1^*)_n$ and $\delta(C_2^*)_n$ and obtain two homogeneous equations in terms of $(B_1)_n$ and $(B_2)_n$. These two equations represent the variational condition on $\hat{e}_{||}$ and $\hat{h}_1$ across the open boundary at $r = a$ as well as zero tangential electric field on the tape. When we set the determinant of these two equations equal to zero we obtain the following dispersion relation for ka vs. $\beta_n a$:

$$\begin{aligned} & \left[tI_n I_n' + \left(\frac{\gamma_n}{k} \right)^2 \left(\frac{\alpha_1 \alpha_2}{\gamma_n a} + t\alpha_1^2 K_n K_n' - t\alpha_2^2 I_n I_n' \right) \right] \\ & \cdot \left[-tK_n K_n' + \left(\frac{\gamma_n}{k} \right)^2 \left(\frac{\alpha_2 \alpha_3}{\gamma_n a} + t\alpha_2^2 K_n K_n' - t\alpha_3^2 I_n I_n' \right) \right] \\ & - \left[\frac{1}{2(\gamma_n a)} + \left(\frac{\gamma_n}{k} \right)^2 \left(-\frac{\alpha_1 \alpha_3 + \alpha_2^2}{2\gamma_n a} - t\alpha_1 \alpha_2 K_n K_n' + t\alpha_2 \alpha_3 I_n I_n' \right) \right]^2 = 0 \end{aligned} \quad (6-55a)$$

where $t = 0.1$ in this instance (Eq. 6-53), $' = \partial/\partial(\gamma_n r)$ at $\gamma_n a$, and

$$\begin{aligned} \alpha_1 &= -\gamma_n a \left[I_n^2 S_1 + \left(\frac{k}{\gamma_n} \right)^2 (I_n')^2 S_2 \right] \\ \alpha_2 &= -\gamma_n a \left[I_n K_n S_1 + \left(\frac{k}{\gamma_n} \right)^2 I_n' K_n' S_2 \right] \\ \alpha_3 &= -\gamma_n a \left[K_n^2 S_1 + \left(\frac{k}{\gamma_n} \right)^2 (K_n')^2 S_2 \right] \end{aligned} \quad (6-55b)$$

and

$$\begin{aligned} S_1 &= \left[\tan \psi - \frac{n\beta_n a}{(\gamma_n a)^2} \right] \\ S_2 &= \left[\cot \psi + \frac{n\beta_n a}{(\gamma_n a)^2} \right]^{-1} \end{aligned} \quad (6-55c)$$

In this form Eq. 6-55 will describe the behavior of any n th mode which predominates in a wave in the region $0 \leq |\beta_n a| \leq 0.5 \cot \psi$ near the first forbidden zone. For narrow tapes such as this one the t terms within the brackets are very small; the important terms dependent upon the tape width are $tI_n I_n'$ and $tK_n K_n'$.

Equations 6-55 with the t terms neglected furnish, in fact, an accurate dispersion relation for narrow tape helices in the range

$$0 < \beta_n a \leq 0.5 \cot \psi$$

Without the t terms the equation reduces to the form of two perfect squares. When we take the square root with the appropriate signs, approximate the $\alpha_{1,2,3}$ by their first terms alone, and take another square root, we obtain just

$$2 \frac{\gamma a}{ka} \gamma a I_n(\gamma a) K_n(\gamma a) S_1 = -1 \quad (6-55')$$

This is plotted in Fig. 6-3 for the $n = +1$ mode.

The solution to dispersion Eq. 6-55 for the $n = +1$ mode in the range $0 \leq \beta_{+1} a \leq 0.5 \cot \psi = 5$ appears in Fig. 6-3 along with the Pierce-Tien estimate [4]. We see that the tape helix dispersion curve does behave as expected: It follows the sheath helix $n = +1$ line down to $\beta_{+1} a = 1.5$ where the structure resonates and then dives toward $ka = 0$ as $\beta_{+1} a$ approaches zero. The region $0 < \beta_{+1} a \leq 1.5$ is the anomalous region, in which the fields are described as follows.

First of all, nothing particularly surprising happens at the cutoff frequency, for which $\beta_{+1} a = 1.5$. The terms in Eq. 6-55 vary smoothly through the cutoff point and give no hint of the resonance condition until we plot ka vs. $\beta_{+1} a$. However, the calculated power flow is very nearly zero at the cutoff point. When we determine the relative amplitudes $(B_1)_1, \dots, (B_4)_1$ by means of the homogeneous equations evaluated for $\beta_{+1} a = 1.5$, $ka = 0.83$ and evaluate the internal power flow

$$P_{av,i} = \frac{1}{2} \operatorname{Re} \int (\hat{e}_r \hat{h}_\theta^* - \hat{e}_\theta \hat{h}_r^*) dS$$

we find that it is nearly zero, the negative component predominating somewhat. The external power flow is also nearly zero, the positive component predominating somewhat. But the total power flow is indeed zero with about a 5% error. However, this does *not* imply that the calculated power flow is in general accurate enough to enable us to calculate the impedance $e_z^2/2\beta^2 P_{av, tot}$ as accurately as the ω - β curve. Indeed, we shall see that the impedance as calculated from the simplest variational procedures is usually in error by a factor of 2 or more. The expressions in Table 6-1 prove useful for calculating these power flows.

As $\beta_{+1} a$ approaches zero in Fig. 6-3 we expect the fields as given by Eqs. 6-51 and 6-52, to become transverse such that $\hat{e}_z$ and $\hat{E}_z$ both disappear. This is indeed so as we may verify from the relative field amplitudes as $\gamma_1 a$ approaches zero faster than $\beta_1 a \simeq ka$ and $I_1(\gamma_1 a) \rightarrow \gamma_1 a/2$, $K_1(\gamma_1 a) \rightarrow 1/\gamma_1 a$. In the limit of $ka = 0$ the predominant field components have the

continuity properties such that $\hat{e}_{z,0} = +\hat{E}_{z,0}$, $\hat{e}_r = -\hat{E}_r$, and $\hat{h}_z = -\hat{H}_z$ on the cylindrical surface $r = a$. Also $\hat{e}_{||} \rightarrow \hat{E}_{||} \rightarrow 0$ and $\hat{h}_{||} = \hat{H}_{||} \rightarrow 0$. In fact, the low-frequency fields approach precisely those of the $n = 1$ sheath helix modes, so that both they and the dispersion curve are well behaved as ka approaches zero.

Table 6-1. Integrals of Modified Bessel Functions

Function	Indefinite Integral
1. ZI_0^2	$ZI_0I_1 - \frac{1}{2}Z^2(I_1^2 - I_0I_2)$
2. $2ZI_1^2$	$Z^2(I_1^2 - I_0I_2)$
3. $Z(I_1')^2$	$-\frac{1}{2}(I_0^2 + I_1^2) + ZI_0I_1 - \frac{1}{2}Z^2(I_1^2 - I_0I_2)$
4. I_1^2/Z	$\frac{1}{2}I_0^2 + \frac{1}{4}ZI_1(I_2 - I_0)$
5. ZI_2^2	$ZI_1I_2 - \frac{1}{2}Z^2(I_1^2 - I_0I_2)$
6. ZI_0I_2	$ZI_0I_1 - I_0^2 - \frac{1}{2}Z^2(I_1^2 - I_0I_2)$
7. $2I_1I_2$	$ZI_1(I_0 - I_2) - I_0^2$
8. ZI_1I_2'	$ZI_1(I_2 - I_0) + I_0^2 + \frac{1}{2}Z^2(I_1^2 - I_0I_2)$
9. $Z(I_2')^2$	$I_0^2 + ZI_1(I_2 - I_0) - I_2^2 + \frac{1}{2}Z^2(I_1^2 - I_0I_2)$
10. I_2^2/Z	$I_1I_2/Z + \frac{1}{4}(I_2^2 - 2I_0I_2 - I_0^2) + \frac{1}{4}ZI_1(I_0 - I_2)$
11. ZK_0^2	$-ZK_0K_1 - \frac{1}{2}Z^2(K_1^2 - K_0K_2)$
12. $2ZK_1^2$	$Z^2(K_1^2 - K_0K_2)$
13. $-Z(K_1')^2$	$\frac{1}{2}(K_0^2 + K_1^2) + ZK_0K_1 + \frac{1}{2}Z^2(K_1^2 - K_0K_2)$
14. K_1^2/Z	$\frac{1}{2}K_0^2 - \frac{1}{4}ZK_1(K_2 - K_0)$
15. $-ZK_2^2$	$ZK_1K_2 + \frac{1}{2}Z^2(K_1^2 - K_0K_2)$
16. $-2K_1K_2$	$ZK_1(K_2 - K_0) - K_0^2$
17. $-ZK_0K_2$	$ZK_0K_1 + K_0^2 + \frac{1}{2}Z^2(K_1^2 - K_0K_2)$

For example, $ZI_0^2 = \frac{d}{dZ} [ZI_0I_1 - \frac{1}{2}Z^2(I_1^2 - I_0I_2)]$

In this table the prime denotes d/dZ .

We have not investigated in detail the other $ka\text{-}\beta_n a$ curves for $|n| > 1$ but each of them ought to be well described by Eq. 6-55 for the appropriate value of n . Each curve has mirror symmetry about the ka axis. Therefore we can calculate rather accurately and straightforwardly the $ka\text{-}\beta_n a$ curves in the fundamental region $-0.5 \cot \psi \leq \beta_n a \leq 0.5 \cot \psi$ and translate them along the βa axis by multiples of $\cot \psi$ to obtain the complete $ka\text{-}\beta a$ diagram. The only modification in this procedure is in the analysis of the wave with predominant $n = 0$ mode in the low-frequency range of $ka \ll 0.25$. For this mode we must make a more accurate variational calculation, discussed in the next subsection.

D. Behavior of the Tape Helix Wave with Predominant $n = 0$ Mode

Dispersion Eq. 6-55 was derived by representing the helix fields of a wave with just the n th TM and TE mode components and by adjusting their amplitudes variationally, subject to the explicit continuity of $\hat{e}_\perp$ and $\hat{h}_\parallel$ across the helix surface at $r = a$. This relation also serves to describe quite well the $n = 0$ mode in the range $ka \geq 0.25$. This is the mode utilized in ordinary traveling wave tube amplifiers for its nearly constant phase velocity characteristic, Fig. 6-4. In order to illustrate the success of our variational technique for thick as well as thin tape helices we shall apply the preceding analysis to the helix measured by Watkins and Siegman [5] with the following parameters:

$$a, \text{ helix radius, } = 0.243 \text{ in.}$$

$$\tan \psi = 0.094$$

$$w, \text{ tape width, } = 0.064 \text{ in.}$$

$$\frac{w}{p} = 0.444$$

$$\text{Tape thickness} = 0.020 \text{ in.}$$

$$t, \text{ tape width parameter of Eq. 6-53, } = \frac{0.44}{0.56} = 0.80$$

The impedance and dispersion curves were measured with a hollow electron beam 0.005 in. thick at a radius of $r/a = 0.825$. The measured

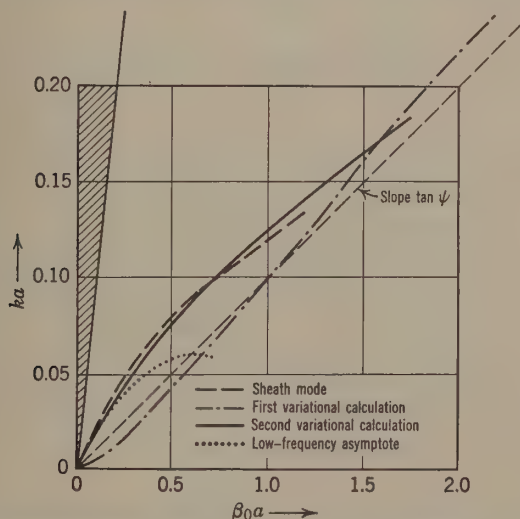


Fig. 6-4. Dispersion of the wave with predominant $n = 0$ mode on a narrow-tape helix. $\cot \psi = 10$ and $t = 0.1$. The serious error in the first variational calculation is apparent.

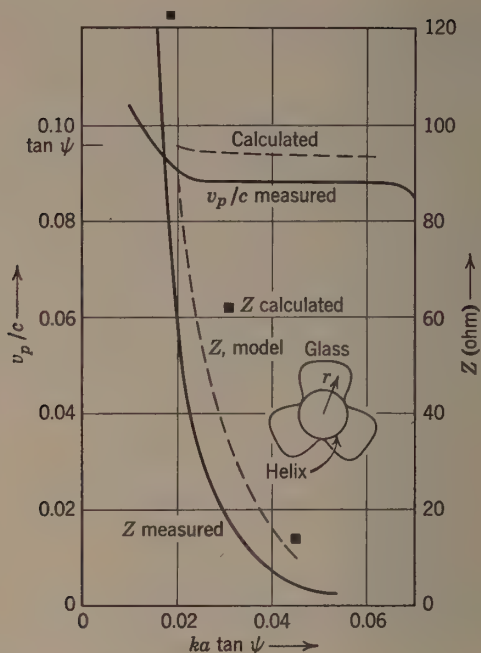


Fig. 6-5. Dispersion and impedance of the wave with predominant $n = 0$ mode on a wide-tape helix. $a/p = 1.7$ and the radial thickness of the tape is 0.020 in. The curve labeled “Z, model” is a correction by Watkins and Siegman to the sheath helix impedance according to the geometry of this structure.

curves for the $n = 0$ mode are shown in Fig. 6-5 along with the curve calculated from Eq. 6-55 for $n = 0$. The calculated $v_p/c = k/\beta_0$ vs. $ka \tan \psi$ curve has very nearly the value of $\tan \psi = 0.094$ over the range indicated and is somewhat higher than the measured curve. We did not calculate the curve in the range $ka \tan \psi > 0.06$ because it lies near point *B* of Fig. 2-4 where the predominant fields are more accurately represented by the $n = -1$ mode of propagation constant $-4 \leq \beta_{-1}a \leq 0$ in this range. Nor do we plot the calculated curve in the range $ka \tan \psi < 0.02$ because serious error develops at low frequencies, as explained in later paragraphs. Our calculation verifies the conjecture of Watkins and Siegman that the $ka - \beta_0 a$ curve is very insensitive to the tape width.

We also determined the relative field amplitudes from the equations preceding Eq. 6-55 and calculated the Pierce impedance at $r/a = 0.825$

$$Z = \frac{|\hat{e}_z(r/a = 0.825)|^2}{2\beta_0^2 \left[\frac{1}{2} \int_0^{\gamma_0 a} (\hat{e}_r \hat{h}_\theta^* - \hat{e}_\theta \hat{h}_r^*) dS + \frac{1}{2} \int_{\gamma_0 a}^\infty (\hat{E}_r \hat{H}_\theta^* - \hat{E}_\theta \hat{H}_r^*) dS \right]} \quad (6-56)$$

at a few points on Fig. 6-5. The calculated points are high by factors of 2 or 3 and demonstrate the usual inability of one or two trial field modes to represent the fields accurately enough for an impedance calculation. To improve on the calculated impedance we could represent the fields of this wave by the $n = \pm 1, 0$ modes in variational Eq. 6-45. After obtaining a set of homogeneous equations for their amplitudes in the usual way we could solve for their relative values easily enough by inserting the values of ka and $\beta_0 a$ known from measurement or the preceding variational calculation. The recalculated impedance Z will then be more accurate.

Now let us calculate at very low frequencies the $n = 0$ dispersion curve for the narrow tape helix of the preceding subsection, with $\cot \psi = 10$, $t = 0.1$ in Eq. 6-55. We obtain the broken dashed line in Fig. 6-4, which approaches the origin with zero slope. This behavior is contrary to measurements which verify the fact that the curve really approaches the velocity of light line $ka = \beta a$ asymptotically. At very low frequencies the fields external to the helix should have the augmented plane wave components E_r and H_θ such that $E_r/H_\theta = \sqrt{\mu/\epsilon}$ of free space. We now inquire into the reason for the failure of determinantal Eq. 6-55 for the $n = 0$ mode at low frequencies.

As ka approaches zero, the field continuities should become those of $\hat{e}_\perp = \hat{E}_\perp$ and $\hat{e}_\parallel = \hat{E}_\parallel = 0$ on the cylindrical surface $r = a$. But if we calculate the relative amplitudes of the fields from the homogeneous equations which led to Eq. 6-55 we find that, as ka approaches zero, $e_\theta/e_z \rightarrow 0$ while $E_\theta/E_z \rightarrow \infty$ at $r = a$. Therefore, $e_\parallel \rightarrow e_z \sin \psi$ and $E_\parallel \rightarrow E_\theta \cos \psi$ simply cannot go to zero properly. Even without knowing the results of measurements we would suspect this first variational calculation to be erroneous at low frequencies and not particularly accurate at higher ones (Fig. 6-4).

In order to make a second improved variational calculation we first recall that we derived Eq. 6-55 with the explicit continuities of $\hat{e}_\perp = \hat{E}_\perp$ and $\hat{h}_\parallel = \hat{H}_\parallel$ on the cylindrical surface $r = a$. Let us relax these conditions and choose between $\hat{e}_\perp = \hat{E}_\perp$ and $\hat{e}_\parallel = \hat{E}_\parallel$ at $r = a$. If we choose the first of these constraints the procedure of Section 6.3C yields a set of three homogeneous equations for three of the amplitudes $(B_1)_0, \dots, (B_4)_0$. The resultant determinantal equation for ka vs. $\beta_0 a$ happens to have the wrong asymptotic dependence upon pitch angle ($\cot^2 \psi$ instead of $\tan^2 \psi$!). Nor does the explicit continuity of $\hat{h}_\parallel = \hat{H}_\parallel$ yield a better dispersion relation. So we choose the second explicit constraint of $\hat{e}_\parallel(\gamma_0 a) = \hat{E}_\parallel(\gamma_0 a)$. With *only* this condition we obtain, by the procedure which led to Eq. 6-55, a proper dispersion relation over the entire frequency range of very low and very high frequencies. Expansion of the 3×3 determinant for $k(\beta_0)$ and straightforward although tedious simplification yields the rather

simple dispersion relation

$$\left(\frac{k}{\gamma}\right)^2 \cot^2 \psi \left[\frac{I_1 K_1}{I_0 K_0} \frac{1}{(\gamma a)^2} - 4 I_1^2 K_1^2 t^2 \right] = \frac{1}{(\gamma a)^2} - 4 I_0 I_1 K_0 K_1 t^2 \quad (6-57)$$

in which t is defined by Eq. 6-53 and $\gamma_0 a$ is the Bessel function argument. The effect of the relative tape width is rather weak since t^2 and not t appears in Eq. 6-57. This relation is plotted in Fig. 6-4 and corresponds closely to the $n = 0$ sheath helix curve given by Eq. 2-7.

As ka approaches zero the more accurate dispersion relation 6-57 takes the form

$$\left(\frac{k}{\gamma_0}\right)^2 = -2 \tan^2 \psi \ln \frac{\gamma_0 a}{2}, \quad \text{with} \quad \gamma_0^2 = \beta_0^2 - k^2 \quad (6-57')$$

independent of the tape width. This is exactly the limiting form of the $n = 0$ sheath helix relation. Equation 6-57' is plotted in Fig. 6-4 as an asymptote and is valid in the range $\beta_0 a < 0.10$.

Examination of the asymptotic expressions for the fields reveals a curious fact: All the external field components apparently have the wrong signs with respect to the sheath helix components! Reversal of their signs does not change the determinantal equation for $ka(\beta_0 a)$ in which the sign of an entire row or column will change. The fact that the external field signs are wrong indicates that the correct explicit continuity to invoke at $r = a$ is $\hat{e}_{\parallel} = -\hat{E}_{\parallel}$ instead of $\hat{e}_{\parallel} = \hat{E}_{\parallel}$. The statement $\hat{e}_{\parallel}(\gamma_0 a) = -\hat{E}_{\parallel}(\gamma_0 a)$ must be interpreted in an averaged sense. The $n = 0$ mode by itself cannot possibly maintain zero electric field tangential to the tape on the $r = a$ surface; rather, this tangential field changes sign across the surface and is zero on the surface.⁶ We find, upon reversing the signs of all the external field components, that the asymptotic fields agree with the asymptotic sheath mode fields within fractional differences on the order of $2t$. We expect our dispersion relation to be accurate, however, even for tape widths as large as $t = 0.2$, or 0.3 because Eq. 6-57' is independent of t and because we do not expect the dispersion to be critical of t at low frequencies.

⁶ A reversal of signs of the fields in the outer region will, in principle, not contradict variational Eq. 6-45 in which the true $\hat{E}_+ \times \hat{n}'$, is 0 on the metal surface and the last two integrals are invariant to the changes $\hat{E}_{\pm} \rightarrow -\hat{E}_{\pm}$, $\hat{H}_{\pm} \rightarrow -\hat{H}_{\pm}$ followed by a conjugation of the whole equation. However, the sign question is properly resolved by changing the explicit boundary condition to $\hat{e}_{\parallel} = -\hat{E}_{\parallel}$ at $r = a$. This is obviously an approximate condition better at lower frequencies and is intended to reduce the number of independent mode amplitudes. It need not be made when employing more modes in the field expansions.

In the intermediate-frequency range $0.25 < ka < 1.5$ the dispersion curve of Fig. 6-4 behaves like that of the sheath wave but approaches the asymptote $k/\gamma_0 = \tan \psi \cong k/\beta_0$ somewhat less rapidly with increasing frequency.

In the high-frequency range $ka > 1.5$ the dispersion curve is essentially $k/\gamma_0 = \tan \psi$, independent of the tape width. The limiting forms of the fields, with the external ones reversed in sign, are also precisely the sheath wave fields, this time independent of the tape width. Therefore the electric field has the asymptotic continuities of $\hat{e}_{\parallel} = \hat{E}_{\parallel} = 0$ and $e_z = +E_z$ on the surface $r = a$. It is interesting to observe that, while we initially specified only continuity of electric field tangential to the tape at $r = a$, the variational procedure actually constrains each of $\hat{e}_{\parallel}$ and $\hat{E}_{\parallel}$ to approach zero at very low and very high frequencies.

There is little doubt experimentally⁷ that the fields in the wave with predominant $n = 0$ mode do actually behave like the sheath mode fields. Cutler measured a large $\hat{e}_z \cong \hat{E}_z$ component of field at $r = a$ —in fact, larger than the sheath mode component—and also observed a tendency of $\hat{e}_r$ to reverse direction across the $r = a$ surface.

In conclusion, we see that our second variational calculation of the tape helix $n = 0$ mode dispersion, with the single explicit condition of $\hat{e}_{\parallel} = \hat{E}_{\parallel}$ at $r = a$, has given us a well-behaved $ka - \beta_0 a$ curve through the entire frequency range of interest. The fields also behave as expected at very low and very high frequencies. The fact that we had to reverse the signs of the calculated external fields is a striking example of the power the variational technique has of yielding an accurate dispersion relation in spite of somewhat ambiguous trial fields.

6-4. The Variational Analysis of Ring Helices

The appropriately modified variational Eq. 6-45 has proven very useful for the study of the behavior of the ring-plane helix [7] shown in Fig. 6-6, along with the interesting portion of the $ka - \beta_0 a$ curve for each of two independent waves. One of the waves, labeled $(++)$, has mirror symmetry of $\hat{e}_z$ across the vertical planes; the less interesting wave for beam interaction purposes is the $(+-)$ wave of antisymmetric $\hat{e}_z$ field. The $(++)$ wave has a Pierce impedance of several hundred ohms and the $(+-)$ wave has an impedance roughly double that of the $(++)$ wave, so this type of helix is a good candidate for a high-power, high-frequency

⁷ C. C. Cutler [6]. Cutler made his measurements at $ka = 0.20$ on a helix of $\cot \psi = 11.5$ with a rather thick wire of $w/p = w \cot \psi / 2\pi a = 0.52$.

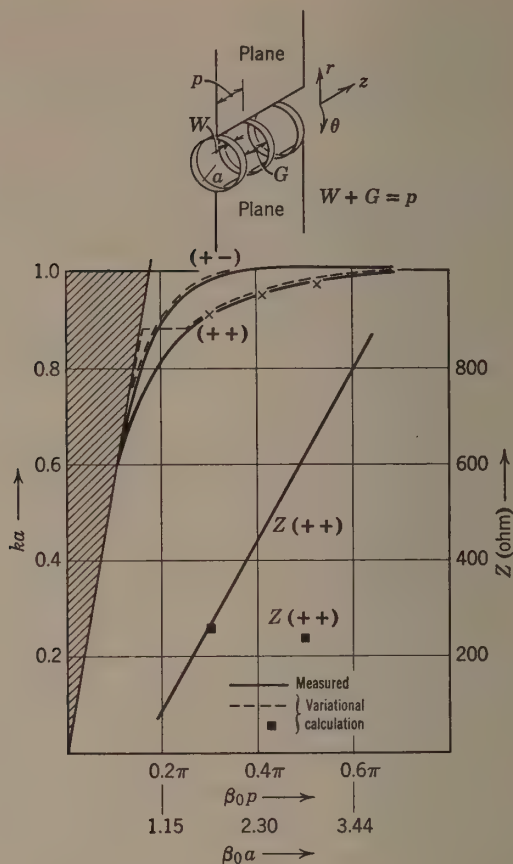


Fig. 6-6. Dispersion and impedance of a ring-plane helix. $(++)$ denotes the wave of symmetric $\hat{E}_z$ field and $(+-)$, the wave of antisymmetric $\hat{E}_z$ field about the planes shown. Both dispersion curves have zero group velocity at $\beta_0 p = \pi$. $a/p = 1.83$, $W/p = 0.5$, and (ring radial thickness)/ $a = 0.073$.

traveling wave tube. The analysis of the $(++)$ wave proceeds as follows.

To represent the $(++)$ wave fields we again employ the modes defined in Eqs. 2-1 through 2-4. Since we are interested in the propagating region $0 \leq \beta_0 p \leq \pi/2$ we choose to ignore the axial harmonics of these modes and let them all propagate as $e^{-j\beta_0 z}$. Therefore, there is just one radial propagation constant, $\gamma_0 = (\beta_0^2 - k^2)^{1/2}$. However, we must include more than one mode of more than one angular variation to represent accurately the fields within the ring region, $r \leq a$.

The field symmetries dictate the following linear combination of $n = 0, \pm 2$ modes in Eqs. 2-1 through 2-4 for the internal trial fields, denoted

again by lower-case letters:

$$\begin{aligned}
 (r \leq a, \quad 0 \leq \theta \leq 2\pi) \\
 \hat{e}_{z+} &= (B_3)_0 I_0(\gamma_0 r) + (B_3)_2 I_2(\gamma_0 r) \cos 2\theta \\
 \hat{e}_{r+} &= (B_3)_0 \frac{j\beta_0}{\gamma_0} I_0' + (B_3)_2 \frac{j\beta_0}{\gamma_0} I_2' \cos 2\theta + (B_1)_2 \frac{j2\omega\mu}{\gamma_0^2 r} I_2 \cos 2\theta \\
 \hat{e}_{\theta+} &= -(B_3)_2 \frac{j2\beta_0}{\gamma_0^2 r} I_2 \sin 2\theta - (B_1)_2 \frac{j\omega\mu}{\gamma_0} I_2' \sin 2\theta \\
 \hat{h}_{z+} &= (B_1)_2 I_2 \sin 2\theta \\
 \hat{h}_{r+} &= (B_3)_2 \frac{j2\omega\varepsilon}{\gamma_0^2 r} I_2 \sin 2\theta + (B_1)_2 \frac{j\beta_0}{\gamma_0} I_2' \sin 2\theta \\
 \hat{h}_{\theta+} &= (B_3)_0 \frac{j\omega\varepsilon}{\gamma_0} I_0' + (B_3)_2 \frac{j\omega\varepsilon}{\gamma_0} I_2' \cos 2\theta + (B_1)_2 \frac{j2\beta_0}{\gamma_0^2 r} I_2 \cos 2\theta \\
 & \quad ' = \frac{\partial}{\partial(\gamma_0 r)}
 \end{aligned} \tag{6-58}$$

There is no I_0 component of axial magnetic field because it is antisymmetric about the planes and therefore zero on the axis. Each field should be multiplied by $e^{-j\beta_0 z}$ to obtain the complete trial field. We represent the fields within the external helix region $r \geq a$ by the $n = \pm 1$ modes, denoted by capital letters:

$$\begin{aligned}
 (r \geq a, \quad 0 \leq \theta \leq \pi) \\
 \hat{E}_{z+} &= (B_4)_1 K_1(\gamma_0 r) \sin \theta \\
 \hat{E}_{r+} &= (B_4)_1 \frac{j\beta_0}{\gamma_0} K_1' \sin \theta - (B_2)_1 \frac{j\omega\mu}{\gamma_0^2 r} K_1 \sin \theta \\
 \hat{E}_{\theta+} &= (B_4)_1 \frac{j\beta_0}{\gamma_0^2 r} K_1 \cos \theta - (B_2)_1 \frac{j\omega\mu}{\gamma_0} K_1' \cos \theta \\
 \hat{H}_{z+} &= (B_2)_1 K_1 \cos \theta \\
 \hat{H}_{r+} &= -(B_4)_1 \frac{j\omega\varepsilon}{\gamma_0^2 r} K_1 \cos \theta + (B_2)_1 \frac{j\beta_0}{\gamma_0} K_1' \cos \theta \\
 \hat{H}_{\theta+} &= (B_4)_1 \frac{j\omega\varepsilon}{\gamma_0} K_1' \sin \theta - (B_2)_1 \frac{j\beta_0}{\gamma_0^2 r} K_1 \sin \theta
 \end{aligned} \tag{6-59}$$

These external fields are defined over the interval $0 \leq \theta \leq \pi$; those in the region $\pi \leq \theta \leq 2\pi$ are obtained by changing the signs of the fields given by Eq. 6-59 evaluated at the θ of interest. The minus trial fields are the same but with independent amplitudes $(C_3)_0$, and so on.

In variational Eq. 6-45, now written for a periodic volume enclosed by

a slab of infinite cross section and axial length p ; the area integrals again vanish. The $\int_W$ integrals are now taken over the surface of a ring while the $\int_D$ integrals are taken over a gap between rings. Then we can write the new variational expression as

$$0 = \int_{Z=0}^W \int_{\theta=0}^{\pi} (\hat{e}_+ \times \hat{h}_-^* - \hat{E}_+ \times \hat{H}_-^*) \cdot \bar{i}_r dS \\ + \frac{1}{2} \int_{W=Z}^{W+G} \int_{\Theta=0}^{\pi} [(\hat{e}_-^* + \hat{E}_-^*) \times (\hat{h}_+ - \hat{H}_+) \\ + (\hat{e}_+ - \hat{E}_+) \times (\hat{h}_-^* + \hat{H}_-^*)] \cdot \bar{i}_r dS \quad (6-60)$$

The ring thickness is W and the gap length is G , such that $W + G = p$, the periodic length in Fig. 6-6. We substitute the above trial fields into Eq. 6-60 and take variations $\delta(C_3)_0$, $\delta(C_3)_2$, $\delta(C_4)_1$, $\delta(C_1)_2$, and $\delta(C_2)_1$, with ω held fixed. No explicit continuities are specified on the surface $r = a$. We obtain five homogeneous equations in terms of the five B amplitudes. The determinant of this array set equal to zero yields the following dispersion relation for $\beta_0 a$ vs. ka :

$$\left[\frac{(\frac{4}{3}F_1)^2}{F_3} + \left(\frac{\pi W}{G} \right)^2 - 2F_4\Lambda \right] \left[\frac{(\frac{2}{3}F_1)^2}{F_3} + \left(\frac{\pi W}{G} \right)^2 \right] \\ + \left[\frac{(\frac{4}{3}F_1)^2}{F_3} + \left(\frac{\pi W}{G} \right)^2 - 4F_3\Lambda \right] \frac{2F_2^2}{F_4} + \frac{16}{3} F_1 F_2 \Lambda - \left(\frac{2\pi W}{G} \right)^2 F_3 \Lambda = 0 \quad (6-61a)$$

where

$$F_1 = \frac{I_2' K_1 - I_2 K_1'}{I_2' K_1'}, \quad F_2 = \frac{I_0 K_1' - I_0' K_1}{I_0' K_1'}, \quad F_3 = \frac{I_2 K_1}{I_2' K_1'}, \\ F_4 = \frac{I_0 K_1}{I_0' K_1'}, \quad \text{and} \quad \Lambda = \left(\frac{\beta_0 a}{\gamma_0 a k a} \right)^2 \\ (\gamma_0 a \text{ is Bessel function argument}) \quad (6-61b)$$

The trial-and-error solution to this equation for equal gap and ring spacing $W/G = 1$, and for rings of infinitesimal radial thickness, is shown by the lower dashed curve of Fig. 6-6. The dispersion relation can only be satisfied in a propagating range above $\beta_0 p \cong 0.25\pi$; in this range there are two βa roots for each ka and we have plotted only the lowest root. The other root must be regarded as spurious. Below the critical point $\beta_0 p \cong 0.25\pi$ the only solution is $\beta a = ka$ on the velocity-of-light line (see Prob. 9). The discontinuous nature of the calculated dispersion relation is indicated by the horizontal dashed line in the figure. We did

not write any explicit field continuities at $r = a$ which might have improved the low-frequency portion of the calculated curve.

It is interesting to observe the decrease of external field as the frequency decreases and the fields take on the character of a fast wave with $\cos 2\theta$ variation of e_z inside the rings. The relative field amplitudes over the range $0.3\pi \leq \beta_0 p \leq 0.5\pi$ are tabulated in Table 6-2, with $(B_2)_1$ taken as reference. These fields should be regarded as qualitative only.

Although our calculation of the Pierce impedance Z (Eq. 6-56) at $\beta_0 p = 0.3\pi$, with $\hat{e}_z$ taken just inside $r = a$ at $\theta = \pi/2$, agreed exactly with the measured value as shown on Fig. 6-6, the calculated value at $\beta_0 p = 0.5\pi$ was below the measured value by more than a factor of 2. We should note that the effective impedance for a hollow electron beam of radius $r = a$ would be estimated by

$$\frac{1}{\pi} \int_0^\pi |\hat{e}_z(a, \theta)|^2 d\theta$$

which is lower than the "point" impedance at $r = a, \theta = \pi/2$. According to a calculation at $\beta_0 p = 0.3\pi$, we estimate the averaged impedance to be lower by a factor of 2.5.

Now we turn our attention to the dispersion curve of the $(+ -)$ wave, which is the upper curve on Fig. 6-6. For this wave we obtain a very accurate dispersion relation if we take the plus trial fields to be (excluding the $\epsilon^{-j\beta_0 z}$ factor),

$$\begin{aligned} (r \leq a, \quad 0 \leq \theta \leq 2\pi) \\ \hat{e}_{z+} &= (B_3)_1 I_1(\gamma_0 r) \sin \theta \\ \hat{e}_{r+} &= (B_3)_1 \frac{j\beta_0}{\gamma_0} I_1' \sin \theta - (B_1)_1 \frac{j\omega\mu}{\gamma_0^2 r} I_1 \sin \theta \\ \hat{e}_{\theta+} &= (B_3)_1 \frac{j\beta_0}{\gamma_0^2 r} I_1 \cos \theta - (B_1)_1 \frac{j\omega\mu}{\gamma_0} I_1' \cos \theta \\ \hat{h}_{z+} &= (B_1)_1 I_1 \cos \theta \\ \hat{h}_{r+} &= -(B_3)_1 \frac{j\omega\epsilon}{\gamma_0^2 r} I_1 \cos \theta + (B_1)_1 \frac{j\beta_0}{\gamma_0} I_1' \cos \theta \\ \hat{h}_{\theta+} &= (B_3)_1 \frac{j\omega\epsilon}{\gamma_0} I_1' \sin \theta - (B_1)_1 \frac{j\beta_0}{\gamma_0^2 r} I_1 \sin \theta \end{aligned} \quad (6-62)$$

The plus fields in the external $r \geq a$ region are obtained by changing $(B_3)_1$, $(B_1)_1$, and I_1 to $(B_4)_1$, $(B_2)_1$, and K_1 , respectively. If we now substitute these fields along with the minus trial functions into variational Eq. 6-60 and proceed to obtain four homogeneous equations in terms of amplitudes $(B_1)_1, \dots, (B_4)_1$ we find that the TE portion of the field is

Table 6-2. Relative Field Amplitudes* of the $(++)$ Mode at Different Frequencies. $(B_2)_1 = 1$

$\beta_0 p$	ka	$(B_1)_2$	$(B_3)_0 \omega \epsilon \gamma_0 a^2$	$(B_3)_2 \omega \epsilon \gamma_0 a^2$	$(B_4)_1 \omega \epsilon \gamma_0 a^2$
0.3π	0.90	-0.77	-0.48	1.85	-3.14
0.4π	0.95	-0.12	-0.24	0.59	-6.75
0.5π	0.975	-0.03	-0.11	0.23	-10.8

* Signs of external fields have not been changed. Rings have infinitesimal radial thickness.

decoupled from the TM portion! We do not obtain a proper dispersion relation at all. But we can remedy this state of affairs by imposing continuity of $\hat{e}_\perp$ and $\hat{h}_\parallel$ at $r = a$ just as we did for the tape helix. For the ring helix this means we set $\hat{e}_z = \hat{E}_z$ and $\hat{h}_\theta = \hat{H}_\theta$ at $r = a$ and thereby eliminate two of the B and C amplitudes in the variational procedure. When we take variations of the remaining two C amplitudes we obtain a two-by-two determinantal equation for ka vs. $\beta_0 a$ of the form

$$k^2 \left[1 + \left(\frac{W}{G} \right)^2 4(\gamma_0 a)^2 I_1(\gamma_0 a) K_1(\gamma_0 a) I_1' K_1' \right] = 4\beta_0^2 \left[1 - \left(\frac{W}{G} \right)^2 \right] (I_1 K_1)^2 \quad (6-63)$$

We may check the fact that $4(\gamma_0 a)^2 I_1 K_1 I_1' K_1'$ is one of those Bessel function combinations which has very nearly a constant value (-1) over the whole range of $\gamma_0 a$. Then the bracket on the left side of Eq. 6-63 cancels the one on the right and we take the square root to obtain

$$ka = 2\beta_0 a I_1(\gamma_0 a) K_1(\gamma_0 a), \quad \gamma_0 = (\beta_0^2 - k^2)^{1/2} \quad (6-64)$$

Equation 6-64 is the dispersion relation for the $(+-)$ wave, and agrees with the measured curve very well indeed on Fig. 6-6. We observe with interest that the relation is independent of the ring-gap ratio, W/G .

We often want to design these ring-plane helices so the electron beam will interact primarily with the $(++)$ wave alone. This is not easy to do, since the $(+-)$ wave is more strongly resonant and will tend to have a high impedance. In fact, we estimated at $\beta_0 p = 0.3\pi$ that the Pierce impedance averaged around the ring just inside its radius is about twice as great for the $(+-)$ wave as for the $(++)$ wave. The situation may be remedied somewhat by cutting notches in the planes between the rings, shown in Fig. 6-7, which have the effect of lowering the ka - $\beta_0 a$ curve for the $(++)$ wave while leaving the $(+-)$ wave virtually unaffected. At first thought we might be inclined to believe that a notch would allow some H_r and H_z components of the $(++)$ wave to cancel on its surface

(see Fig. 6-7a), reduce the magnetic stored energy near the notch, and thus raise the dispersion curve of this wave. But the opposite effect is true, and we now outline a general procedure whereby the correct effect can be predicted.

First of all, it is clear that a notch will not affect the $(+ -)$ wave with continuous field components across the plane at $\theta = 0$ (Eq. 6-62 written for the external region). However, the $(+ +)$ wave has the discontinuous field components pictured in Fig. 6-7a. After we cut a notch the $\hat{E}_\theta$ field fringes and looks like $\hat{E}'$ of Fig. 6-7b. We need not know the details of the associated $\hat{H}'$ field in the considerations to follow. Now, we should like to have a relation between the frequency k before the notch is cut and frequency k' (at the same value of β_0) after it is cut, in terms of the fields on the notch. Such a relation is Green's theorem, Eq. 3-43b, written for the infinite cross-sectional slab between the two rings of Fig. 6-7b. In this theorem, we let $\bar{A} = \hat{E}'$, the electric field existing after the notch is cut and $\bar{B} = \hat{E}^*$, the conjugate of that field before the notch is cut. Then we have the relation

$$\int_V (\hat{E}' \cdot \nabla \times \nabla \times \hat{E}^* - \hat{E}^* \cdot \nabla \times \nabla \times \hat{E}') dV = \oint_S (\hat{E}^* \times \nabla \times \hat{E}' - \hat{E}' \times \nabla \times \hat{E}^*) \cdot d\bar{S} \quad (6-65)$$

Since $\hat{E}'$ satisfies the same equation as $\hat{E}$ but at a different frequency,

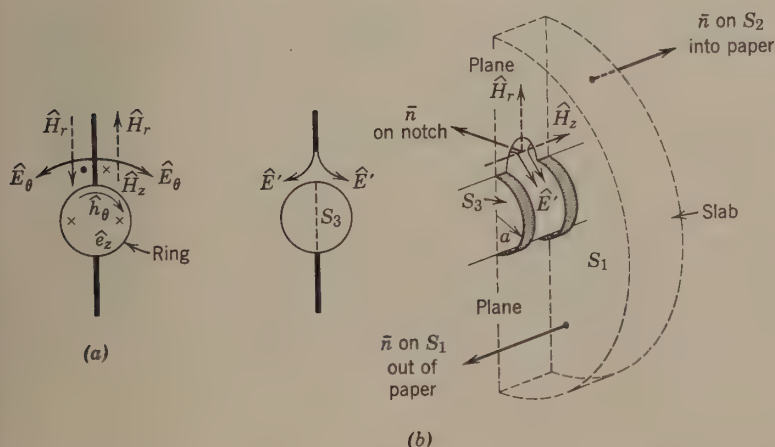


Fig. 6-7. Perturbation of the $(+ +)$ wave by notches in the planes. (a) Field components before cutting the notches, (b) $\hat{E}'$ field present when the notches are cut. Surface S_3 between the rings in the $r \leq a$ region is effectively a magnetic short.

namely $\nabla \times \nabla \times \hat{E}' - (k')^2 \hat{E}' = 0$, we can write the left side as

$$[k^2 - (k')^2] \int_{\text{slab}} \hat{E}' \cdot \hat{E}^* dV$$

The sign of $k^2 - (k')^2$ is the quantity of interest. Now, if the notch is small it has only a local effect on $\hat{E}$, and so $\hat{E}' \cong \hat{E}$ throughout most of the slab. Certainly this approximation is good enough to say that the left side of Eq. 6-65 is $[k^2 - (k')^2](+ \text{number})$.

In the right side of Eq. 6-65 $\hat{E}$ and $\hat{E}'$ are both perpendicular to the metal plates so the integrand is zero on them. Moreover, the integral over surface S_1 of the slab in Fig. 6-7b cancels the integral over surface S_2 a period away because the phase shift factors cancel and normal $\bar{n}$ reverses direction. Finally, the integral over inner surface S_3 cancels because tangential $\hat{h}_r$ and $\hat{h}_z$ are zero there (see Eq. 6-58). Therefore the right side of Eq. 6-65 is simply

$$\begin{aligned} \int_{\text{notch}} (-j\omega'\mu\hat{E}^* \times \hat{H}' - j\omega\mu\hat{E}' \times \hat{H}^*) \cdot \bar{n} dS \\ = -j\omega\mu \int_{\text{notch}} \hat{E}' \times \hat{H}^* \cdot \bar{n} dS \quad (6-66) \end{aligned}$$

where we have used the fact that $\hat{E}$ is parallel to $\bar{n}$ on the notch surface.

Our relation between the frequencies and fields on the notch is now

$$[k^2 - (k')^2](+ \text{number}) = -j\omega\mu \int_{\text{notch}} \hat{E}' \times \hat{H}^* \cdot \bar{n} dS \quad (6-67)$$

We know $\hat{H}$ on the notch surface before it is cut, but we must estimate $\hat{E}'$. Let us say it is just a perturbed $\hat{E}_0$ of the same amplitude but different flow lines, shown on Fig. 6-7b. Then we can see that $\hat{E}' \times \hat{H}_r^*$ portion of the integral will be zero because the z component of $\hat{E}'$ reverses direction across the notch. Therefore the sign of the right side of Eq. 6-67 depends upon the sign of $\hat{E}_r' \times \hat{H}_z^* \cdot \bar{n}$. We have the amplitudes of $\hat{E}_0$, that is, of $\hat{E}'$, and of $\hat{H}_z$ in Table 6-2. With them, the right side of Eq. 6-67 turns out to be a positive real number over $0 < \beta_0 p < 0.5\pi$. Therefore we have deduced that $k^2 - (k')^2 > 0$, which means that the notches tend to depress the $ka - \beta_0 a$ curve for the $(++)$ wave. We do not have sufficient faith in the calculated amplitudes of $\hat{E}_0$ and $\hat{H}_z$ to warrant a reliable calculation of the magnitude of this depression.

6-5. Contrawound Helices

Figure 6-8a shows one type of countrawound helix considered by Chodorow and Chu [8], and Fig. 6-8b shows a more practical structure

discussed theoretically and experimentally by Birdsall and Everhart [7]. These helices are appropriate for high-voltage, high-power traveling wave tubes and have higher forward wave (predominant $n = 0$ mode) and considerably lower backward wave (predominant $n = -1$ mode) impedances than do the single-tape helices.

Chodorow and Chu have shown that the form of $\hat{e}_z$, or $\hat{h}_z$, appropriate to the symmetry is

$$\begin{aligned} \begin{pmatrix} \hat{e}_z \\ \hat{E}_z \end{pmatrix} &= \sum_{\ell=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left\{ \frac{(B_3)_{\ell m} I_{\ell}(\gamma_{\ell m} r)}{(B_4)_{\ell m} K_{\ell}(\gamma_{\ell m} r)} \right\} e^{jm\theta - j\beta_{\ell m} z} \\ \beta_{\ell m} &= \beta_{00} + \frac{2\pi}{lp} (\ell + 2m) \\ \gamma_{\ell m}^2 &= \beta_{\ell m}^2 - k^2 > 0 \quad \text{for all } \ell, m \end{aligned} \quad (6-68)$$

A linear combination of these mode-harmonics for $\hat{e}_z$ and $\hat{E}_z$ and a similar combination for $\hat{h}_z$ and $\hat{H}_z$ determines the other field components through Maxwell's equations. All expansions may then be inserted into variational Eq. 6-45, where now the periodic volume may be taken to be an infinite slab of axial width $p/2$ rather than p because of the form of Eq. 6-68. The ω - β diagram for either contrawound helix differs only in detail from that of Fig. 2-4 for the single-tape helix, but now a series of points on the various branches of an ω - β curve represents an infinite number of mode-harmonics for different values of ℓ and m which correspond to the same value of β_{00} instead of β_0 .

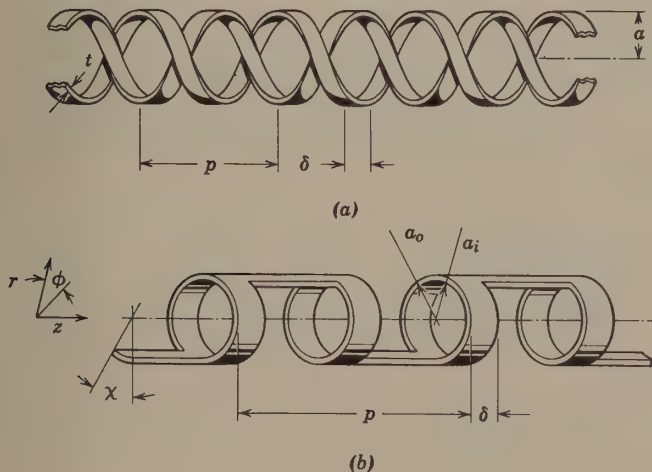


Fig. 6-8. Contrawound helices (from Birdsall and Everhart [7]). Compare with the bifilar helix of Fig. 2-5. (a) Structure analyzed by Chodorow and Chu ([8]), (b) structure which has been built and tested (Ref. [7]).

Let us now turn to the ring-bar contrawound helix of Fig. 6-8*b* and compare it with the ring-plane helix of Fig. 6-6. If we cut wide notches between the rings of the ring-plane helix and alternate them between the planes, and then cut off the remaining extensions of the planes, we can perturb this helix into the ring-bar contrawound helix. This will have the effect of lowering the $(++)$ curve of Fig. 6-6 so that it follows roughly the $n = 0$ curve of the single-tape helix. The $(+-)$ curve of the ring-plane helix in the range $0 \leq \beta_0 p \leq \pi$ becomes, so to speak, the $n = -1$ curve of the contrawound helix in the range $0 \leq \beta_{-1} a \leq 0.5 \cot \psi$. But now β_{-1} has associated with it many values of ℓ and m in Eq. 6-68. The remaining portions of the $ka-\beta a$ curves for the contrawound helix waves with predominant $n = 0, \pm 1$ mode-harmonic components are determined by the symmetry and translational properties of the $\omega-\beta$ diagram, Fig. 2-4.

6-6. The Variational Analysis of the Ridged Waveguide

We now wish to apply our variational technique to the calculation of the first $\omega-\beta_0$ curve of a ridged waveguide like the one discussed in Section 2-6. Now, however, we consider the practical closed structure shown in Fig. 6-9, which has been studied by Slater [9] and by Chu and Hansen [10] and is well understood experimentally. These workers neglected the fin thickness in their calculations of the π -cutoff frequency; the difference between the measured curve for $\frac{1}{32}$ -in. thick fins and the calculations for zero-thickness fins is almost unnoticeable when dimensions a is more than $b/2$. We shall develop our variational analysis so as to account for fins of any thickness in the calculation of the entire $\omega-\beta_0$ curve in the range $0 \leq \beta_0 L \leq \pi$. But first we want to summarize the elegant Chu-Schwinger method [10] of calculating upper and lower bounds to the π cutoff frequency.

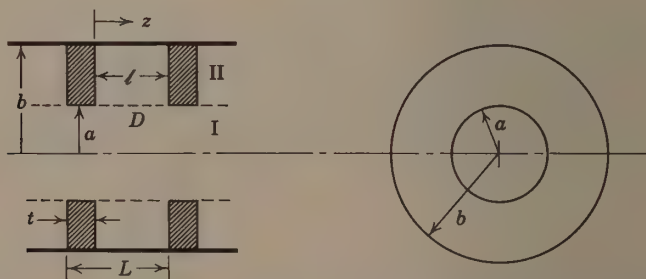


Fig. 6-9. A ridged waveguide. I denotes the inner region and II denotes the outer region, separated by surface D .

The Chu-Schwinger method starts by guessing a reasonable E_z field along discontinuity surface D which separates regions I and II in Fig. 6-9. From this E_z field we calculate by means of a Green's function defined in region I the H_θ field (which is the only magnetic field component in this passband) on D . Let us call this field $H_{\theta I}$. We then calculate from this same E_z the magnetic field, $H_{\theta II}$ on D from the appropriate Green's function in region II. Generally these two H_θ will not be equal, but Schwinger's method asserts that if we choose $k_\pi = \omega_\pi/c$, the π cutoff frequency, so as to make

$$\int_D E_z(H_{\theta I} - H_{\theta II}) dz = 0$$

then this value of k_π will approach the correct one as E_z approaches the correct field. Furthermore, the calculated k_π will always be *lower* than the correct one, and its error will be quadratic in the error of the assumed E_z .

We can reverse the procedure and guess a distribution of H_θ on D , whereupon we calculate E_{zI} and E_{zII} by means of new Green's functions defined within regions I and II. Then if we choose the frequency k_π' so as to obtain the condition

$$\int_D (E_{zI} - E_{zII}) H_\theta dz = 0$$

we will obtain a k_π' which is always *higher* than the correct one and with an error which is quadratic in the error of H_θ . With this procedure, Chu and Hansen quote bounds on the values of $k_\pi a$ which, for five relatively simple trial functions chosen, differ by no more than 0.02.

Elegant as the Chu-Schwinger method is, the calculations are tedious because of the slow convergence of the Green's function expressions for the fields. Furthermore, one of the functions must be redefined if the fins have appreciable thickness in order that E_{zI} remain zero on the fin edges. For these reasons we now adopt the variational analysis of the preceding sections to the ridged waveguide structure, even though it will not give us upper and lower bounds to the frequency corresponding to a given value of β_0 .

Let us assume for the moment the fins in the Fig. 6-9 structure do have zero thickness and that the dimensions are as shown in Fig. 1-10, for definiteness. We can employ variational Eq. 6-45 with the periodic volume now taken over axial distance L . We can simplify matters by employing a set of trial functions each of which satisfies Maxwell's equations identically within region I or II. Then the volume integrals in

Eq. 6-45 vanish. The q th trial field in region I, denoted by small letters, is taken to be

$$\left. \begin{aligned} e_{za} &= A_a \frac{1}{\omega \epsilon} I_0(\gamma_a r) \\ e_{ra} &= j A_a \frac{\beta_a}{\omega \epsilon \gamma_a} I_1(\gamma_a r) \\ h_{\theta a} &= j A_a \frac{1}{\gamma_a} I_1(\gamma_a r) \end{aligned} \right\} \epsilon^{-j\beta_a z} \quad \left. \begin{aligned} \gamma_a &= +\sqrt{\beta_a^2 - k^2} \\ (|\beta_a| \text{ assumed } \geq k) \\ \beta_a &= \beta_0 + \frac{2q\pi}{L} \end{aligned} \right\} \quad (6-69)$$

If one of the $|\beta_q|$ is less than k , which is often the case for β_0 , then we must replace I_0 and I_1 by $+J_0(\gamma_q, r)$ and $+J_1(\gamma_q, r)$, respectively, and define γ_q , as $+(k^2 - \beta_q^2)^{1/2}$. In region II we define the m th trial function, denoted by capital letters, as

$$\begin{aligned} E_{zm} &= -j B_m \frac{1}{\omega \epsilon} \cos \frac{m\pi z}{\ell} R_0(\eta_m r) \\ E_{rm} &= -j B_m \frac{1}{\omega \epsilon} \frac{m\pi}{\ell} \sin \frac{m\pi z}{\ell} \frac{R_1(\eta_m r)}{\eta_m}, \quad \eta_m = + \left| k^2 - \left(\frac{m\pi}{\ell} \right)^2 \right|^{1/2} \\ H_{\theta m} &= B_m \cos \frac{m\pi z}{\ell} \frac{R_1(\eta_m r)}{\eta_m}, \quad m = 0, 2, 4, \dots \end{aligned} \quad (6-70)$$

where $\ell = L$, the periodic length, since the fins have zero thickness. If $k \geq |m'\pi/\ell|$ for one of the m' we define $R_0(\eta_m, r)$ so that $E_{zm} = 0$ at $r = b$:

$$R_0(\eta_{m'} r) = J_0(\eta_{m'} r) - \frac{J_0(\eta_{m'} b)}{N_0(\eta_{m'} b)} N_0(\eta_{m'} r)$$

whereupon

$$R_1(\eta_{m'} r) = +J_1(\eta_{m'} r) - \frac{J_0(\eta_{m'} b)}{N_0(\eta_{m'} b)} N_1(\eta_{m'} r) \quad (6-71a)$$

Here J_0 and J_1 are the ordinary Bessel functions of the first kind and N_0, N_1 are the Neumann functions or Bessel functions of the second kind. If, on the other hand, $k \leq |m''\pi/\ell|$ for one of the m'' , we define R_0 as

$$R_0(\eta_{m''} r) = I_0(\eta_{m''} r) - \frac{I_0(\eta_{m''} b)}{K_0(\eta_{m''} b)} K_0(\eta_{m''} r)$$

whereupon

$$R_1(\eta_{m''} r) = +I_1(\eta_{m''} r) + \frac{I_0(\eta_{m''} b)}{K_0(\eta_{m''} b)} K_1(\eta_{m''} r) \quad (6-71b)$$

Here I_0 and I_1 are the modified Bessel functions of the first kind, and K_0, K_1 are the modified functions of the second kind.

When we use these trial functions for the plus fields in Eq. 6-45, and a

similar set with independent amplitudes A_{q-} and B_{m-} for the minus trial fields, the variational expression reduces to

$$0 = \int_D (\hat{e}_{z-}^* \hat{H}_{\theta+} + \hat{E}_{z+} \hat{h}_{\theta-}^*) dz \quad (6-72)$$

because of the locations of the j factors in the trial fields and because the fins have no thickness.

For a first calculation of the ω - β_0 curve let us take the simplest possible trial fields: the one $q = 0$ harmonic of Eq. 6-69 for the internal region and the one $m = 0$ harmonic of Eq. 6-70 for the external region. Upon varying A_{0-} in Eq. 6-72 with ω or k held fixed we obtain this simple dispersion relation:

$$\frac{R_0(ka)I_1(\gamma_0 a)}{R_1(ka)I_0(\gamma_0 a)} = \frac{\gamma_0 a}{ka} \quad (6-73)$$

where R_0 and R_1 are given by Eq. 6-71a for $m' = 0$. This same relation obtains by matching $\hat{e}_z/\hat{h}_\theta$ to $\hat{E}_z/\hat{H}_\theta$ at $r = a$. This relation is not very accurate throughout the passband, as we see in Fig. 6-10, but it is very accurate at the cutoff frequency. Nevertheless, we may not know this until we make the next variational calculation.

For the next calculation let us employ the $q = -1, 0, +1$ harmonics for region I and the $m = 0, 1, 2$ harmonics for region II. Our first impulse is to take the $m = 0, 2, 4$ harmonics of even E_z symmetry in region II

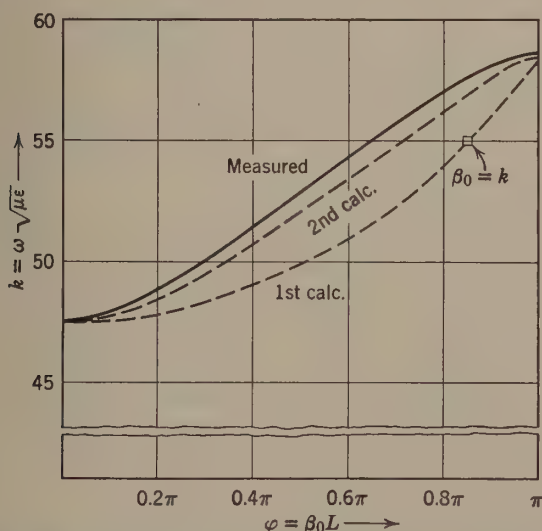


Fig. 6-10. The first dispersion curve of the disk-loaded waveguide of Fig. 1-10. Shown also are several variational calculations.

because the field has this symmetry at both cutoff frequencies. However, the $m = 1$ harmonic is excited at frequencies within the passband and is responsible for most of the correction to the dispersion curve. The new calculated curve on Fig. 6-10 is obtained by trial-and-error solution of a 3×3 determinantal equation resulting from variations of either the A_{q-} 's or the B_{m-} 's. This $k-\beta_0 L$ curve is very much improved over the first one; presumably the next variational calculation which employs the $q = -2, \dots, +2$ and $m = 0, \dots, 4$ harmonics will furnish an extremely accurate curve. Note that we always take the same number of harmonics in each region so as to obtain a square determinant set equal to zero as our dispersion relation.

If, now, the fins have thickness which is not small compared to L , we are advised to proceed as follows. We first determine the A_q amplitudes in terms of the B_m 's so as to make $e_z = 0$ along the fin edges:

$$\sum_{q=-\infty}^{\infty} A_q \begin{Bmatrix} I_0(\gamma_q a) \\ J_0(\gamma_q a) \end{Bmatrix} \epsilon^{-j\beta_q z} = \begin{cases} 0 & \text{over } -t \leq z \leq 0 \\ -j \sum_m B_m \cos \frac{m\pi z}{\ell} R_0(\eta_m a) & \text{over } 0 \leq z \leq \ell \end{cases} \quad (6-74)$$

Then if we multiply both sides by $\epsilon^{j\beta_q z}$, integrate over the interval $(0, L)$, and use the orthogonality of the exponentials, we obtain

$$A_q \begin{Bmatrix} I_0(\gamma_q a) \\ J_0(\gamma_q a) \end{Bmatrix} = -j \frac{\ell}{L} \sum_m B_m R_0(\eta_m a) \epsilon^{+j\beta_q \ell/2} \frac{\beta_q^2}{\beta_q^2 - (m\pi/\ell)^2} \begin{cases} \frac{\sin(\beta_q \ell/2)}{\beta_q \ell/2} & \text{if } m \text{ even} \\ j \frac{\cos(\beta_q \ell/2)}{(\beta_q \ell/2)} & \text{if } m \text{ odd} \end{cases} \quad (6-75)$$

Then our variational Eq. 6-72 reads, with $e_z = E_z$ at $r = a$ and $e_z = 0$ on the fin edges,

$$0 = \int_0^\ell \hat{E}_{z-}^* \cdot (\hat{h}_\theta - \hat{H}_\theta) dz = \frac{j}{\omega \epsilon} \int_0^\ell \sum_m B_{m-}^* \cos \frac{m\pi z}{\ell} R_0(\eta_m a) \cdot \left[j \sum_q A_{q+} \frac{1}{\gamma_q} \begin{Bmatrix} I_1(\gamma_q a) \\ +J_1(\gamma_q a) \end{Bmatrix} \epsilon^{-j\beta_q z} - \sum_{\text{all } n} B_{n+} \cos \frac{n\pi z}{\ell} \frac{R_1(\eta_n a)}{\eta_n} \right] dz \quad (6-76)$$

This must be true for $\delta B_{1-}^* \neq 0$, $\delta B_{2-}^* \neq 0, \dots$ successively and so we obtain a set of homogeneous equations for the B_{m+} 's alone when we substitute Eq. 6-75 for A_{q+} . The determinant of the matrix of coefficients is Hermitian in the sense that its transpose is also its conjugate. This means its expansion will yield only real terms in the dispersion relation.

6-7. Variational Expressions for $j\omega$ and k^2 of a Periodic Cavity Chain

We now wish to write variational expressions for the general symmetrical cavity chain shown in Fig. 6-11 from which we may verify the dispersion relations derived in Chapters 4 and 5. These expressions will justify the idea of averaging the mode expressions for a field on either side of a coupling hole in order to obtain a good representation of that field right *on* the hole. We represent the fields by the resonant cavity modes used in Chapters 4 and 5, the amplitudes of which are the variational parameters. A trivial modification of one of the variational expressions will enable us to derive a well-behaved dispersion relation for certain resonant-slot and resonant-loop coupled structures, to be discussed in Section 7-5.

Equation 6-42 is a variational expression for $j\omega$ of a periodic cavity chain provided the area integrals are changed to integrals over a periodic volume. That volume includes one coupling hole across which we let the amplitudes of the resonant cavity modes phase shift abruptly by $\epsilon^{-j\varphi}$. Some of their field patterns may also reverse direction (see Fig. 4-1c,d). Alternatively the trial fields may have discontinuities across one cavity midplane in the periodic volume, as in the resonant-slot coupled structure. In any case we assume in this section only *one* such discontinuity surface in the volume, either on the coupling surfaces or on the cavity midplanes. The integrands in Eq. 6-42 do not vanish identically as they conveniently did for the helix analyses because each resonant cavity mode used as a trial function satisfies Maxwell's equations only at its resonant frequency and not at the *operating* frequency ω . Furthermore, Eq. 6-42 must include a surface integral on the coupling hole in order to account for the discontinuities there *unless* we choose to redefine the open- and short-circuit cavity modes for continuity in the immediate vicinity of the hole. For example, we can redefine the electric field $v_1\bar{e}_1$ of the first open-circuit cavity mode shown in Fig. 4-1c, so that its amplitude v_1 changes smoothly from v_1 on one side of the hole to $v_1\epsilon^{-j\varphi}$ on the other side, and so that pattern $\bar{e}_1$ changes smoothly from $\bar{e}_1^{(0)}$ in reference cavity (0) to $\bar{e}_1^{(+1)}$ in the next cavity. The result of these redefinitions of the modes near the holes is the appearance of certain volume integrals within a thin pillbox enclosing the hole [3]. But these integrals are actually surface integrals right on the hole and are entirely analogous to the integrals which we have already introduced on discontinuity surfaces of the helices and ridged waveguide. Therefore we shall modify Eq. 6-42 by the appropriate surface integrals.

It is not difficult to prove that the variational expression for $j\omega$ should read

$j\omega =$

$$\frac{\left\{ \int_V (\hat{E}_-^* \cdot \nabla \times \hat{H}_+ - \hat{H}_-^* \cdot \nabla \times \hat{E}_+) dV + \frac{1}{2} \int_D [(\hat{E}_-^{(+)*} + \hat{E}_-^{(-)*}) \cdot \bar{i}_z \times (\hat{H}_+^{(+)} - \hat{H}_+^{(-)}) - (\hat{H}_-^{(+)*} + \hat{H}_-^{(-)*}) \cdot \bar{i}_z \times (\hat{E}_+^{(+)} - \hat{E}_+^{(-)})] dS \right\}}{\int_V (\epsilon \hat{E}_-^* \cdot \hat{E}_+ + \mu \hat{H}_-^* \cdot \hat{H}_+) dV} \quad (6-77)$$

in which V is a periodic volume which includes only one discontinuity or coupling surface D perpendicular to $\bar{i}_z$, and the $(+)$ superscript refers to a trial field on the $+z$ side of D while $(-)$ refers to the $-z$ side. The trial fields $\hat{E}_-$ and $\hat{H}_-$ are independent of $\hat{E}_+$ and $\hat{H}_+$, but all amplitudes phase shift by $\epsilon^{-j\varphi}$ from the $(-)$ to the $(+)$ cavity. Equation 6-77 to be variational must employ trial functions with certain properties which are obtained by expanding the minus set of fields with the open-circuit cavity modes defined by Eq. 5-10 (with $\mu_r = \epsilon_r = 1$) and the plus set of fields with the short-circuit cavity modes of Eq. 5-5. Then we may prove that, if $\delta\{j\omega\} = 0$ for every trial field variation, the resultant set of equations will determine fields which will satisfy Maxwell's equations within V and have the following continuities across surface D :

$$(\hat{E}_-^{(+)} - \hat{E}_-^{(-)}) \times \bar{i}_z = 0 \quad \text{on } D \quad (6-78a)$$

if $\hat{H}_+ \times \bar{i}_z$ is not identically zero on D ;

$$(\hat{H}_+^{(+)} - \hat{H}_+^{(-)}) \times \bar{i}_z = 0 \quad \text{on } D \quad (6-78b)$$

if $\hat{E}_- \times \bar{i}_z$ is not identically zero on D .

We do not prove these statements but rather prove the corresponding ones for the next variational expression. It is interesting to observe that the minus fields composed of a linear combination of open-circuit modes represent only the tangential electric and normal magnetic fields on D ; the plus fields composed of short-circuit modes represent the other two field components.

We can form a variational expression in terms of electric field alone by dot-multiplying the wave equation for $\hat{E}_+$ by $\hat{E}_-^*$:

$$\hat{E}_-^* \cdot (\nabla \times \nabla \times \hat{E}_+ - k^2 \hat{E}_+) = 0$$

and by integrating this over the periodic volume V . The necessary surface integrals on coupling surface D are such as to make the variational

expression for k^2 read

$$k^2 = \frac{\int_V \hat{E}_-^* \cdot \nabla \times \nabla \times \hat{E}_+ dV + \frac{1}{2} \int_D (\hat{E}_-^{(+)*} + \hat{E}_-^{(-)*}) \cdot \bar{i}_z \times (\nabla \times \hat{E}_+^{(+)} - \nabla \times \hat{E}_+^{(-)}) dS}{\int_V \hat{E}_-^* \cdot \hat{E}_+ dV} \quad (6-79)$$

The notation is the same as before. Volume V is assumed to include only one discontinuity surface D . The factor k^2 is associated with a specified value of phase shift φ . This expression is variational if we also expand $\hat{E}_-$ with the open-circuit cavity modes and $\hat{E}_+$ with the short-circuit modes. Let us now prove why.

It is easy to see with the help of rule 6-17 that if $\delta k^2 = 0$ for every variation $\delta \hat{E}_-$ within V and on D , the $\hat{E}_+$ field must satisfy the wave equation within V and have continuous tangential $\nabla \times \hat{E}_+$ component on D :

$$(\nabla \times \hat{E}_+^{(+)} - \nabla \times \hat{E}_+^{(-)}) \times \bar{i}_z = 0 \quad \text{on } D,$$

provided $\hat{E}_- \times \bar{i}_z$ is not identically zero on D . We guarantee this condition by expanding $\hat{E}_-$ with the open-circuit modes. Now suppose we set $\delta k^2 = 0$ with respect to an arbitrary variation $\delta \hat{E}_+^{(-)}$ on the $-z$ side of D . Then we must invert the first integral by means of Green's theorem, Eq. 3-43b, to read

$$\begin{aligned} & \int_V \hat{E}_-^* \cdot \nabla \times \nabla \times \delta \hat{E}_+^{(-)} dV \\ &= \int_{V-} \delta \hat{E}_+^{(-)} \cdot \nabla \times \nabla \times \hat{E}_-^* dV \\ &+ \oint_S (\delta \hat{E}_+^{(-)} \times \nabla \times \hat{E}_-^* - \hat{E}_-^* \times \nabla \times \delta \hat{E}_+^{(-)}) \cdot \bar{n} dS \quad (6-80) \end{aligned}$$

The surface integral over S will have a portion over the left cross-sectional boundary of the volume and another over the coupling hole D . However, the first portion cancels a similar surface integral on that boundary a period L away. This follows from the fact that variation $\delta \hat{E}_+$ must be periodic, whereupon $\delta \hat{E}_+^{(-)} \times \nabla \times \hat{E}_-^* \cdot (-\bar{i}_z)$ on the left cross-sectional boundary cancels $\delta \hat{E}_+^{(-)} \epsilon^{-j\varphi} \times \nabla \times \hat{E}_-^* \epsilon^{+j\varphi} \cdot \bar{i}_z$ on the boundary a period away. A similar statement holds for the second portion of the surface integral in Eq. 6-80. Also we know that $\delta \hat{E}_+^{(-)} \times \bar{n} = 0$ on D because $\hat{E}_+$ is expanded in short-circuit modes. Therefore, Eq. 6-80 reduces to

$$\begin{aligned} & \int_V \hat{E}_-^* \cdot \nabla \times \nabla \times \delta \hat{E}_+^{(-)} dV \\ &= \int_V \delta \hat{E}_+^{(-)} \cdot \nabla \times \nabla \times \hat{E}_-^* dV - \int_D \hat{E}_-^{(-)*} \times \nabla \times \delta \hat{E}_+^{(-)} \cdot \bar{i}_z dS \quad (6-81) \end{aligned}$$

Therefore, when we take a variation $\delta\hat{E}_+^{(-)}$ on the $-z$ side of discontinuity surface D in Eq. 6-79 and use Eq. 6-81, we obtain, with the aid of rule 6-17,

$$\begin{aligned} 0 = & \int_V \delta\hat{E}_+^{(-)} \cdot (\nabla \times \nabla \times \hat{E}_-^* - k^2 \hat{E}_-^*) dV \\ & - \int_D \hat{E}_-^{(-)*} \times \nabla \times \delta\hat{E}_+^{(-)} \cdot \bar{i}_z dS \\ & + \frac{1}{2} \int_D (\hat{E}_-^{(+)*} + \hat{E}_-^{(-)*}) \cdot \bar{i}_z \times (-\nabla \times \delta\hat{E}_+^{(-)}) dS \quad (6-82) \end{aligned}$$

Rearrangement of the last two integrals shows they can be written as

$$\frac{1}{2} \int_D (\hat{E}_-^{(-)*} - \hat{E}_-^{(+)*}) \times \bar{i}_z \cdot \nabla \times \delta\hat{E}_+^{(-)} dS$$

Since $\delta\hat{E}_+^{(-)}$ is arbitrary on the $-z$ side of D we see that $\hat{E}_-$ will be constrained to satisfy the wave equation within V and to possess continuity of tangential component on D , provided $(\nabla \times \delta\hat{E}_+) \times \bar{i}_z$ is not identically zero on D . But we have guaranteed this condition by expanding $\hat{E}_+$ in the short-circuit mode set.

The proof of the variational properties for an arbitrary variation $\delta\hat{E}_+^{(+)}$ on the $+z$ side of D is entirely analogous. The $\hat{E}_+$ and $\hat{E}_-$ fields each represent one component of the true electric field on D , as mentioned just after Eq. 6-78. According to the reasoning in Section 5-4 we may expand $\hat{E}_+$ and $\hat{E}_-$ with the solenoidal modes alone and neglect the irrotational ones which do not determine the "cold" circuit dispersion.

Now let us use Eq. 6-79 to derive the single-passband Eq. 4-11 and 4-18, which were obtained by expanding the fields of Maxwell's equations with the one short-circuit mode of the passband. For the n th passband we let $\hat{E}_- = v_n \bar{e}_n$, $\hat{E}_+ = V_n \bar{E}_n$, and take the periodic volume V to include coupling surface ($M-$) on Fig. 6-11. Then Eq. 6-79 reads

$$\begin{aligned} & \left\{ \int_{V-} (v_n \epsilon^{j\varphi})^* \bar{e}_n \cdot \nabla \times \nabla \times V_n \epsilon^{j\varphi} \bar{E}_n dV \right. \\ & \quad \left. + \int_{V+} v_n \bar{e}_n \cdot \nabla \times \nabla \times V_n \bar{E}_n dV \right. \\ & \quad \left. + \frac{1}{2} \int_{M-} (v_n^* \bar{e}_n^{(0)} + v_n^* \epsilon^{-j\varphi} \bar{e}_n^{(-1)}) \cdot \bar{i}_z \times (\nabla \times V_n \bar{E}_n^{(0)} - \nabla \times V_n \epsilon^{j\varphi} \bar{E}_n^{(-1)}) dS \right\} \\ k^2 = & \frac{\quad}{\int_V \hat{E}_-^* \cdot \hat{E}_+ dV} \quad (6-83) \end{aligned}$$

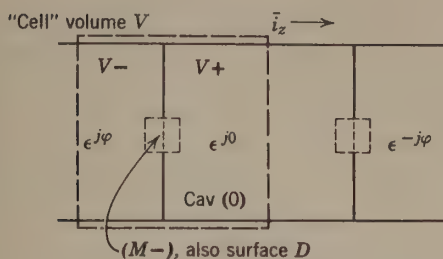


Fig. 6-11. Notation for the general symmetrical cavity chain.

Here $V-$ denotes that portion of the volume to the left of $(M-)$ in cavity (-1) and $V+$ the remaining portion of the volume to the right of $(M-)$ in reference cavity (0) . Because of the periodicity, we may as well replace the $\int_{V-}$ integral by one over the remainder of cavity (0) since the phases cancel. Then we can write just one volume integral over cavity (0) . We recall from the mode definitions that $\nabla \times \nabla \times \vec{E}_n = P_n^2 \vec{E}_n$ and $\nabla \times \vec{E}_n = P_n \vec{H}_n$. According to Eqs. 5-1 and 5-2 $\vec{e}_n$ changes direction across $(M-)$ for an even mode but maintains direction for an odd one. However, $\vec{H}_n$ reverses direction across $(M-)$ only for an odd mode. Therefore we can write k^2 as

$$k^2 = \frac{v_n^* V_n P_n^2 \int_V \vec{e}_n \cdot \vec{E}_n dV + v_n^* V_n P_n (1 \mp \cos \varphi) \int_{M-} \vec{e}_n^{(0)} \cdot \vec{i}_z \times \vec{H}_n^{(0)} dS}{v_n^* V_n \int_V \vec{e}_n \cdot \vec{E}_n dV} \quad (6-84)$$

(upper sign, even mode-pair; lower sign, odd mode-pair).

If we set $\delta k^2 = 0$ for a variation δv_n^* or δV_n we obtain, with the definition of coupling coefficient M_{nn} in Eqs. 4-5 and 4-16,

$$(P_n^2 - k^2) \frac{1}{\tau} \int_V \vec{e}_n \cdot \vec{E}_n dV = P_n M_{nn} (1 \mp \cos \varphi) \quad (6-85)$$

which is dispersion Eq. 4-11 or 4-18 of Chapter 4. Furthermore, the variational derivation has removed the ambiguity of relating the two expressions $V_n \vec{E}_n$ and $v_n \vec{e}_n$ for electric field within the cavity. The proper relation between the V_n amplitudes in a short-circuit mode expansion of the fields, as excited by tangential electric field on the coupling holes, is

$$V_n = \sum_m T_{nm} v_m, \quad T_{nm} = \frac{1}{\tau} \int_V \vec{e}_m \cdot \vec{E}_n dV \quad (6-86)$$

To obtain the general dispersion equation we expand $\hat{E}_-$ as $\Sigma v_n \bar{e}_n$ and $\hat{E}_+$ as $\Sigma V_n \bar{E}_n$ in Eq. 6-79 and obtain one homogeneous equation for each δv_n^* or δV_n . The determinant of the set of δv_n^* equations set equal to zero is just the complex conjugate of the determinant we obtain from the mode expansion procedure. But the conjugation has no effect on the determinantal equation, all terms of which are real. The determinant of the set of δV_n equations in terms of the v_n 's set equal to zero is identical to the determinant we obtain from the mode expansion procedure. We may also say that conjugation has the effect of changing φ to $-\varphi$, both of which belong to a given k^2 .

In analogy to Eq. 6-79 we can write a variational expression for k^2 in terms of magnetic field alone as

$$k^2 = \frac{\int_V \hat{H}_-^* \cdot \nabla \times \nabla \times \hat{H}_+ dV + \frac{1}{2} \int_D (\hat{H}_-^{(+)*} + \hat{H}_-^{(-)*}) \cdot \bar{I}_z \times (\nabla \times \hat{H}_+^{(+)} - \nabla \times \hat{H}_+^{(-)}) dS}{\int_V \hat{H}_-^* \cdot \hat{H}_+ dV} \quad (6-87)$$

To guarantee variational properties we expand $\hat{H}_-$ with the short-circuit modes and $\hat{H}_+$ with the open-circuit modes. All the modes have the same characteristics as they did for the preceding expression. Verification of the variational nature of Eq. 6-87 is dual to the reasoning following Eq. 6-79. By representing $\hat{H}_-$ as $I_n \bar{H}_n$ and $\hat{H}_+$ as $i_n \bar{h}_n$ at a frequency in the narrow n th passband we may derive the single-passband Eq. 4-42. The variational derivation removes the ambiguity, Eq. 4-39, in relating these two representations of magnetic field. The proper general relation is

$$i_n = \sum_m U_{mn} I_m, \quad U_{mn} = \frac{1}{\tau} \int_V \bar{H}_m \cdot h_n dV \quad (6-88)$$

More generally, we may expand $\hat{H}_-$ as $\Sigma I_n \bar{H}_n$ and $\hat{H}_+$ as $\Sigma i_n \bar{h}_n$ in Eq. 6-87, in solenoidal modes alone according to the reasoning in Section 5-4. Upon setting $\delta k^2 = 0$ with respect to every δi_n we obtain a homogeneous set of equations for the I_n 's, the determinant of which is the extended version of Eq. 5-56 or 5-69. The determinant from the set of δI_n^* equations is the complex conjugate of this. As we just said, however, the complex conjugate is trivial.

The preceding variational equations depended upon the use of trial function fields with discontinuities across just one surface, D , in the periodic volume V . Certain resonant-slot and resonant-loop coupled structures have one passband characterized by resonant modes which phase shift across the slot- or loop-bisecting planes and another passband

characterized by resonant modes which phase shift across the cavity midplanes. We discussed the properties of one such resonant-slot coupled structure briefly in Section 4-3. In order to obtain a proper dispersion relation in terms of the four resonant modes of these two passbands, we could use Eq. 6-77. It is more convenient, however, to work with the electric field alone, in which case we must modify Eq. 6-79 to account for more general discontinuities. The properly generalized expression is

$$k^2 = \frac{\left\{ \int_V \hat{E}_-^* \cdot \nabla \times \nabla \times \hat{E}_+ dV + \frac{1}{2} \int_D (\hat{E}_-^{(+)*} + \hat{E}_-^{(-)*}) \cdot \hat{i}_z \times (\nabla \times \hat{E}_+^{(+)} - \nabla \times \hat{E}_+^{(-)}) dS - \frac{1}{2} \int_D (\hat{E}_+^{(+)} - \hat{E}_+^{(-)}) \cdot \hat{i}_z \times (\nabla \times \hat{E}_-^{(+)*} + \nabla \times \hat{E}_-^{(-)*}) dS \right\}}{\int_V \hat{E}_-^* \cdot \hat{E}_+ dV} \quad (6-89)$$

In this expression D denotes one or more surfaces across which there may be one or more discontinuities of $\hat{E}_\pm \times \hat{i}_z$ and $(\nabla \times \hat{E}_\pm) \times \hat{i}_z$. This expression has all the variational properties of Eq. 6-79 plus a few more. If $\delta k^2 = 0$ for all arbitrary variations of $\hat{E}_\pm$ within V and on D , not only do $\hat{E}_\pm$ satisfy the vector wave equation within V ,

$$\nabla \times \nabla \times \hat{E}_\pm - k^2 \hat{E}_\pm = 0 \quad (6-90)$$

but also $\hat{E}_\pm \times \hat{i}_z$ and $\hat{i}_z \times \nabla \times \hat{E}_\pm$ are constrained to be continuous across any discontinuity surface D provided $\hat{i}_z \times \nabla \times \hat{E}_\mp$ and $\hat{E}_\mp \times \hat{i}_z$, respectively, are not identically zero on D . The proof of these statements is not difficult with the discussion following Eq. 6-79 as a guide.

We employ Eq. 6-89 in the discussion of the resonant-slot and resonant-loop coupled structures in Sections 7-5 and 7-6.

6-8. Critique of the Variational Analyses

The variational analyses of the helical and ridged waveguide structures originated with variational Eq. 6-45, and the analyses of the cavity chains stemmed from any one of the expressions just quoted for $j\omega$ or k^2 . These analyses have both desirable and undesirable properties, which we shall now summarize.

Among the desirable properties, the most valuable one is the ability of the variational technique to describe very general helices and cavity chains straightforwardly, without becoming involved in complicated boundary-matching procedures. These analyses have the property of yielding an

accurate dispersion relation with judicious choice of just a few modes as trial fields. Sometimes we can improve the calculations by writing known explicit field continuities on the discontinuity surfaces, as we did for the $(+ -)$ ring-plane helix wave in Section 6-4. We can also straightforwardly analyze the effects of geometrical perturbations, which appear as modifications to the integrals in the variational expressions. For example, the effect of a thin dielectric rod of scalar permittivity $\epsilon \neq \epsilon_0$ outside of and parallel to a helix shows up as an additional integral in Eq. 6-45,

$$\int_{\text{rod}} \hat{E}_-^* \cdot [-\{j\omega\}(\epsilon - \epsilon_0)] \hat{E}_+ dS$$

if the trial fields for $\hat{E}_{\pm}$ are each defined so as to satisfy Maxwell's equations at frequency ω with respect to ϵ_0 over the cross section. The variational method has further merit in the fact that a calculation of the dispersion may very easily be improved by including more modes to represent the fields.

Among the undesirable properties, perhaps the worst is the inability to tell how accurate a dispersion relation is, without recourse to measurement, before deriving an improved relation. In Section 12-4 we outline a proof that Eq. 6-45 actually is an upper-bound expression for $\omega(\beta_0)$ of a general helix; and calculations made for a ring-plane helix indicate that reasonable trial fields yield a tight upper bound for certain helices. However, we cannot make this latter statement for general helices, nor have we found an associated lower-bound expression, also presented in Section 12-4, to be useful. Equation 6-45, moreover, does not furnish an upper-bound expression for $\omega(\varphi)$ of a cavity chain with abrupt changes of phase in the trial fields across the discontinuity surfaces. Nor do Eqs. 6-77, 6-79, or 6-87 furnish either an upper or a lower bound to $\omega(\varphi)$ of a cavity chain.

Certain helpful explicit continuities on discontinuity surfaces are not always obvious, as was the case when we had to decide whether to write explicit continuity of $\hat{e}_{\perp}$ or of $\hat{e}_{\parallel}$ for the tape helix $n = 0$ mode analysis in Section 6-3D. Lastly, we must always remember that a variationally calculated dispersion curve is invariably more accurate than the Pierce impedance calculated from the corresponding trial fields at a given frequency on that curve. We found that the first estimates of impedance could be erroneous by factors of 2 or more.

Problems

1. Show that Eqs. 6-34 and 6-40 are variational in the sense that small errors in the voltages and currents will create errors in $j\omega$ or ω^2 proportional to the squares of the voltage and current errors.

2. Determine numerically the first π cutoff frequency of the transmission line of Prob. 5, Chapter 4, using Eq. 6-34. Let $I_+ = I_0 + I_2 \cos 2\pi z/L$, $I_- = I_0' + I_2' \cos 2\pi z/L$, $V_+ = V_1 \sin \pi z/L$, and $V_- = V_1' \sin \pi z/L$, where $I_0, \dots, V_1'$ are variational amplitudes. Remember $\int C(z) dz$ across C_L is equal to C_L . The exact answer is 1.50 mKc.
3. Show that Eq. 6-45 is variational with respect to $\delta \hat{e}_+$, for example, and that we cannot reverse the signs of $\hat{E}_\pm$ and $\hat{H}_\pm$ and still have a variational expression.
4. Verify the remark at the end of Section 6-3B to the effect that the modified Eq. 6-45 does yield the correct dispersion relation for the sheath helix.
5. Verify by using Eq. 6-55 that the dispersion of the tape helix wave with predominant $n = +1$ mode near the first forbidden region is very insensitive to the relative tape width t .
6. Verify the remark on page 179 to the effect that the enforced continuity $\hat{e}_\perp = \hat{E}_\perp$ for the tape helix wave of predominant $n = 0$ mode leads to the wrong asymptotic dependence of the dispersion upon pitch angle ψ .
7. Verify the fact that each mode in Eq. 6-58 and 6-59 is a proper linear combination of sheath modes. How would the expressions be augmented to include harmonics of axial propagation constant $\beta_1 = \beta_0 + 2\pi/p$?
8. Develop a variational expression suitable for obtaining $\omega(\beta_0)$ for the tape ladder line shown in Fig. 6-12. Assume the slot is nearly a half wavelength long so that $e_z = \sin \pi y/\ell$ on the slot surface and zero on the metal surface. For trial fields use a set of mode-harmonics in regions I and I' with their



Fig. 6-12

amplitudes evaluated by the slot field and a compatible set of trial fields with unknown amplitudes in regions II, II', III, and III'. Then make ω stationary for variations in these latter amplitudes and determine the dispersion relation.

9. Demonstrate the fact that below $ka \cong 0.88$ dispersion Eq. 6-61 for the $(++)$ wave of the ring-plane helix has only the solution $ka = \beta a$. $W/G = 1$.
10. Extend the analysis of Section 6-6 to include the $q = \pm 2$ and $m = 3, 4$ harmonics in the analysis of the fin-loaded waveguide of zero-thickness fins (Fig. 6-9).
11. Show that the single mode-pair dispersion relation obtained from Eq. 6-77 (with the fields expanded according to remarks following it) agrees with Eq. 6-85.
12. Modify the numerator of Eq. 6-79 so as to obtain a variational expression

for the resonant frequency of a cavity in which a small, perfectly conducting object is a perturbation. Then derive the Slater perturbation formula

$$\frac{k^2 - P_0^2}{P_0^2} = \frac{1}{\tau} \int_{\Delta V} (\bar{H}_0^2 - \bar{E}_0^2) dV$$

in terms of P_0 , $\bar{H}_0$, and $\bar{E}_0$ of the resonant mode in the absence of the object. $\Delta V \ll \tau$ is the volume of the object. (See J. C. Slater [11].) Is this expression accurate in terms of just this resonant cavity mode?

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Chapter VII

Resonant-Slot and Resonant-Loop Coupled Structures

7-1. Synopsis

Now that we have treated a wide variety of open helical structures we turn to some of their counterparts—the resonant-slot and resonant-loop coupled cavity chains, including the centipede and cloverleaf structures. The resonant-slot coupled chain discussed earlier in Section 4-3 is distinguished by having one of its first two passbands characterized by a pair of “slot” modes defined within a unit cell centered around each slot. At the cutoff frequency of the open-circuit “slot” mode each slot is very nearly half a free-space wavelength long. The other passband is described by the usual “cavity” modes. The method of Allen and Kino is invoked to derive the dispersion relation for both these passbands. In this method the fields within the cavities proper are expanded with the first solenoidal short-circuit cavity mode (and perhaps a few irrotational modes for the magnetic field) as excited by tangential electric field on the slots. This latter electric field propagates along the slot like a voltage along a transmission line, excited by tangential magnetic field of the cavity modes. The resultant dispersion relation is corrected for the presence of centerholes. The derivation yields the relative proportions of the cavity solenoidal and irrotational modes with which the interaction parameter $E_z(0)^2/W_s$ can be quite accurately calculated.

We next use the one solenoidal cavity mode and several irrotational modes to represent the fields in a structure with successive slots rotated 90° . At frequencies in the “slot” passband the irrotational modes are by no means negligible. We obtain from a variational expression in terms of slot voltage alone a dispersion curve for this passband which is closely aligned with the measured one.

The propagation in a resonant-slot or resonant-loop coupled structure (with just a few loops) can be described with the aid of a variational expression for electric field alone, in which we employ both the "slot" and the "cavity" modes of the first two passbands. We first derive a proper dispersion relation for the slot-coupled structure which has all the desirable properties except that it neglects the crosspower between both kinds of modes in the relation, power flow equals stored energy per unit length times group velocity. It does, however, describe both ω - φ curves of the first two passbands quite well in terms of five parameters, several of which are difficult to calculate and should be chosen so as to "fit" the theoretical and measured curves at several points. The dispersion relation shows that if the first dispersion curve is characterized by the "cavity" modes it will be depressed somewhat because of the coupling with the higher-frequency "slot" modes.

The aforementioned dispersion relation also describes the propagation in a chain of cavities coupled by U-shaped loops. A. F. Pearce labels this coupling "positive mutual" because the magnetic field predominates on the coupling surfaces near the outer cavity walls. However, a structure coupled by S-shaped loops has a better dispersion curve for beam interaction purposes in the cavity-mode passband. So we describe the properties of the four resonant modes in a chain coupled by S-shaped loops and derive an alternate dispersion relation. This relation shows that if the first dispersion curve is characterized by the "cavity" modes, it will be depressed somewhat by the effect of the higher-frequency "loop" modes.

This dispersion relation for S-loop structures describes quite well the measured ω - φ curves for several of Pearce's 2 and 4 loop-pair chains. However, it fails to describe the 8 and 12 loop-pair chains, including the centipede; for all of these we use a modified dispersion relation.

Pearce's curves of $\Delta f/f$ proportional to the interaction parameter $E_z(0)^2/W_s$ are presented for various numbers of pairs of S-loops.

The centipede chain has twelve or so loop-pairs in a complete ring around the centerhole near the outer walls. It is designed for noninversion; that is, a first passband described by "cavity" modes at both the $\varphi = 0$ and π cutoff frequencies. We justify a modified dispersion relation in terms of only five parameters by the close agreement between theory and measurement for typical structures.

Lastly we describe the character of the cloverleaf slow wave structure, in which the "cavity" modes interact with the higher-frequency "slot" modes in such a way that the first dispersion curve is suitable for broad-band microwave amplifiers in a frequency range centered near the $\varphi = 2\pi/3$ frequency.

7.2. A Resonant-Slot Coupled Chain

In Section 4-3 we discussed qualitatively the dispersive properties of the cavity chain coupled by long slots, the length of which is about half a free-space wavelength in the frequency range of the first two passbands. Figure 4-2 on page 104 shows one such structure, in which the open- and short-circuit "slot" modes defined between magnetically and electrically shorted *cavity midplanes*, respectively, resonate at the cutoff frequencies of the "slot" passband. Figure 7-1 shows another such chain. According to the developments of Sections 4-2 and 4-6 either of these modes represents, in the single mode-pair approximation, the fields at frequencies within the "slot" passband. At these frequencies, we can quite accurately say the field amplitudes phase shift abruptly by $\epsilon^{\pm j\phi}$ across the cavity midplanes, provided we average the mode expressions on both sides of each midplane in order to obtain accurate fields *on* the midplanes. In the Fig. 7-1 structure the slots are relatively long, so that the narrow "slot" passband, one end of which coincides with the half-wavelength frequency of a slot, lies below the "cavity" passband (Fig. 7-4). The "cavity" passband is so labeled because the pair of resonant modes defined between electrically and magnetically shorted slots and centerholes (Fig. 4-1*b, c*) describes the fields and qualitative shape of this passband according to the discussion of Sections 4-2 and 4-6. In other such structures the slots are shorter, the "slot" passband lies above the "cavity" passband, and the first dispersion curve is not only considerably widened by the presence of the slots but also shifted to the right so as to favor interaction with an electron beam which can be kept nearly synchronous with the "cold" circuit over a wider portion of the dispersion curve.

Despite the fact that we can, in principle, analyze the dispersive behavior of the typical symmetrical structure of Fig. 7-1 with *either* the open- and short-circuit "cavity" modes *or* the open- and short-circuit "slot" modes

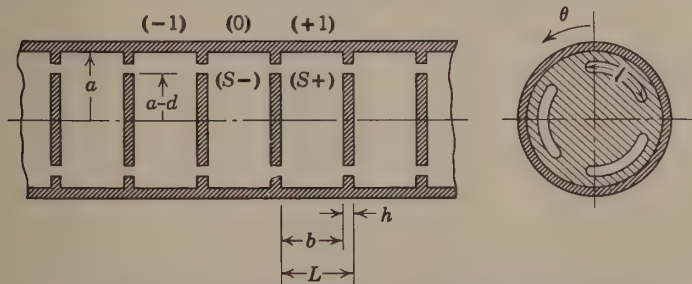


Fig. 7-1. A resonant-slot coupled cavity chain (from Allen and Kino [1]).

(excluding the irrotational modes in either case according to the discussion of Section 5-4) it proves more expedient to determine the curves as follows. We expand the fields within each cavity proper with the short-circuit "cavity" modes, excited on the slots and centerholes by tangential electric field. Practical structures utilize long narrow slots, along which the tangential electric field may be regarded as propagating just as voltage propagates along a TEM transmission line. The tangential magnetic field of the short-circuit "cavity" modes excites the slot tangential electric field much as external current drives the transmission line. The tangential electric field on the centerholes may be represented in the usual way by the average of the "cavity" open-circuit mode expressions for it on either side. The slot modes are too weak on the centerholes to represent tangential electric field there. In this way we can obtain as many amplitude relations as there are "cavity" modes, and the consistency of these relations furnishes the dispersion equation for both passbands. In this procedure the irrotational "cavity" modes for the magnetic field are excited and, in turn, help to excite the near-slot electric field. Fortunately, the influence of these modes upon the "cavity" dispersion curve is rather small, but we usually must include a few of them in order to obtain an accurate "slot" curve. Presently we use them to advantage in our treatment of the slot-coupled chain with consecutive slots rotated 90° along the chain.

With this survey of the situation let us now proceed to calculate the dispersion relation for the structure of Fig. 7-1 following the method of Allen and Kino [1] but with some changes in their notation. Initially we assume no centerholes present and later introduce a corrective term for them. The slots are aligned along the chain so that the periodicity operation is just a translation by L . Within the cavity of reference phase (0) we expand the fields of Maxwell's equations with the short-circuit "cavity" modes as in Eq. 5-19, each mode defined by Eq. 5-5 (with $\mu_r = \epsilon_r = 1$) and obtain Eqs. 5-20 and 5-21 in which the symbol ($M\pm$) is to be replaced by ($S\pm$) designating the slots. The expressions for the cavity solenoidal fields excited by tangential $\hat{e}$ on the slots are therefore

$$\hat{E} = \sum V_n \bar{E}_n, \quad V_n = - \frac{P_n}{P_n^2 - k^2 \tau} \frac{1}{\tau} \int_{S\pm} \hat{e} \times \bar{H}_n^{(0)} \cdot \bar{n} dS \quad (7-1a)$$

$$\hat{H} = \sum I_n \bar{H}_n, \quad I_n = \frac{j\omega\epsilon V_n}{P_n} \quad (7-1b)$$

These equations do not correspond identically to Allen and Kino's Eqs. (4) and (5), because their normal modes are defined differently. From Eq. 5-21 we also obtain the irrotational magnetic field (there is no

irrotational electric field in the short-circuit mode representation) as

$$\hat{H}^i = \sum I_n^i \bar{H}_n^i, \quad I_n^i = -(j\omega\mu\tau)^{-1} \int_{S\pm} \hat{e} \times \bar{H}_n^{i(0)} \cdot \bar{n} dS \quad (7-2)$$

With the cavity fields proper thus determined by tangential $\hat{e}$ on the slots we now proceed to solve for this $\hat{e}$ as excited by tangential cavity magnetic field, $\hat{H} + \hat{H}^i$ along the slots. A narrow slot of $d \ll \lambda_0/2$ in Fig. 7-2 behaves like a TEM transmission line in the x direction of its long dimension. With the definition of slot voltage as $\phi(x) \triangleq \int \hat{e}_y(x, y) dy$, the differential equation for voltage $\phi_{S-}(x)$ on the left or ($S-$) slot turns out to be

$$\left(\frac{\partial^2}{\partial x^2} + k^2 \right) \phi_{S-}(x) = -j\omega L_0 (\hat{H}_{x, \text{tot}}^{(0)} - \hat{H}_{x, \text{tot}}^{(-1)}), \quad k^2 = \omega^2 \mu \epsilon \quad (7-3)$$

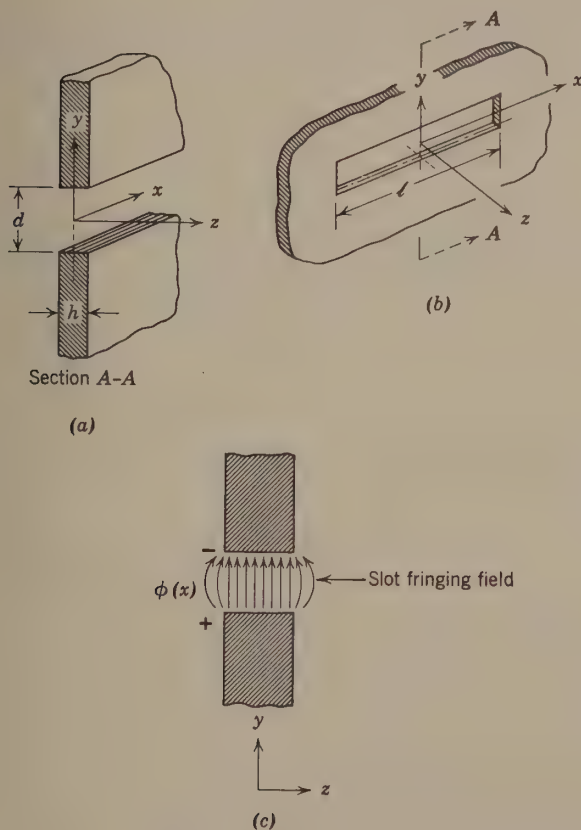


Fig. 7-2. Slot geometry and voltage $\phi(x)$ along the slot (from Allen and Kino [1]).

with an analogous expression for $\phi_{S+}(x)$ on the $(S+)$ slot. L_0 is the slot TEM inductance per unit length in the x direction. The slot capacitance per unit length, C_0 , is easier to estimate from an analytic expression for the voltage of the thin-walled slot of Fig. 7-3. The analytic expression for potential $u(\eta, \xi) \equiv \phi$ in the plane of the paper is [2]

$$u + jv = \cos^{-1} \left(\frac{\eta + j\xi}{d} \right)$$

The potential difference between the $\eta = 0$ plane (perfectly conducting) and the vertical wall is $\pi/2$. The charge per unit length into the paper on the $\eta = 0$ plane is

$$\begin{aligned} q &= \epsilon \int_{-md}^{md} -\frac{\partial u}{\partial \eta} d\xi = \epsilon \int \frac{\partial v}{\partial \xi} d\xi = \epsilon [v(0, md) - v(0, -md)] \quad (7-4) \\ &= 2\epsilon \sinh^{-1} m \end{aligned}$$

Here m measures the effective slot width in the ξ direction; $m = 8$ is a good practical value. Hence the capacitance per unit length of this slot as a TEM line is

$$C_0 = \frac{4}{\pi} \epsilon_0 \sinh^{-1} m \quad (\eta \geq 0 \text{ region of Fig. 7-3}) \quad (7-5a)$$

Now if the slot is more than several slot widths from a cavity wall, as it often is, the $\eta = 0$ plane of Fig. 7-3 merely serves as a symmetry plane for the potential; the charge q on either slot plane remains the same but the potential between them doubles.

Then C_0 halves:

$$C_0 = \frac{2}{\pi} \epsilon_0 \sinh^{-1} m$$

(no conducting plane in Fig. 7-3)

(7-5b)

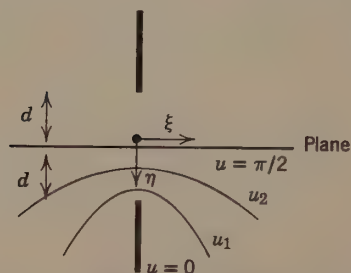


Fig. 7-3. Electrostatic potential $u(\eta, \xi)$ near a slot.

For $m = 8$ a good estimate of C_0 for this latter slot is $C_0 \cong 5.6 \epsilon_0/\pi$ farad/m. Knowing C_0 we find L_0 for Eq. 7-3 as $L_0 = \mu_0 \epsilon_0 / C_0$.

Now we have, in principle, enough information in Eqs. 7-1 through 7-5 with which to obtain the dispersion relation for both "cavity" and "slot" passbands. Allen and Kino show that we can obtain an excellent relation for most structures of interest in which the long slots lie along the θ coordinate of Fig. 7-1, by making these assumptions: (a) The cavity field is adequately represented by the first short-circuit $n = 1$ solenoidal mode in Eq. 7-1, (b) the irrotational

field of Eq. 7-2 may be ignored, and (c) the $(\bar{H}_1)_\theta$ mode pattern is uniform across width d of the slots. For a narrow slot, $d \ll \ell$ and the θ direction is synonymous with the $-x$ direction in Figs. 7-1 and 7-2.

We now derive the approximate dispersion relation. The normalized $(\bar{H}_1)_\theta = -\bar{H}_{1x}$ pattern is $\sqrt{3.71}J_1(P_1r_s)$, r_s being the mean slot radius, which does not change direction from one slot surface to the next according to Fig. 4-1b. Equation 7-3 therefore reads

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right)\phi_{S-}(x) = -j\omega L_0 I_1 \bar{H}_{1x}(1 - \epsilon^{+j\varphi}) \quad (7-6)$$

which is satisfied by

$$\phi_{S-}(x) = +j\omega L_0 I_1 \bar{H}_{1x}(1 - \epsilon^{j\varphi}) \frac{\cos kx - \cos(k\ell/2)}{k^2 \cos(k\ell/2)} \quad (7-7)$$

This correctly reduces to zero at the slot ends, $x = \pm\ell/2$. The expression for $\phi_{S+}(x)$ is the same but with $\epsilon^{-j\varphi} - 1$ in place of $1 - \epsilon^{j\varphi}$.

With $\phi_{S\pm}(x)$ known we can evaluate the integral on the (S-) slot of mean radius r_s as

$$\begin{aligned} \int_{S-} \hat{e} \times \bar{H}_1^{(0)} \cdot \bar{n} dS \\ = + \int_y \int_x \hat{e}_y(x, y) \bar{H}_{1x} dS = H_{1x} \int_{-\ell/2}^{\ell/2} \phi_{S-}(x) dx \\ = +j\omega L_0 I_1 H_{1x}^2(r_s)(1 - \epsilon^{j\varphi})k^{-3} \left(2 \tan \frac{k\ell}{2} - k\ell\right) \end{aligned} \quad (7-8)$$

The expression for

$$\int_{S+} \hat{e} \times \bar{H}_1^{(0)} \cdot \bar{n} dS$$

on slot (S+) replaces $1 - \epsilon^{j\varphi}$ by $-\epsilon^{-j\varphi} + 1$.

Thus, Eq. 7-1 yields a relation for magnetic field amplitude I_1 in terms of itself as

$$I_1 = -\frac{j\omega\epsilon}{(P_1^2 - k^2)\tau} \left[j\omega L_0 I_1 H_{1x}^2(1 - \epsilon^{j\varphi} - \epsilon^{-j\varphi} + 1)k^{-3} \left(2 \tan \frac{k\ell}{2} - k\ell\right) \right] S \quad (7-9)$$

Here S stands for the number of slots in a given slot plane of Fig. 7-1, three in this instance. Equation 7-9 rearranges into the following dispersion relation for $k(\varphi)$,

$$\begin{aligned} \frac{(k/P_1)[1 - (k/P_1)^2]}{\tan(k\ell/2) - k\ell/2} &= \alpha_\pi \sin^2 \frac{\varphi}{2} \\ \alpha_\pi &= \frac{8H_{1x}^2(r_s)(L_0/\mu_0)S}{P_1^3\tau} \end{aligned} \quad (7-10)$$

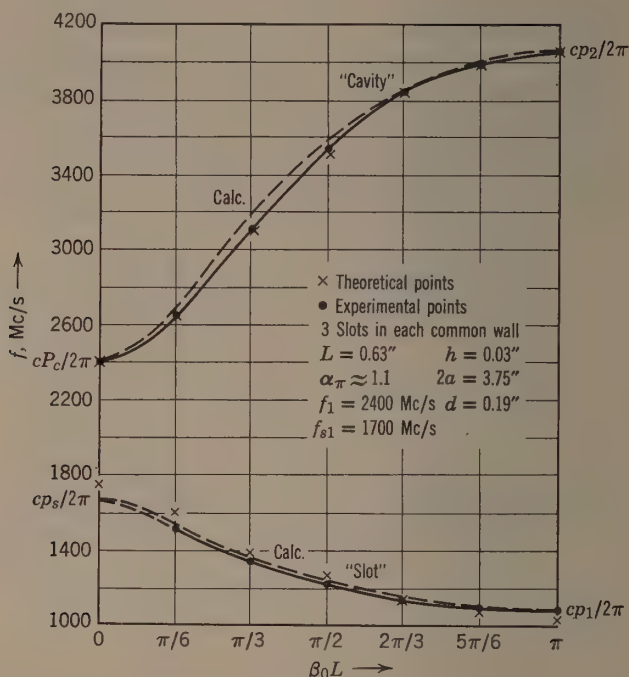


Fig. 7-4. The first two dispersion curves for a particular resonant-slot coupled chain. The theoretical points were calculated by Eq. 7-10. The calculated curves (dashed) were obtained by Eq. 7-26, with the m_s parameter chosen to "fit" the upper $2\pi/3$ point.

Here τ is the cavity volume, as always, and the other symbols have been introduced. Equation 7-10 is equivalent to Allen and Kino's Eq. (31). If each slot of Fig. 7-1 is cut very near the outer circumference, then $r_s \cong a$, and $H_{10} \cong \sqrt{3.71} J_1(2.40) \cong 1.0$, and the expression for coefficient α_π simplifies to

$$\alpha_\pi = 0.18 \frac{L_0}{\mu_0} \frac{a}{L} S = 0.18 \frac{Z_0}{\sqrt{\mu_0/\epsilon_0}} \frac{a}{L} S$$

A typical pair of dispersion curves for the chain appears in Fig. 7-4. The dimensions are identified in Figs. 7-1 and 7-2, and α_π has been estimated with the aid of Eq. 7-5b for $m = 8$. There f_1 denotes the cutoff frequency $cP_1/2\pi$ and f_{s1} is the frequency at which each slot is half a free-space wavelength long.

Equation 7-10 describes the curves of Fig. 7-4 very well. The agreement is particularly good in the "cavity" passband because the next pair of open- and short-circuit "cavity" modes has resonant frequencies high enough so they contribute little to the fields at frequencies in the first two

passbands. For instance, the next short-circuit "cavity" mode in the expansion of Eq. 7-1 has a uniform $\bar{E}_{2z}$ pattern in the axial direction which varies radially as $J_2(P_2r)$; its resonant frequency is therefore $(P_2a/P_1a)f_1 = 5.52$ kMc. Discrepancies between theory and experiment in the lower "slot" band are due primarily to the neglect of irrotational modes by earlier assumption (b). Electron beams are, however, usually designed to interact with the cavity fields at frequencies near the π cutoff frequency of the "cavity" passband where we can calculate the group velocity from Eq. 7-10 and the interaction parameter $E_z(0)^2/W_s$, W_s being twice the stored electric energy, in terms of the cavity solenoidal modes.

We shall now make a correction to Eq. 7-1 for the presence of small centerholes in the structure of Fig. 7-1; their effect on the ω - φ curve of the cavity passband is quite small compared to the effect of the slots. In Eqs. 7-1 V_1 now includes an additional term to represent the hole coupling,

$$V_1 = -P_1(P_1^2 - k^2)^{-1}\tau^{-1}\left(\int_{S\pm} \hat{e} \times \bar{H}_1^{(0)} \cdot \bar{n} dS + \int_{H\pm} \hat{e}' \times \bar{H}_1^{(0)} \cdot \bar{n} dS\right) \quad (7-11)$$

where $(H\pm)$ symbolizes the holes bounding reference cavity (0). In the second integral $\hat{e}'$ can be approximated, as usual, by the open-circuit cavity mode expression

$$\hat{e}'(H\pm) = \frac{1}{2}(v_1\bar{e}_1^{(0)} + v_1\epsilon^{\mp j\varphi}\bar{e}_1^{(\pm 1)})_{\text{on}(H\pm)}$$

(see Eq. 5-40). The open-circuit mode can be conveniently assumed to resonate at the π cutoff frequency of the "cavity" passband with the slots and centerholes magnetically shorted. Again, $V_1 \cong v_1$ because both the open- and short-circuit modes are equivalent for electric field far from the slots, in particular, on the centerholes where the slot electric field is very weak. This approximation worsens as frequency increases toward the π cutoff frequency because the next pair of resonant cavity modes contributes more and more to the fields. According to the discussion of

Section 4-2 the $\int_{H\pm}$ integrals of Eq. 7-11 can be developed in terms of V_1 and φ so that the equation reads

$$\begin{aligned} V_1[1 - P_1M_{11}'(P_1^2 - k^2)^{-1}(1 - \cos \varphi)] \\ = -\frac{P_1}{(P_1^2 - k^2)\tau} \int_{S\pm} \hat{e} \times \bar{H}_1^{(0)} \cdot \bar{n} dS \quad (7-12) \\ M_{11}' \triangleq \tau^{-1} \int_{H-} \bar{e}_1^{(0)} \times \bar{H}_1^{(0)} \cdot \bar{i}_z dS < 0 \end{aligned}$$

The term M_{11}' is primed to denote the fact that it measures field on the holes and not both the holes and the slots. We can now see that the

suitably modified dispersion Eq. 7-10 is

$$\frac{[1 - P_1 M_{11}'(P_1^2 - k^2)^{-1}(1 - \cos \varphi)](k/P_1)[1 - (k/P_1)^2]}{\tan(k\ell/2) - k\ell/2} = \alpha_\pi \sin^2 \frac{\varphi}{2} \quad (7-13)$$

for both passbands. The assumptions are still those of (a)–(c) listed earlier.

7.3. Calculation of the Interaction Parameter $E_z(0)^2/W_s$

This parameter is closely related to the *impedance* parameter $E_z(0)^2/2\beta_0^2 P_{av}$ of the cavity chain operated as a traveling wave tube, where $E_z(0)$ is the axial electric field, W_s is the total time-average “cold” circuit stored energy in a periodic length, and P_{av} is the time-average “cold” circuit power flow. As we proved in Section 1-4, $P_{av} = (\partial\omega/\partial\beta_0)W_s/L$. Thus, an accurate dispersion relation gives us group velocity $\partial\omega/\partial\beta_0$ and enables us to find the impedance parameter from either the calculated interaction parameter or the value as measured by one of the methods described in Appendix B.

Allen [3] has obtained an accurate expression for the interaction parameter of the Fig. 7-1 structure with the slots aligned from cavity to cavity. He evaluates the amplitudes of all the TM_{0n0} modes from the slot voltages

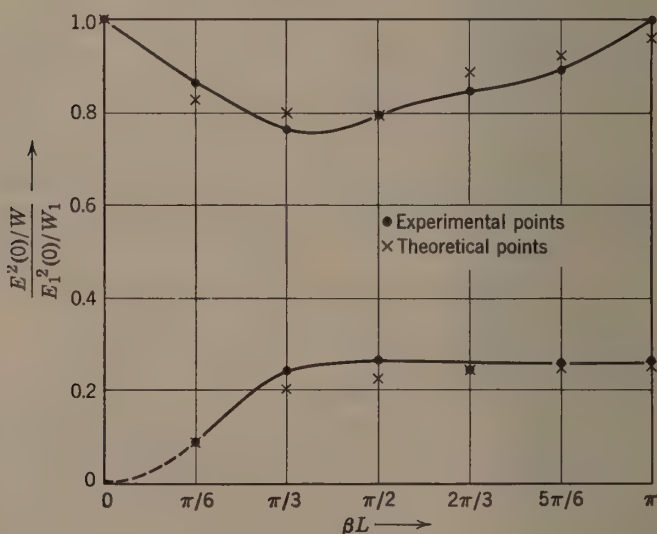


Fig. 7-5. $\hat{E}^2(0)/W_s$ as measured along the axis of a particular resonant-slot coupled chain, relative to the value calculated by the first short-circuit resonant cavity mode alone (from Allen and Kino [1]).

and sums the contributions to $E_z(0)$ and to $W_s = 2W_e$, W_e being the time-average electric stored energy per cavity in terms of the solenoidal modes alone. The results for one such structure with very narrow slots appear in Fig. 7-5; the excellent agreement with measurement in the "slot" passband is somewhat fortuitous in view of the fact that Allen estimated the slot voltages as determined by the first solenoidal cavity mode alone, whereas the irrotational cavity modes are by no means negligible at frequencies in this passband. The more accurate dispersion relation would include at least the effect of the first few irrotational modes upon the slot voltages. These modified slot voltages would then determine somewhat different solenoidal TM_{0n0} mode amplitudes as well as irrotational modes in the expression for $E_z(0)^2/W_s$. Apparently the fractional error in the numerator canceled much of that in the denominator.

7-4. A Slot-Coupled Chain with Successive Slots Rotated

We now dwell upon the behavior of the lower or "slot" passband of a structure like that shown in Fig. 7-1 but with only two slots in each slot plane. If the successive slots are aligned along the chain so that the structure has mirror symmetry about any cavity midplane or slot plane, then the dispersion curves are labeled *A* in Fig. 7-6. The "slot" passband is very narrow and therefore conveniently described by the single mode-pair approximation. Now, let us rotate the successive slots by 90° , as shown in Fig. 7-7, whereupon the dispersion curves change to those labeled *B* in Fig. 7-6. We now offer an explanation of the striking change in the lower curve.

Rather than develop the fields of a given slot in terms of the radiation fields of neighboring slots or in terms of the "slot" modes, which are rotated 90° from one slot plane to the next and are therefore difficult to visualize on the cavity midplanes, we extend the analysis of the preceding section so as to include the irrotational modes. Fortunately, the first few terms of the irrotational magnetic field expansion of Eq. 7-2 are sufficient to yield a good estimate of the "slot" curve.

The "cavity" irrotational modes excited in the Fig. 7-7 structure are angularly varying, for the nmq th such normalized mode (subscripts nmq replace the single one n) which satisfies Eqs. 5-13 and 5-17 is

$$\bar{H}_{nmq}^i = -\nabla \Psi_{nmq} \quad (7-14a)$$

$$\Psi_{nmq} = N_{nmq} J_n(p_{nq} r) \begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix} \cos \frac{m\pi z}{L} \quad (7-14b)$$

where $(\partial/\partial r)J_n(p_{nq}r)|_{r=a} = 0$; $m = 0, 1, \dots$; and normalization factor

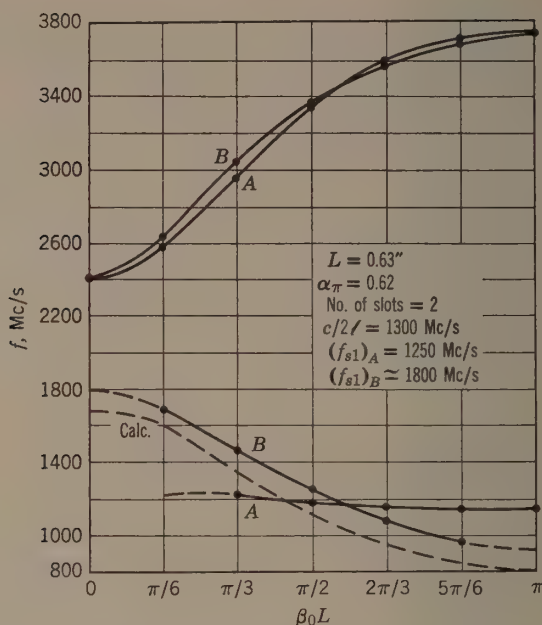


Fig. 7-6. The first two measured dispersion curves in a resonant-slot coupled chain with (A) successive slots aligned, and (B) each slot plane rotated 90° with respect to its neighbors, Fig. 7-7. The calculated curve for the "slot" passband was obtained from Eq. 7-19.

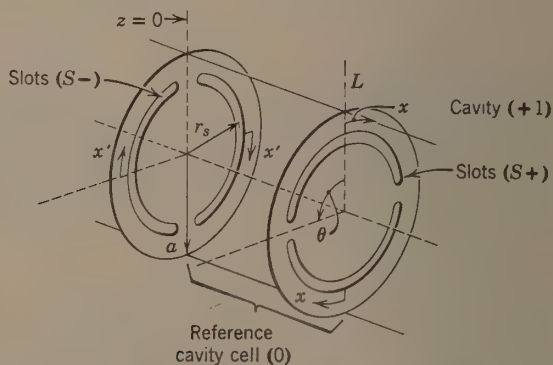


Fig. 7-7. A resonant-slot coupled chain with a rotation of 90° from one slot plane to the next one. x' is the coordinate measuring position along the slots at $z = 0$ and x , the coordinate at $z = L$.

N_{nmq} is given by

$$N_{nmq}^2 = \frac{4}{[(m\pi/L)^2 + p_{nq}^2](1 + \delta_{m0})(1 + \delta_{n0})[1 - (n/p_{nq}a)^2]J_n^2(p_{nq}a)} \quad (7-14c)$$

$$\delta_{m0} = 1 \quad \text{if } m = 0, \quad 0 \quad \text{if } m \neq 0$$

It is generally safer to employ modes of this form which satisfy the boundary conditions of $\vec{H}_{nmq}^i \cdot \vec{n} = 0$ on the cavity walls and slot surfaces rather than modes developed from Allen and Kino's ψ_{nm} :

$$\psi_{nm} = A_{nm} \cosh \beta_n z J_n(\beta_n r) \sin m\theta$$

because these latter modes do not satisfy the proper boundary conditions at $z = \pm L$ of Fig. 7-7 where $(\vec{H}_{nm})_z \sim \sinh \beta_n L \neq 0$.

For a simple first calculation of the slot dispersion curve B of Fig. 7-6 it is tempting to say the cavity solenoidal field predominates. If this were the case we would proceed as follows. Starting with a reference amplitude I_1 for the "cavity" solenoidal mode, we would determine the slot voltages by Eq. 7-3. These voltages would then quite accurately determine the irrotational I_{nmq}^i amplitudes in Eq. 7-2. The irrotational field would then furnish a correction to the magnetic field on the slots and a correction to the slot voltages by Eq. 7-3. Finally, these improved slot voltages would determine amplitude I_1 with which we started. But in this process of deriving a dispersion relation we find that the irrotational field is comparable to the solenoidal field at frequencies in this passband. The "dispersion" relation obtained without it does not even qualitatively check with measurement.

We can utilize the physical fact that the slot is half a free-space wavelength long at a frequency near the center of the passband. Thus, the predominant slot voltage will vary as $\cos(\pi x/\ell)$ along each slot in Fig. 7-7. These voltages will determine amplitudes I_1 and I_{nmq}^i as we just outlined. To minimize the effect of errors in these cavity fields we can utilize a variational expression for $k^2(\varphi)$ which is formulated in terms of slot voltage alone, such that the error in k^2 will tend to be proportional to the square of the voltage errors. We shall develop this expression after we evaluate amplitudes I_1 and I_{nmq}^i .

Since pattern $\vec{H}_{10} = -\vec{H}_{1x}$ is circularly symmetric, the successive slot rotations in Fig. 7-7 do not change Eqs. 7-1 and 7-8, from which I_1 is obtained as

$$I_1 = -\frac{j\omega\epsilon S}{(P_1^2 - k^2)\tau} \left[\int_{-\ell/2}^{\ell/2} \phi_{S-}(x') H_{1x'} dx' - \epsilon^{-j\varphi} \int_{-\ell/2}^{\ell/2} \phi_{S-}(x) H_{1x} dx \right] \quad (7-15)$$

Here we regard the $(S+)$ slot voltage as having reference phase $\epsilon^{-j\varphi}$

relative to the ($S-$) slot voltage; S is the number of slots in a plane, 2 in our case. To obtain I_{nmq}^i we first need the pattern $(\bar{H}_{nmq}^i)_\theta = -(\bar{H}_{nmq}^i)_x$ along the slots. From Eq. 7-14 it is

$$-(H_{nmq}^i)_x = n \frac{N_{nmq} J_n(p_{nq} r_s)}{r_s} \begin{Bmatrix} \cos n\theta \\ -\sin n\theta \end{Bmatrix} \cos \frac{m\pi z}{L} \quad (7-16)$$

Let us fix this pattern in cavity (0) of Fig. 7-7 relative to the θ coordinate shown; in the neighboring cavities it will be rotated 90° and its amplitude will be $I_{nmq}^i \epsilon^{\mp j\varphi}$. Now in this lowest passband we expect each slot voltage to be symmetric about the slot center and also the slot voltages in a given slot plane to be invariant to a rotation of the whole plane by π . This is the symmetry of the first cavity solenoidal mode. The irrotational modes with this symmetry will couple to the first cavity mode and to themselves on the slots and determine the ω - φ curve of the first passband. Those with other symmetries will couple to the angular varying solenoidal modes of higher resonant frequencies and determine the higher-frequency dispersion curves. For this reason we take the $\cos n\theta$ variation of the reference cavity patterns in Eq. 7-16 as well as the even values of n so as to obtain the invariance in a given slot plane of a rotation by π . All positive values of q and m are, however, allowed: $q = 1, 2, \dots$, and $m = 0, 1, \dots$.

From these remarks we know it does not matter whether we rotate pattern $\bar{H}_{nmq}^i$ by $\pm 90^\circ$ from the reference cavity to cavity (± 1) provided we also phase shift amplitude I_{nmq}^i . We can now find this amplitude from Eq. 7-2; it is

$$I_{nmq}^i = -\frac{S}{j\omega\mu\tau} \left[\int_{-\ell/2}^{\ell/2} \phi_{S-}(x') \bar{H}_{nmq,x'} \left(r = r_s, \theta = -\frac{x'}{r_s} + \frac{\pi}{2}, z = 0 \right) dx' \right. \\ \left. + \epsilon^{-j\varphi} \int_{S+} \phi_{S-}(x) \cdot -\bar{H}_{nmq,x} \left(r = r_s, \theta = \frac{x}{r_s}, z = L \right) dx \right], \quad S = 2 \quad (7-17)$$

Here coordinate x' on the ($S-$) slot is distinguished from coordinate x on the ($S+$) slot, but ϕ_{S-} serves for both because we have written explicitly its phase shift of $\epsilon^{-j\varphi}$ from ($S-$) to ($S+$).

Now that we know the mode amplitudes in terms of voltage $\phi_{S-}(x)$ we present a variational expression for $k^2(\varphi)$ in the spirit of Allen and Kino's work, as developed from the basic equation

$$\left(\frac{\partial^2}{\partial x^2} + k^2 \right) \phi_{S-}(x) = -j\omega L_0 \left[I_1 \bar{H}_{1x} + \sum_{nmq} I_{nmq}^i \bar{H}_{nmq,x}^{i(0)}(z = 0^+) \right. \\ \left. - I_1 \epsilon^{j\varphi} \bar{H}_{1x} - \epsilon^{j\varphi} \sum_{nmq} I_{nmq}^i \bar{H}_{nmq,x}^{i(-1)}(z = 0^-) \right] \quad (7-18)$$

The expression is obtained by multiplying this equation by $\phi_{S-}(x)^*$, integrating over the ($S-$) slot, and substituting Eqs. 7-15 and 7-17 to get

$$\begin{aligned} \int_{S-} \phi_{S-}(x)^* \left(\frac{\partial^2}{\partial x^2} + k^2 \right) \phi_{S-}(x) dx = & -\frac{k^2(L_0/\mu_0)S}{\tau(P_1^2 - k^2)} (1 - \epsilon^{-j\varphi})(1 - \epsilon^{j\varphi}) \\ & \cdot \int_{S-} \phi_{S-}(x)^* \bar{H}_{1x} dx \int_{S-} \phi_{S-}(x) \bar{H}_{1x} dx + \frac{L_0}{\mu\tau} \sum_{nmq} \left[\frac{nN_{nmq}J_n(p_{nq}r_s)}{r_s} \right]^2 \\ & \cdot \left[\cos \frac{n\pi}{2} - \cos(m\pi)\epsilon^{-j\varphi} \right] \left[\cos \frac{n\pi}{2} - \cos(m\pi)\epsilon^{j\varphi} \right] \int_{S-} \phi_{S-}(x)^* \cos \frac{nx}{r_s} \\ & \cdot dx \int_{S-} \phi_{S-}(x) \cos \frac{nx}{r_s} dx \end{aligned} \quad (7-19)$$

We can easily demonstrate that if ϕ^* is in error by $\delta\phi^*$ and ϕ satisfies Eq. 7-18 in conjunction with Eqs. 7-15 through 7-17, then the linear error $\delta(k^2)$ is zero. In addition, if ϕ is in error by $\delta\phi$ and ϕ^* satisfies Eq. 7-18 conjugated, $\delta k^2 = 0$. The n and m terms in Eq. 7-19 account for the rotational effect; the $\int_{S-}$ integrals are taken over the one slot ($S-$) at mean radius r_s in which we specify the reference voltage distribution by a trial function.

For the simplest estimate of the lower ω - φ curve B in Fig. 7-6 let us use $\phi_{S-}(x) = \cos(\pi x/\ell)$. We also take $m = 0$ because the N_{nmq} coefficients of the $m \neq 0$ terms are rather small, because of the factor $[(m\pi/L)^2 + p_{nq}^2] \sim [(m\pi a/L)^2 + (p_{nq}a)^2]$ in the denominator. In this structure $(p_{21}a)^2$ of the predominant irrotational term is 9.3 while $(\pi a/L)^2 \cong 89$. We find that the $n = 2, q = 1$ term is predominant, with the $n = 2, q = 2$ term about 12% as large. The $n = 4, q = 1$ term is negligible as is the $n = 2, m = 1, q = 1$ term. With these two irrotational modes the dispersion relation for the structure with the dimensions listed on Fig. 7-6 boils down to

$$0.035[(k\ell)^2 - \pi^2] = -\frac{0.91}{(P_1/k)^2 - 1} (1 - \cos \varphi) + 0.115(1 + \cos \varphi) \quad (7-20)$$

in which P_1 is the "cavity" mode cutoff frequency corresponding to 2.4 kMc and $k\ell = \pi$ occurs at $f = 1.30$ kMc. The calculated dispersion curve shown by the series of x 's is qualitatively correct but low by about 0.2 kMc over the whole range $0 \leq \varphi \leq \pi$.

This discrepancy of 0.2 kMc is due largely to the slow convergence of the summation over the irrotational modes in Eq. 7-19. It is not due to neglect of the next solenoidal cavity mode which resonates at 5.52 kMc, as we have observed. Nor does the small $\cos(3\pi x/\ell)$ component of slot

voltage account for more than 1% of the discrepancy. For if we let $\phi(x) \cos(\pi x/\ell) + A \cos(3\pi x/\ell)$, with A a variational coefficient, and solve for A such that k^2 is an extremum at each phase shift, we find that A lies in the range 0.02–0.04.

Equation 7-20 explains qualitatively why the “slot” passband narrows considerably when the slots are aligned with their neighbors. Then the irrotational I_{nmq}^i amplitudes (for $m = 0$) are excited exactly like the I_1 amplitude. There is no change in sign of $\bar{H}_{2mq}^i$ from one slot to the next due to the $\cos(2\theta)$ factor in Eq. 7-17; the predominant $n = 2$, $q = 1$ irrotational last term of Eq. 7-20 has a $1 - \cos\varphi$ dependence just like that of the first solenoidal term and the two terms of Eq. 7-20 tend to cancel each other throughout the entire “slot” passband. In this case $k\ell$ will remain nearly π and the lower dispersion curve A of Fig. 7-6 results.

7-5. Variational Description of the Dispersion in Resonant-Slot and Resonant-Loop Coupled Structures

In this section we demonstrate the fact that the dispersion relation for the first two passbands of the resonant-slot coupled structure of Fig. 7-1 can be formulated variationally, in terms of just the four cutoff frequencies and one other parameter which measures certain volume integrals of the mode patterns. We treat only those structures with mirror symmetry; that is, chains in which the periodic operation is a translation without a rotation. Since the dispersion relation will be obtained independently of the precise nature of the (symmetric) coupling at the slots, it will also apply, with slight modification, to resonant-loop coupled chains. In these latter chains one of the passbands is characterized by a pair of “loop” modes which are open- and short-circuit, respectively, on the cavity midplanes.

It is convenient to employ variational Eq. 6-89 for $k^2(\varphi)$ in terms of electric field alone. This expression is valid for discontinuities of both $\hat{E}$ and $\nabla \times \hat{E}$ on discontinuity surfaces within the periodic volume:

$$k^2 = \frac{\left[\int_V \hat{E}_-^* \cdot (\nabla \times \nabla \times \hat{E}_+) dV + \frac{1}{2} \int_D (\hat{E}_-^{(+)*} + \hat{E}_-^{(-)*}) \cdot \hat{i}_z \times (\nabla \times \hat{E}_+^{(+)} - \nabla \times \hat{E}_+^{(-)}) dS - \frac{1}{2} \int_D (\hat{E}_+^{(+)} - \hat{E}_+^{(-)}) \cdot \hat{i}_z \times (\nabla \times \hat{E}_-^{(+)*} + \nabla \times \hat{E}_-^{(-)*}) dS \right]}{\int_V \hat{E}_-^* \cdot \hat{E}_+ dV} \quad (7-21)$$

Here V denotes any periodic volume, D is any discontinuity surface (perpendicular to the z axis) within V , of which we will have two in these structures, the plus superscript denotes a field measured on the $+z$ side of D and the minus superscript, a field on the $-z$ side. Both $\hat{E}_-$ and independent field $\hat{E}_+$ phase shift by $e^{-j\varphi}$ from a point (x, y, z) in cavity (n) to the point $(x, y, z + L)$ in cavity $(n + 1)$. The variational properties have been mentioned at the end of Section 6-7.

To represent fields $\hat{E}_-$ and $\hat{E}_+$ in the resonant-slot coupled chain we employ the first open- and short-circuit "cavity" modes of Fig. 4-1b, c, reproduced in Fig. 7-8b, c. Their amplitudes phase shift across the *slots* and *centerholes*. These modes describe, in the first approximation, the fields at frequencies in the "cavity" passband, according to the discussion of Section 4-2. We also employ the "slot" modes of Fig. 4-2, reproduced as Fig. 7-8d, e, and let their amplitudes phase shift across the *cavity midplanes* ($B\pm$) in Fig. 7-8a. These latter modes describe, in the single mode-pair approximation, the fields at frequencies in the "slot" passband. The "slot" modes may be formally defined for any number of slots in each slot plane and are convenient to use in symmetrical structures because each represents the physical field at one of the cutoff frequencies. The somewhat unnatural notation for the modes is chosen to conform with the literature [4].

Now the question arises: How should we represent $\hat{E}_-$ and independent field $\hat{E}_+$ with these modes? We naturally associate two of them with $\hat{E}_-$ and the other two with $\hat{E}_+$, but not every choice leads to a well-behaved dispersion relation. It turns out that only one choice is allowed, the one for which

$$\begin{aligned}\hat{E}_- &= V_c \bar{E}_c + v_s \bar{e}_s \\ \hat{E}_+ &= v_2 \bar{e}_2 + v_1 \bar{e}_1\end{aligned}\tag{7-22}$$

The representation with $\hat{E}_-$ and $\hat{E}_+$ interchanged leads to the same dispersion relation. Equation 7-22 actually designates the fields in the common volume τ' of the cavity of reference phase zero in Fig. 7-8. To obtain the field at any other point we must repeat the "slot" mode patterns $\bar{e}_s$ and $\bar{e}_1$ as well as $\bar{h}_s$, and $\bar{h}_1$ from "slot" cell to "slot" cell, the "cavity" mode patterns $\bar{E}_c$, $\bar{e}_2$, and so on, from one "cavity" cell to the next, and then phase shift each amplitude by the number of "slot" or "cavity" cells between the reference point and the point under consideration.

Notice that these fields have the requisite properties on the discontinuity surfaces ($M\pm$) and ($B\pm$), as listed at the end of Section 6-7. Tangential $\hat{E}_-$ and $\nabla \times \hat{E}_+$ are, by construction, continuous across ($M\pm$). Tangential $\hat{E}_+$ and $\nabla \times \hat{E}_-$ are constrained to be continuous across ($M\pm$) because tangential $\nabla \times \hat{E}_-$ and $\hat{E}_+$, respectively, are not identically zero on

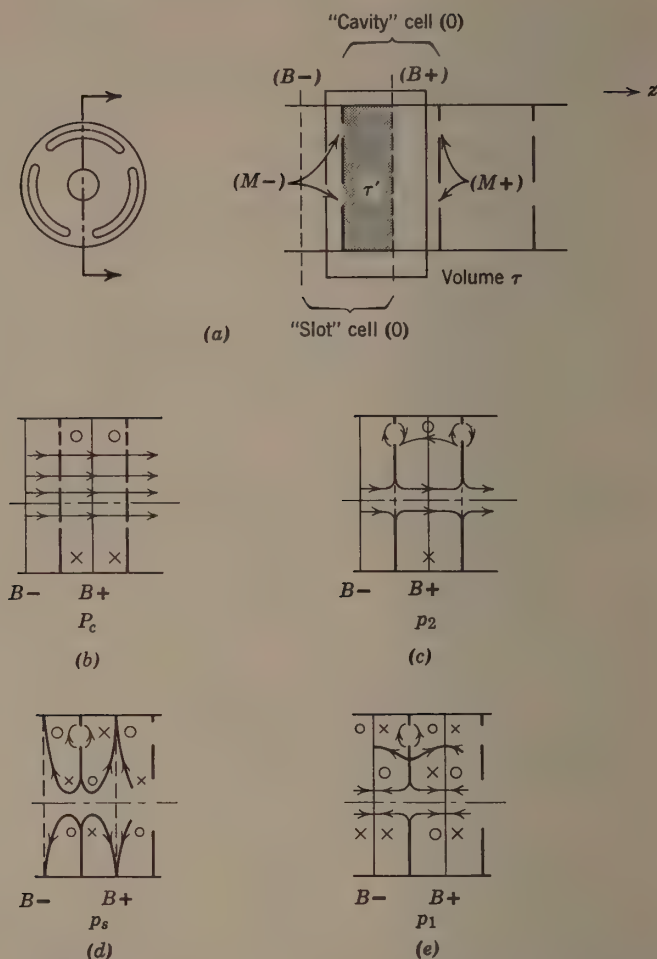


Fig. 7-8. A resonant-slot coupled chain and its first four resonant modes. (a) Schematic view of the structure showing a typical periodic volume τ and a subvolume τ' in common with the "cavity" cell and the "slot" cell. (b) The "cavity" short-circuit mode of frequency P_c in Fig. 7-4. (c) The "cavity" open-circuit mode of frequency p_2 . (d) The "slot" open-circuit mode of frequency p_s . (e) The "slot" short-circuit mode of frequency p_1 in Fig. 7-4. The solid lines denote the electric field patterns while the circles and crosses denote the magnetic field patterns.

$(M\pm)$. On $(B\pm)$, $\hat{E}_-$ and $\nabla \times \hat{E}_+$ are properly constrained by the variational procedure because $\nabla \times \hat{E}_+$ and $\hat{E}_-$ have nonvanishing tangential portions. Tangential $\hat{E}_+$ and $\nabla \times \hat{E}_-$ are continuous there.

We now substitute the trial fields of Eqs. 7-22 into Eq. 7-21, observe the directions of the mode patterns in Fig. 7-8 as well as the phase shifts of

their amplitudes, and evaluate the integrals as

$$\begin{aligned}
 0 = & \int_{\tau'} (V_c^* \bar{E}_c + v_s^* \bar{e}_s) \cdot [(p_2^2 - k^2)v_2 \bar{e}_2 + (p_1^2 - k^2)v_1 \bar{e}_1] dV \\
 & + \int_{B+ \text{ to } M+} (V_c^* \bar{E}_c + v_s^* \epsilon^{j\varphi} \bar{e}_s) \cdot [(p_2^2 - k^2)v_2 \bar{e}_2 + (p_1^2 - k^2)v_1 \epsilon^{-j\varphi} \bar{e}_1] dV \\
 & + \frac{1}{2} \int_{B+} v_s^* \bar{e}_s^{(0)} (\epsilon^{j\varphi} + 1) \cdot \bar{i}_z \times [p_1 v_1 \bar{h}_1^{(0)} (-\epsilon^{-j\varphi} - 1)] dS \\
 & - \frac{1}{2} \int_{M-} v_2 \bar{e}_2^{(0)} (1 + \epsilon^{j\varphi}) \cdot \bar{i}_z \times [P_c V_c^* \bar{H}_c (1 + \epsilon^{-j\varphi})] dS \quad (7-23)
 \end{aligned}$$

(Mode patterns $\bar{E}_c$, $\bar{H}_c$, and the others are pure real quantities.) Now we set variation $\delta k^2 = 0$, first for a variation δV_c^* and then for variation δv_s^* . We refer the second integral to the integral over τ' by the symmetry of the mode patterns in Fig. 7-8. The result is two equations for v_2 and v_1 which are cast into matrix form for later use:

$$\begin{bmatrix} (p_2^2 - k^2)T_{22} + P_c M_{22}(1 + \cos \varphi) & (1 - \epsilon^{-j\varphi})(p_1^2 - k^2) \frac{1}{\tau} \int_{\tau'} \bar{E}_c \cdot \bar{e}_1 dV \\ (1 - \epsilon^{j\varphi})(p_2^2 - k^2) \frac{1}{\tau} \int_{\tau'} \bar{e}_2 \cdot \bar{e}_s dV & (p_1^2 - k^2)T_{11} - p_1 M_{11}(1 + \cos \varphi) \end{bmatrix} \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} = 0 \quad (7-24)$$

in which

$$\begin{aligned}
 T_{22} &= \frac{1}{\tau} \int_{\tau} \bar{E}_c \cdot \bar{e}_2 dV & M_{22} &= \frac{1}{\tau} \int_{M-} \bar{e}_2^{(0)} \times \bar{H}_2^{(0)} \cdot \bar{i}_z dS \\
 T_{11} &= \frac{1}{\tau} \int_{\tau} \bar{e}_s \cdot \bar{e}_1 dV & M_{11} &= \frac{1}{\tau} \int_{B-} \bar{e}_s^{(0)} \times \bar{h}_1^{(0)} \cdot \bar{i}_z dS
 \end{aligned}$$

(Subscripts c and 2 denote the "cavity" modes; subscripts s and 1, the "slot" modes.)

The determinant of Eq. 7-24 set equal to zero furnishes us with a dispersion relation with the properties we want. First, it says $k = p_2$ and p_1 at $\varphi = \pi$. When $\varphi = 0$ the two off-diagonal terms are zero and k is correctly P_c and p_s . To see this we invoke Green's theorem, Eq. 3-43b, first in terms of $\bar{E}_c$ and $\bar{e}_2$ within the "cavity" cell and then in terms of $\bar{e}_2$ and $\bar{e}_1$ within the "slot" cell. With the preceding definitions of the T 's and M 's and the relations $\nabla \times \nabla \times \bar{E}_c = P_c^2 \bar{E}_c$, and so on, we obtain

$$\begin{aligned}
 (p_2^2 - P_c^2)T_{22} + 2P_c M_{22} &= 0 \\
 (p_1^2 - p_s^2)T_{11} - 2p_1 M_{11} &= 0 \quad (7-25)
 \end{aligned}$$

which prove our statement that $k = P_c$ and p_s at $\varphi = 0$. This dispersion relation also predicts zero group velocity at $\varphi = 0, \pi$ because $k^2(\varphi) = k^2(-\varphi)$.

The determinant of Eq. 7-24 set equal to zero describes well the dispersion in typical resonant-slot coupled structures, but we are not unduly surprised because the four cutoff frequencies have been constrained to be correct. Three of them, with the exception of p_s , can be measured by shorting a single "slot" cell at the cavity midplanes. There are, however, five parameters to evaluate if we express M_{22} and M_{11} in terms of P_o and p_s by Eq. 7-25 and absorb the T_{22} and T_{11} factors into the single cross-coupling coefficient conveniently defined as

$$m_s = \frac{4}{\tau^2 T_{11} T_{22}} \int_{\tau'} \vec{E}_o \cdot \vec{e}_1 dV \int_{\tau'} \vec{e}_2 \cdot \vec{e}_s dV \quad (7-26a)$$

In this way we can write the dispersion relation in the form

$$[\frac{1}{2}P_o^2(1 + \cos \varphi) + \frac{1}{2}p_2^2(1 - \cos \varphi) - k^2][\frac{1}{2}p_s^2(1 + \cos \varphi) + \frac{1}{2}p_1^2(1 - \cos \varphi) - k^2] - m_s(p_2^2 - k^2)(p_1^2 - k^2) \sin^2 \frac{\varphi}{2} = 0 \quad (7-26b)$$

Then in order to obtain the additional parameters p_s and m_s for calculation we could short a four-section chain and measure the $\pi/3$ and $2\pi/3$ resonant frequencies in the "cavity" passband, as well as the cutoff frequencies P_o , p_2 , and p_1 .

Equation 7-26b tells us that the "slot" modes of a "slot" passband which lies below the "cavity" passband will tend to raise the ω - φ curve of the latter. To verify this we use Green's theorem Eq. 3-43b, with $\vec{A} = \vec{e}_2$ and $\vec{B} = \vec{e}_s$ in the volume τ' , and refer to the directions of the patterns in Fig. 7-8 to show that m_s is negative because one of its integrals

$$\int_{\tau'} \vec{e}_2 \cdot \vec{e}_s dV$$

is negative. Since the second bracketed expression and the factor $p_2^2 - k^2$ are both positive it follows that k^2 in the second ("cavity") passband will be higher due to the $\sin^2 \varphi$ coupling with the "slot" modes of the first passband. Now, if the slots are shortened so as to move the "slot" passband just above the "cavity" passband, the Green's relation indicates that

$$\int_{\tau'} \vec{e}_2 \cdot \vec{e}_s dV$$

goes positive and m_s goes positive. However, because of the new arrangement of resonant frequencies, Eq. 7-26b predicts that k^2 in the first or "cavity" passband will fall somewhat lower because of the coupling with the higher-frequency "slot" modes.

In order to justify further this dispersion Eq. 7-26 for a large class of structures we now show the extent to which it verifies the relation: Power

flow equals stored energy per unit length times group velocity. To obtain this relation we first premultiply the matrix Eq. 7-24 by row matrix (V_c^*, v_s^*) , where the equations for *these* amplitudes are obtained by taking variations δv_1 and δv_2 in Eqs. 7-21 and 7-23. We now differentiate this new matrix with respect to φ and obtain three matrix terms. The $(\delta V_c^*/\partial\varphi, \delta v_s^*/\partial\varphi)$ term is obviously still zero, and so is the

$$\begin{pmatrix} \frac{\partial v_2}{\partial \varphi} \\ \frac{\partial v_1}{\partial \varphi} \end{pmatrix}$$

term, as we discover by transposing this particular triple-matrix product, whereupon row matrix $(\partial v_2/\partial\varphi, \partial v_1/\partial\varphi)$ multiplies a matrix expression exactly like that of Eq. 7-24 but in terms of amplitudes V_c^* and v_s^* . The only remaining term has the differentiated bracketed expression of Eq. 7-24. Upon multiplying out the matrices in this term, rearranging the result, and taking the real part, we obtain

$$\begin{aligned} 2k \frac{\partial k}{\partial \varphi} \operatorname{Re} & \left[V_c^* T_{22} v_2 + V_c^* \left(\frac{1}{\tau} \int_{\tau'} \bar{E}_c \cdot \bar{e}_1 dV \right) (1 - \epsilon^{-j\varphi}) v_1 \right. \\ & \left. + v_s^* \left(\frac{1}{\tau} \int_{\tau'} \bar{e}_s \cdot \bar{e}_2 dV \right) (1 - \epsilon^{j\varphi}) v_2 + v_s^* T_{11} v_1 \right] \\ = \operatorname{Re} & \left[V_c^* (-P_c M_{22} \sin \varphi) v_2 + v_s^* p_1 M_{11} \sin \varphi v_1 \right. \\ & + V_c^* (p_1^2 - k^2) \left(\frac{1}{\tau} \int_{\tau'} \bar{E}_c \cdot \bar{e}_1 dV \right) j \epsilon^{-j\varphi} v_1 \\ & \left. - v_s^* (p_2^2 - k^2) \left(\frac{1}{\tau} \int_{\tau'} \bar{e}_s \cdot \bar{e}_2 dV \right) j \epsilon^{j\varphi} v_2 \right] \quad (7-27) \end{aligned}$$

This is in the form group velocity times stored energy per unit length equals power flow, apart from a factor of $\frac{1}{2}\omega\mu$. Now

$$\frac{\partial k}{\partial \varphi} = \frac{\partial \omega \sqrt{\mu \epsilon}}{\partial \beta_0 L} = \left(\frac{\sqrt{\mu \epsilon}}{L} \right) \frac{\partial \omega}{\partial \beta_0},$$

this latter derivative being the group velocity. On the left side $V_c^* T_{22} v_2$ measures the time-average electric energy stored by the cavity modes:

$$\epsilon \int_{\tau} (V_c^* \bar{E}_c) \cdot v_2 \bar{e}_2 dV$$

The next term measures part of the crossenergy

$$\int_{\tau'} (V_c \bar{E}_c)^* \cdot (v_1 \bar{e}_1) dV + \int_{B+ \text{ to } M+} (V_c \bar{E}_c)^* \cdot (v_1 \epsilon^{-j\varphi} \bar{e}_1) dV$$

when we refer the second volume integral of the patterns to the integral over τ' by the symmetries of $\bar{E}_c$ and $\bar{e}_1$. The remaining two terms on the left side measure the remainder of this crossenergy and the self-energy of the "slot" modes, respectively. The first term on the right side of Eq. 7-27 measures the time-average power flow in the cavity-mode portion of the field. That power flow across either ($M \pm$) surface was given by Eq. 4-26 as

$$P_{av} = -\frac{1}{4} j v_2 I_c^* M_{22} \tau \sin \varphi \quad \text{in the present notation} \quad (7-28)$$

where now the magnetic field is obtained from the electric field as $\nabla \times \hat{E} = -j\omega\mu\hat{H}$ or $P_c V_c = -j\omega\mu I_c$ within the cavity. Analogously, the $v_s^* \cdot v_1$ term on the right side measures the power flow across plane ($B-$) in terms of the "slot" mode fields. The very last two terms in Eq. 7-27 contribute no net real power. This may be verified by examining the first of Eq. 7-24 multiplied by V_c^* and one of the associated equations for V_c^* in terms of v_s^* multiplied by v_2 .

Upon multiplying Eq. 7-27 by $\frac{\tau}{4\omega\mu}$ we obtain precisely this statement:

Time-average stored energy per unit length times group velocity equals time-average power flow across surface ($M-$) plus time-average flow across ($B+$) in the structure shown in Fig. 7-8.

This is not quite correct, even if the other resonant "cavity" and "slot" modes are negligibly excited. In addition to the question whether the stored energy is represented accurately enough, we also know that the total power flow at frequencies in the "cavity" passband is, more exactly,

Time-average power flow across ($M-$) carried by the "cavity" modes plus crosspower flow across ($M-$) due to the interaction with the (weakly excited) "slot" modes,

with the self-power carried by the "slot" modes very small. At frequencies in the "slot" passband the total power flow consists of

Time-average power flow across ($B+$) carried by the "slot" modes plus crosspower flow across ($B+$) due to the coupling with the (weakly excited) "cavity" modes,

with the self-power carried by the "cavity" modes now very weak. These three power flow statements inform us that the variational treatment has neglected the crosspower flow at frequencies in both passbands.

Figure 7-4 shows the ω - q curve calculated from Eq. 7-26 with the normalized frequencies $P_c \sim 2.4$ kMc, $p_2 \sim 4.04$ kMc, and $p_1 \sim 1.1$ kMc considered known, p_s chosen as $(2\pi/c)(1.70$ kMc), and m_s determined by the measured $\pi/2$ point in the "slot" passband. We may conclude—for this structure at least—that the errors in the dispersion relation are not serious despite the inherent errors in the expressions for stored energy and power flow. The effects of the neglected irrotational modes for electric field also appear to be small, a situation favored by our constraint of the four cutoff frequencies to be correct. The irrotational magnetic field modes do not appear, of course, in this variational formulation in terms of electric field alone.

The form of dispersion Eq. 7-26 is independent of the precise nature of the "slot" coupling and therefore will describe, with suitable modification, structures coupled by U-shaped loops. This is true if just a few loops couple neighboring cavities, but the dispersion relation quickly becomes invalid as the number of loops is increased beyond a critical number. Then we must use a different dispersion relation. These matters are discussed in the next section.

7-6. Loop-Coupled Cavity Chains; the Centipede

Figure 7-9 shows a chain in which adjacent cavities are coupled by U-shaped loops through long slots, as well as by the centerholes. The four relevant modes for the first two passbands resemble those of Fig. 7-8 with some modifications near the loops. Assuming that the "loop" passband lies below the "cavity" passband we can verify by resketching

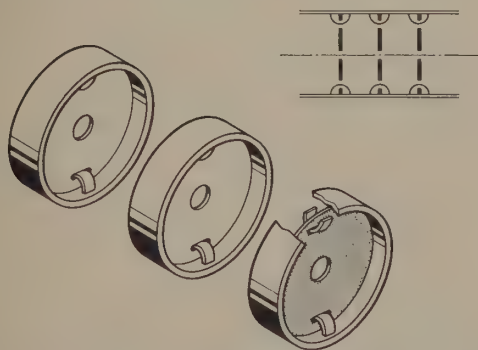


Fig. 7-9. A loop-coupled cavity chain, with U-shaped loops (from Chodorow and Craig [6]).

these patterns on the slots through which the loops pass that the dispersion curves retain the forms shown in Fig. 7-4. Pearce [5] has measured these two curves on one such twelve-loop structure and they are shown as broken lines on Fig. 7-10. The lower $\varphi = 0$ cutoff point is extrapolated by the dashed line because this mode will not resonate in a finite structure terminated with shorting planes on the cavity midplanes (see Fig. 7-8*d*). He calls this a "positive mutual" structure but in our formalism the single

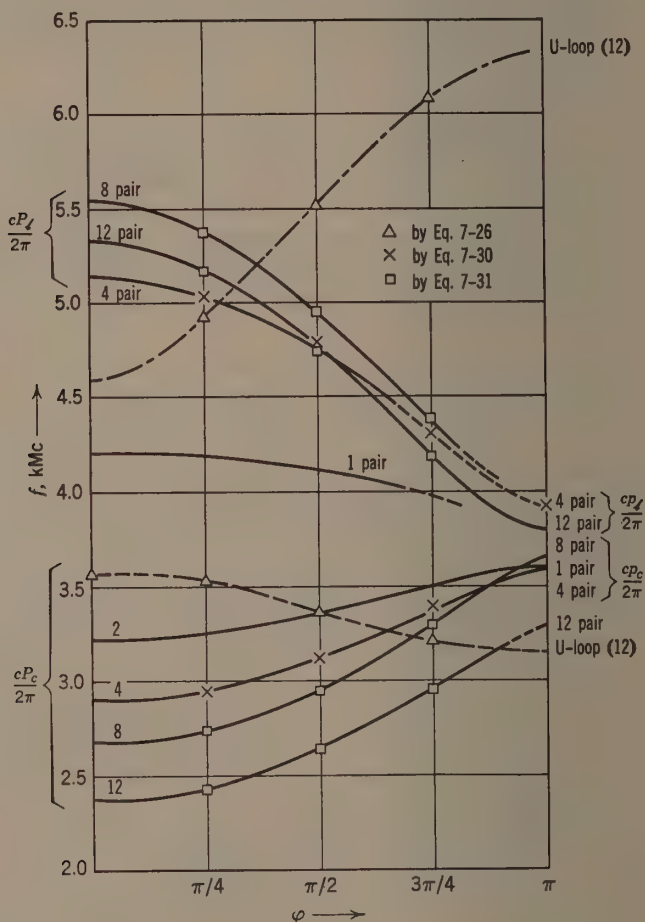


Fig. 7-10. The first two dispersion curves for the U-loop chain of Fig. 7-9, and the S-loop chains of Fig. 7-11. For each chain P_c and p_c are the resonant frequencies of the short- and open-circuit resonant "cavity" modes, respectively, and P_ℓ and p_ℓ are the resonant frequencies of the short- and open-circuit resonant "loop" modes, respectively. These modes are shown in Fig. 7-12.

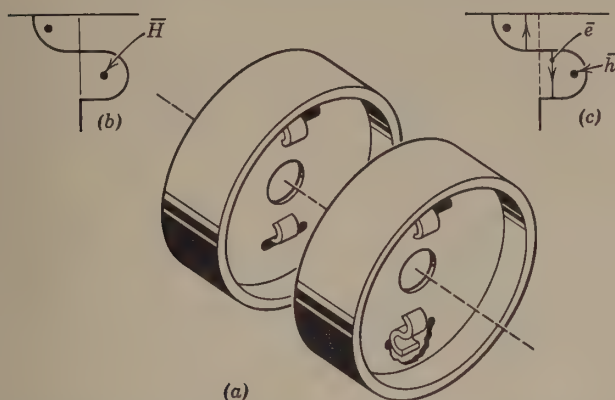


Fig. 7-11. A loop-coupled cavity chain, with S-shaped loops (from Chodorow and Craig [6]).

mode-pair behavior of the fields in each passband is characterized by a passband parameter $M < 0$, just as for the ordinary centerhole- or slot-coupled chain. The loop coupling has the effect of raising the π cutoff frequency in the "cavity" passband, with an attendant increase in passband width to 30% or so of the midband frequency. Accompanying this considerable passband width is a high interaction parameter; nevertheless, these U-loop structures are not as practical for microwave amplifiers as those in which the two dispersion curves are interchanged, the chief reason being the better shape of dispersion curve we can attain in the latter case.

By redesigning the loops to have S shapes, Fig. 7-11, we may shift the "loop" passband above the "cavity" passband, so that the "loop" modes will tend not only to depress the $\varphi = 0$ cutoff frequency of the "cavity" passband considerably more than the $\varphi = \pi$ cutoff frequency but also to straighten the ω - φ curve in the $2\pi/3$ region for better synchronism with an electron beam. The loops may be added in pairs, as designated by the dashed lines of Fig. 7-12, so as to augment this effect. It is true the S loops no longer preserve the mirror symmetry about a cavity-bisecting plane but at low frequencies in the first two passbands the effect of the asymmetry can be ignored. The patterns of the four relevant "loop" and "cavity" modes for these structures are sketched in Fig. 7-12; P_c and p_c denote the short- and open-circuit "cavity" mode frequencies while P_ℓ and p_ℓ refer to the resonant frequencies of the short- and open-circuit "loop" modes which resonate at $\varphi = 0$ and π , respectively. These frequencies are also labeled on the ω - φ curves of Fig. 7-10. Observe in Fig. 7-12 that the "cavity" modes are defined with respect to electrically

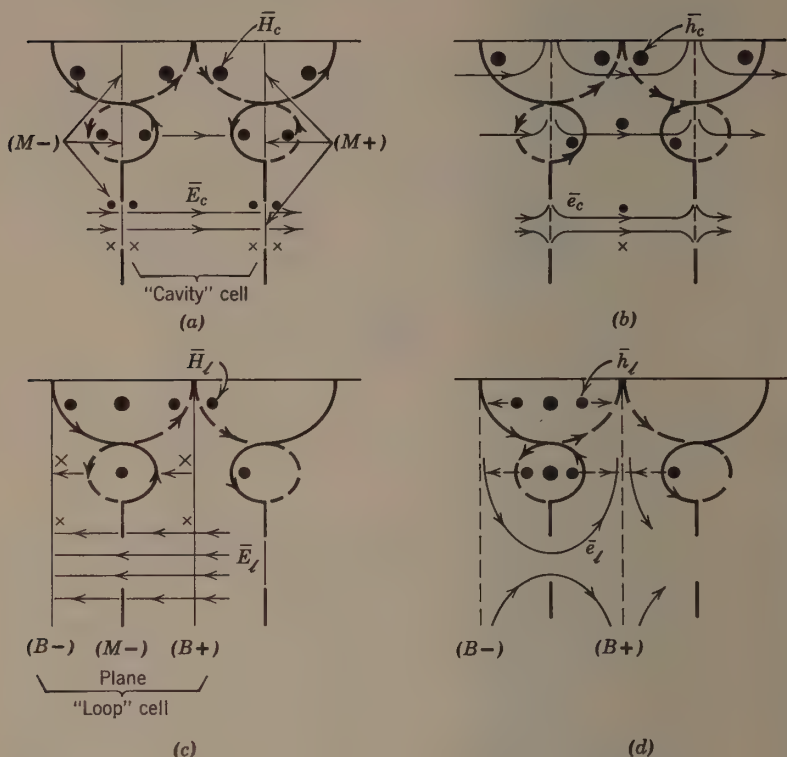


Fig. 7-12. The resonant modes for a loop-coupled cavity chain. (a) Electric and magnetic field patterns for the "cavity" short-circuit mode resonating between electrically shorted coupling surfaces $(M\pm)$, (b) patterns for the "cavity" open-circuit mode resonating between magnetically shorted surfaces $(M\pm)$, (c) patterns for the "loop" short-circuit mode resonating between electrically shorted planes $(B\pm)$, (d) patterns for the "loop" open-circuit mode resonating between magnetically shorted surfaces $(B\pm)$. Arrows on the loops denote current flows.

and magnetically shorted centerholes and loop-bisecting planes, across which their amplitudes phase shift, while the "loop" modes are defined relative to electric and magnetic shorts on the *cavity midplanes*, across which *their* amplitudes phase shift.

Let us study the effect upon the first or "cavity" passband of inserting an additional pair of S loops. Regarding the P_1 mode, Fig. 7-12a, the effect of cutting a slot and inserting a pair of loops is to displace magnetic field away from the plane of the slot toward the interior of the cavity at the $\varphi = 0$ frequency where surfaces $(M\pm)$ are shorted. Therefore, the additional loop-pair will lower the $\varphi = 0$ cutoff frequency. Probably the

most reliable way of verifying this fact utilizes a Green's relation, Eq. 6-65, in which we let $\bar{E}_c$ be the electric field of the short-circuit mode before the perturbation and $\bar{E}_c'$ be that field pattern after inserting the slot and loop-pair. The expression yields

$$[P_c^2 - (P_c')^2] \int_V \bar{E}_c \cdot \bar{E}_c' dV = \oint_{\text{loops}} \bar{E}_c \times \bar{H}_c' \cdot \bar{dS}$$

where P_c' is the new $\varphi = 0$ cutoff frequency and $\bar{dS}$ points into each loop surface. According to Fig. 7-12a, the loop integral is positive because of the perturbed magnetic field set up by the loop currents, and so $P_c' < P_c$.

On the other hand, the π cutoff frequency p_c of the first passband is little changed by the addition of more loop-pairs, up to ten or so pairs. This effect is difficult to predict, but we can substantiate it partially by showing that the passband width increases with the number of loop-pairs. Figure 7-12b shows the relevant π mode, and we see that the $\bar{e}_c$ lines flow on the loop-bisecting planes just the same way as on the centerholes, relative to $\bar{H}_c$ of Fig. 7-12a. Therefore the loops augment the effect of the centerholes and increase the passband width.

The ω - φ curves for the Fig. 7-11 structure are shown in Fig. 7-10 as a function of the number of S-shaped loop-pairs, and they verify the preceding remarks. It is an experimental fact that the "cavity" passband width measured as a fraction of the $\pi/2$ frequency increases nearly as the square root of the number of loop-pairs. Thus the cavity mode coupling per loop-pair diminishes as the number of loop-pairs increases. The single-passband analysis of the "loop" passband with these "loop" modes qualitatively verifies its behavior. Our first impression is that the relatively strong $\bar{E}_\ell$ pattern in Fig. 7-12c is drawn in the wrong direction near the axis. That it is, in fact, correct is one of the conclusions of the variational analysis, to be outlined next.

The reasoning which led us to dispersion Eq. 7-26 for the resonant-slot and the so-called "positive mutual" loop-coupled structures with similar dispersion curves must be modified for the loop-coupled chains now under discussion in order to account for the altered symmetries of the "loop" mode patterns in Fig. 7-12. We now represent the trial fields for variational Eq. 7-21 as

$$\begin{aligned} \hat{E}_- &= v_c \bar{e}_c + v_\ell \bar{e}_\ell \\ \hat{E}_+ &= V_c \bar{E}_c + V_\ell \bar{E}_\ell \end{aligned} \quad (7-29)$$

and derive the dispersion relation as before. However, the $\sin^2(\varphi/2)$ coupling term in Eq. 7-26 resulted from the fact that the axial components of the $\bar{e}_s$ and $\bar{e}_1$ patterns of the "slot" modes (Fig. 7-8) were antisymmetric

about midplane ($B+$). This made integral

$$\int_{B+ \text{ to } M+} \vec{E}_c \cdot \vec{e}_1 dV$$

in Eq. 7-23 equal to

$$-\int_{r'} \vec{E}_c \cdot \vec{e}_1 dV,$$

for example. Examination of the "loop" mode patterns in Fig. 7-12 shows $\vec{E}_\ell$ and $\vec{e}_\ell$ to have symmetric axial components on each side of plane ($B+$). In the dispersion relation obtained by taking variations δv_o^* and δv_ℓ^* in Eq. 7-23 written in terms of the new $\vec{E}_-$ and $\vec{E}_+$ we get a $\cos^2(\varphi/2)$ instead of a $\sin^2(\varphi/2)$ coupling term. The new dispersion relation is conveniently written in the form

$$\begin{aligned} & [\tfrac{1}{2}P_c^2(1 + \cos \varphi) + \tfrac{1}{2}p_c^2(1 - \cos \varphi) - k^2][\tfrac{1}{2}P_\ell^2(1 + \cos \varphi) \\ & + \tfrac{1}{2}p_\ell^2(1 - \cos \varphi) - k^2] - m_\ell(P_c^2 - k^2)(P_\ell^2 - k^2)\cos^2\frac{\varphi}{2} = 0 \quad (7-30a) \end{aligned}$$

with

$$m_\ell \triangleq \frac{4}{T_{cc}T_{\ell\ell}\tau^2} \int_{r'} \vec{E}_c \cdot \vec{e}_\ell dV \int_{r'} \vec{e}_c \cdot \vec{E}_\ell dV \quad (7-30b)$$

$$T_{cc} = \frac{1}{\tau} \int_{\tau} \vec{E}_c \cdot \vec{e}_c dV \quad (7-30c)$$

$$T_{\ell\ell} = \frac{1}{\tau} \int_{\tau} \vec{E}_\ell \cdot \vec{e}_\ell dV \quad (7-30d)$$

This relation describes the S-shaped loop-coupled chains in which the "loop" passband lies above the "cavity" passband but not the U-loop structure in which the passbands are interchanged.

We can verify dispersion Eqs. 7-30 with respect to the experimental data of Fig. 7-10 but only for the one- and four-pair sets of curves. The agreement with the four-pair curves is shown by the series of x 's near them. To calculate these points we took P_c , p_c , and P_ℓ as experimentally known and chose p_ℓ and m_ℓ according to the two $\pi/2$ points. The "loop" π mode will not resonate in a series of cavities electrically shorted on the cavity midplanes (see Fig. 7-12d). The quantity m_ℓ is negative for these curves, showing that

$$\int_{r'} \vec{e}_c \cdot \vec{E}_\ell dV < 0$$

which verifies our directions of the $\vec{E}_\ell$ lines near the axis in Fig. 7-12c.

Agreement between theory and measurement was also good for the two-pair structure (curves not shown).

Equation 7-30 shows that the first or "cavity" dispersion curve is depressed by the effect of the "loop" modes resonating at higher frequencies. This effect was also observed in resonant-slot coupled structures. To prove it, note that at frequencies in the first passband the second bracketed expression in Eq. 7-30 is positive, whereas $(P_c^2 - k^2)(P_\ell^2 - k^2)$ is negative, as is m_ℓ . Hence k^2 decreases a bit below its single mode-pair value at which the first bracketed expression would be zero.

We found that the preceding theory failed decisively for the eight-pair structure in that it predicted far too low an upper π cutoff frequency, p_ℓ . For eight or more loop-pairs we use the following empirical dispersion equation

$$\begin{aligned} & [\tfrac{1}{2}P_c^2(1 + \cos \varphi) + \tfrac{1}{2}P_c^2(1 - \cos \varphi) - k^2][\tfrac{1}{2}P_\ell^2(1 + \cos \varphi) \\ & + \tfrac{1}{2}P_\ell^2(1 - \cos \varphi) - k^2] - K \sin^2 \varphi = 0 \quad (7-31) \end{aligned}$$

Each bracketed expression alone set equal to zero is the single mode-pair dispersion relation for one of the passbands and coupling coefficient K is implicitly dependent only on the geometry. Coefficient K as well as p_ℓ must be chosen in accord with measured data. Points have been calculated and plotted for the eight- and twelve-pair curves of Fig. 7-10, for which we took the four cutoff frequencies as known and determined K so as to "fit" the lower $\pi/2$ point. Agreement with measurement is very good.

We do not know the precise reasons for this breakdown and only call attention to the fact that our theory begins to fail just before the inversion of the two π points which occurs when about ten pairs of loops occupy each loop plane. That is, Pearce has found that as the number of loop-pairs increases the two π points approach each other and cross over when the number of loop-pairs increases from eight to twelve. In the inverted condition, shown for the twelve-pair structure in Fig. 7-10, the cavity open-circuit mode resonates at the *upper* π cutoff frequency and is therefore observable in a structure terminated with the proper symmetry planes. The "loop" open-circuit mode resonates at the *lower* π cutoff frequency. Of course the two dispersion curves never actually cross. The inverted condition is to be avoided, because the fields at frequencies near the lower π cutoff frequency are described by the "loop" open-circuit mode which has no axial electric field for beam interaction work.

Curves of a parameter $\Delta f/f$ vs. $\varphi = \beta_0 L$ are reproduced from Pearce's article [5] in Fig. 7-13 for the structures with the curves of Fig. 7-10. There Δf refers to the change in resonant frequency induced by a long rod on the axis. In Appendix B we list the important relations between the

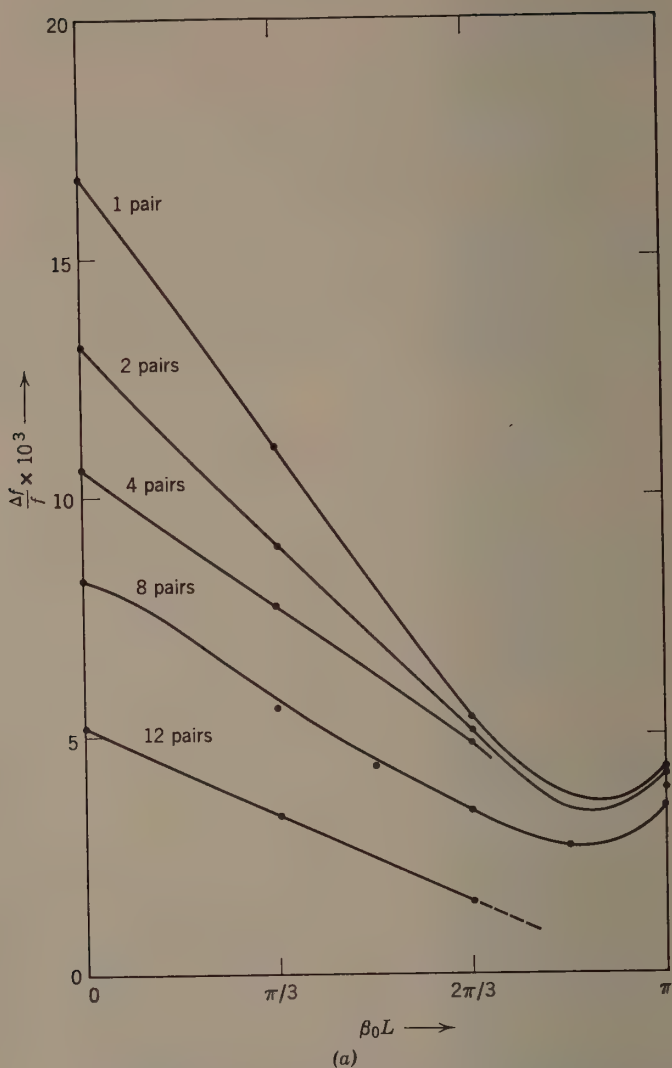


Fig. 7-13. $\Delta f/f$ proportional to the interaction parameter $(\hat{E}_z^2)_{\text{eff}}/W_s$ by Eq. 7-32 (from Pearce [5]). (a) For the first or "cavity" passband of several S-loop coupled chains.

measured $\Delta f/f$ and the interaction parameter E_z^2/W_s for various perturbing objects. Pearce's relation is

$$\frac{(E_z^2)_{\text{eff}}}{W_s} = -\frac{\Delta f}{f} \frac{480 c}{(\epsilon_r - 1)R^2} D = -\frac{\Delta f}{f} 1.03 \times 10^{16} D \quad (\text{mks units}) \quad (7-32)$$

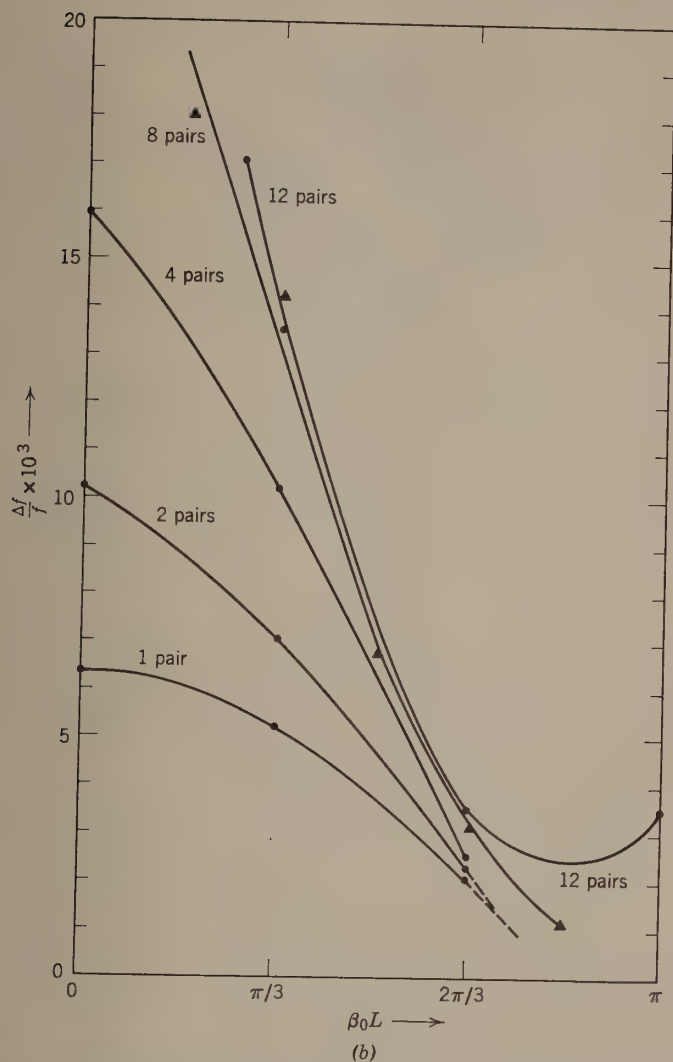


Fig. 7-13 (continued). (b) For the second or "loop" passband of those chains.

in which

c = light velocity

R = sapphire rod radius, 0.050 in.

ϵ_r = relative permittivity, taken as 10

and D is a factor which converts the measured E_z^2 to the effective $(E_z^2)_{\text{eff}}$ seen by an electron beam. In Appendix B, Section B-5, we show that D

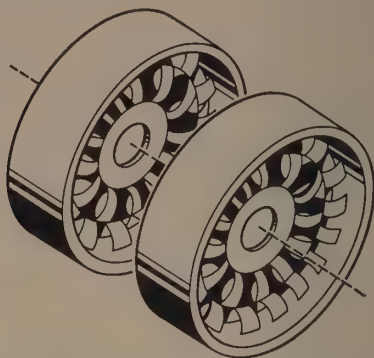


Fig. 7-14. A centipede cavity chain (from Chodorow and Craig [6]).

decreases from the value 1 for a beam in synchronism with the circuit near $\varphi = 0$ to the value 0.5 for a beam in synchronism near $\varphi = \pi$. An additional factor $I_0^2(\gamma b) - I_1^2(\gamma b) \cong 1 - (\gamma b)^2/4$ may be inserted into Eq. 7-32 to average E_z^2 over the beam cross section, where the beam has radius b .

The interaction parameter measured by $\Delta f/f$ tends to rise in both passbands as φ approaches zero because $\hat{E}_c$ tends to resemble the "straight-through" $\bar{E}_c$ pattern at frequencies in the "cavity" passband and the similar $\bar{E}_l$ pattern at frequencies in the "loop" passband. The inverted twelve-pair structure has no $\hat{E}_z(0) \sim \bar{e}_{lz}(0)$ on the axis at $\varphi = \pi$ in the "cavity" passband, as we observed earlier.

When the number of S loops fills the cavity circumference, we have a so-called centipede chain [6], shown in Fig. 7-14. The modes of the first two passbands are depicted by Fig. 7-12; also, there is another passband between these two in which the fields are angularly varying. The cutoff modes of this passband do not, however, couple to those of the other two passbands. Typical ω - φ curves for several choices of dimensions appear in Fig. 7-15. Although there are a dozen or so loop-pairs on the circumference the centipede is designed for noninversion of the π points. Equation 7-31 is the dispersion relation for the centipede chain, by which we have calculated points for each pair of curves in Fig. 7-15 by selecting P_c , P_ℓ , and p_c in accord with measurement and by choosing p_ℓ (the upper π cutoff frequency) and K so as to curve-fit the two $3\pi/4$ points. The calculated curves for the "loop" passband are subsequently extrapolated to the π point.

We can describe qualitatively the effects of various geometrical perturbations such as changes in loop height and cavity diameter by determining their effects upon the mode cutoff frequencies and thence on the

7-7. The Cloverleaf Chain

Figure 7-16 shows an exploded view of the cloverleaf chain which has dispersion curves like those shown on Fig. 7-10 for the chain of twelve U loops but resonant modes of different character. The periodicity operation in the cloverleaf chain is a translation by L and a rotation of $\pm 45^\circ$, as the more detailed Fig. 7-17a shows. A single cavity has four leaves and eight slots plus a centerhole in each slot plane. The neighbors are rotated 45° , as illustrated by the dashed leaf B_r (r for reference) of the cavity just behind the reference cavity shown.

Let us define and examine suitable cutoff "cavity" modes in order to discover why this structure has a positive group velocity and a wide dispersion curve over the range $0 \leq \varphi \leq \pi$. The flow of the magnetic field in the centerplane of the reference cavity is shown by the dashed lines in Fig. 7-17a. Because of the rotation of neighboring cavities the component of magnetic field parallel to the long axis of reference slot S_r in one cavity tends to cancel that component in the adjacent cavity. This means the tangential magnetic field along each slot will be quite small at $\varphi = 0$. Let us approximate the situation by placing magnetic shorting planes on the outer portions of the slots in Fig. 7-17b, with electric shorts on the inner ends of the slots and the centerholes where the electric field has the "straight-through" character. These planes define a hybrid open-circuit-short-circuit mode which we label simply the $\varphi = 0$ mode. At $\varphi = \pi$, the slot magnetic fields reinforce on both sides of each slot but cancel on both sides of each centerhole, so we define the $\varphi = \pi$ mode, Fig. 7-17c, by electric shorts on the outer ends of the slots and magnetic shorts on the inner ends and the centerholes.

These properties enable us to understand the effect of the slots on the "cavity" (lower) dispersion curve. Let us write Green's relation, Eq. 3-43b, in terms of the $\varphi = 0$ and π modes so as to obtain an exact relation

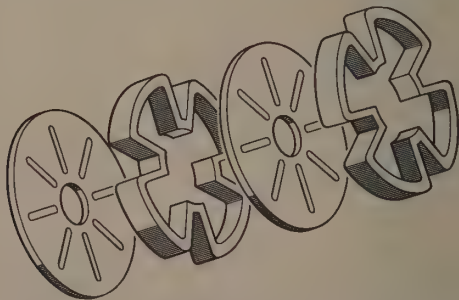


Fig. 7-16. A cloverleaf slow wave structure (from Chodorow and Craig [6]).

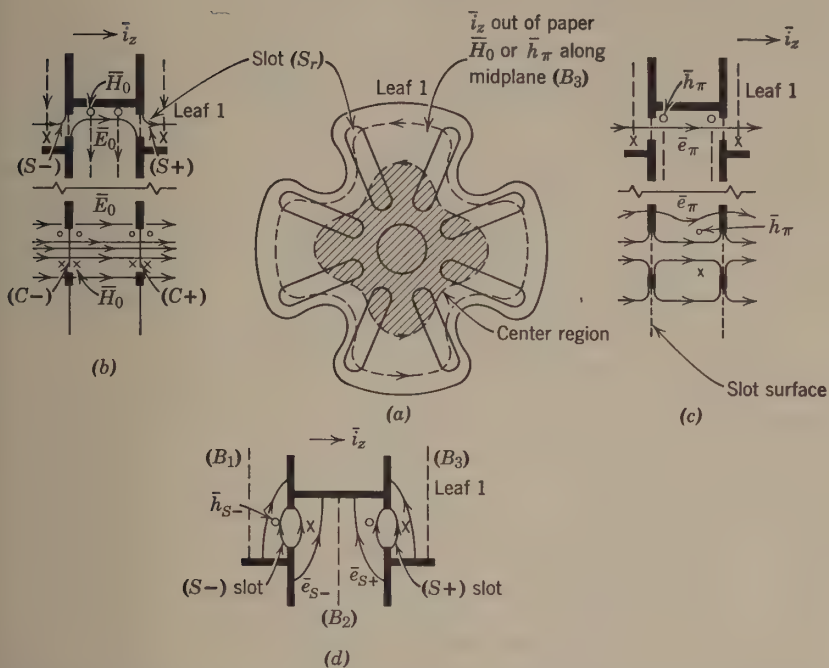


Fig. 7-17. The resonant modes of the cloverleaf cavity chain. (a) Geometry, showing the field patterns $\vec{H}_0$ and $\vec{h}_\pi$ of the resonant "cavity" modes, (b) the $\varphi = 0$ resonant "cavity" mode, (c) the $\varphi = \pi$ resonant "cavity" mode, (d) the resonant "slot" mode.

between P_0 , the resonant frequency of the $\varphi = 0$ mode and nearly the true $\varphi = 0$ cutoff frequency, and p_π , the π mode resonant frequency and very nearly the true π cutoff frequency. With $\vec{A} = \vec{E}_0$, $\vec{B} = \vec{e}_\pi$, $\nabla \times \vec{e}_\pi = p_\pi \vec{h}_\pi$, $\nabla \times \vec{h}_\pi = p_\pi \vec{e}_\pi$, $\nabla \times \vec{E}_0 = P_0 \vec{H}_0$, and $\nabla \times \vec{H}_0 = P_0 \vec{E}_0$, we obtain

$$\begin{aligned}
 (p_\pi^2 - P_0^2) \int_{\text{cav}(0)} \vec{E}_0 \cdot \vec{e}_\pi dV &= \oint_S (\vec{e}_\pi \times P_0 \vec{H}_0 - \vec{E}_0 \times p_\pi \vec{h}_\pi) \cdot \vec{n} dS \\
 &= -2P_0 \int_{\text{center region}} \vec{e}_\pi \times \vec{H}_0 \cdot \vec{i}_z dS + 2p_\pi \int_{S-}^{\text{slots}} \vec{E}_0^{(0)} \times \vec{h}_\pi^{(0)} \cdot \vec{i}_z dS \quad (7-33)
 \end{aligned}$$

Since the volume integral $\int \vec{E}_0 \cdot \vec{e}_\pi dV$ is positive we see that $p_\pi > P_0$, not only because the integral over the centerhole alone has its usual negative value but also because the slot integral is *positive* (see the pattern directions on Fig. 7-17b, c). It follows that the π cutoff frequency, with the slots effectively shorted, is only slightly higher than the π cutoff frequency of an equivalent centerhole-coupled chain, while the zero cutoff frequency is considerably lower because of the slot coupling. Such deductions based upon Green's relation are safer and easier than equivalent statements on

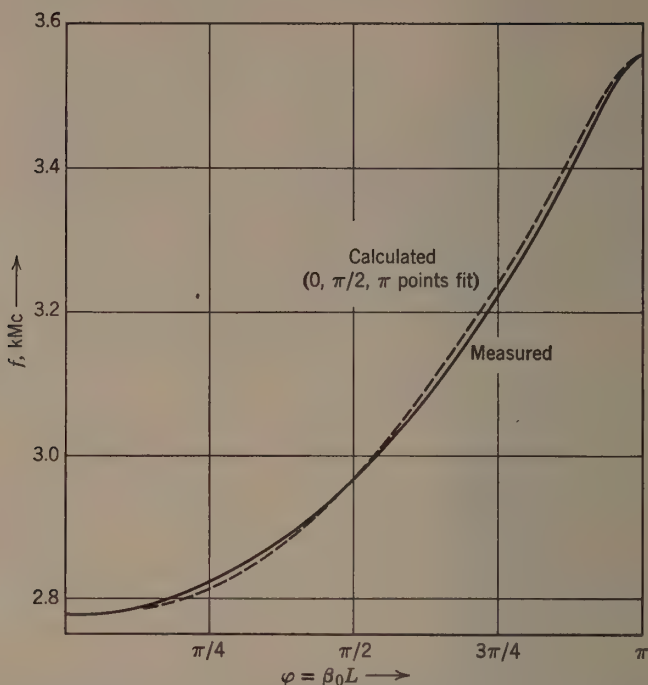


Fig. 7-18. Dispersion curve for a typical cloverleaf chain. The calculated curve was obtained from Eq. 7-34 in the manner explained in the text. The slots are half a free-space wavelength long at 4.1 kMc.

the basis of displaced electric and magnetic stored energies, particularly in a geometry of this sort.

The ω - φ curve for the “cavity” passband of a typical cloverleaf chain appears in Fig. 7-18. The “slot” passband lies above, with a π cutoff frequency of about 4.1 kMc. In order to develop an appropriate dispersion relation using the modes just introduced we define the resonant slot mode as a convenient means of representing tangential electric field on these relatively wide slots. This mode is defined between magnetically shorted cavity midplanes (B_1) and (B_2), and the $\bar{e}$ pattern has the configuration shown in Fig. 7-17d. We must distinguish the ($S-$) slot mode from its neighbor at the ($S+$) slot, also shown, because of the rotational effect.

In Appendix C we form a dispersion relation by representing the fields deep within the cavity by the “cavity” mode; this mode will resemble closely the fields in beam tubes at usual operating frequencies of about $2\pi/3$ “cold” circuit phase shift per cavity. This mode is excited by tangential electric field on the slots, given by the “slot” modes, and also tangential *magnetic* field on the centerholes, approximated best by the

“cavity” $\varphi = 0$ mode. The “slot” mode, in turn, is excited by tangential magnetic field on the midplanes, as expressed by the “cavity” $\varphi = \pi$ mode at the frequencies of interest. The result is the following dispersion relation for the “cavity” dispersion curve alone.

$$(p_s^2 - k^2)[(p_\pi^2 - k^2)U_c + p_\pi M_{cs}(1 + \cos \varphi) - k^2 m_{cs}^2(1 + \cos \varphi)] = 0 \quad (7-34a)$$

with p_s = the half-wavelength resonant frequency of a slot

p_π = the “cavity” π mode resonant frequency, taken to be the actual π cutoff frequency

$$U_c = \frac{1}{\tau} \int_V \vec{H}_0 \cdot \vec{h}_\pi dV$$

$$M_{cs} = \frac{1}{\tau} \int_{\text{center region}} \vec{e}_\pi^{(0)} \times \vec{H}_0 \cdot \vec{i}_z dS \quad (7-34b)$$

$$m_{cs}^2 \triangleq \frac{2S}{\tau^2} \int_{S-} \vec{e}_{S-} \times \vec{h}_\pi^{(0)} \cdot \vec{i}_z dS \int_{B2} \vec{e}_{S-} \times \vec{h}_\pi \cdot \vec{i}_z dS > 0$$

S = number of slots in a slot plane

This dispersion relation was used to calculate the dashed curve of Fig. 7-18, where p_s corresponds to 4.1 kMc, the slot half-wavelength frequency, which is very nearly the π cutoff frequency of the upper passband, p_π is taken as known, and M_{cs}/U_c and m_{cs}^2/U_c are chosen to “fit” the $\varphi = 0$ and $\pi/2$ points. The validity of Eq. 7-34 will now be further justified by calculation of the interaction parameter $E_z(0)^2/W_s$.

There is no difficulty in describing $E_z(0)$ along the axis at operating frequencies near the first π cutoff; these cavities are narrow enough to guarantee a nearly sinusoidal field pattern across the cavity (see Fig. 7-17c). However, calculation of W_s , the total time-average stored energy per cavity, is less straightforward because of the considerable overlap of the mode patterns of these wide slots with the “cavity” mode patterns. We therefore choose to calculate W_s as P_{av}/v_π , where P_{av} is the total power flow across a slot plane in terms of “slot” and “cavity” modes separately,

$$P_{av} = \frac{1}{2} \text{Re} \left[S \int_{S-} v_{S-} \vec{e}_{S-} \times \frac{1}{2} \vec{i}_\pi^* (1 - \epsilon^{-j\varphi}) \vec{h}_\pi^{(0)} \cdot \vec{i}_z dS \right. \\ \left. + \int_{\text{center region}} v_\pi \vec{e}_\pi^{(0)} \times \frac{1}{2} \vec{I}_0^* (1 + \epsilon^{-j\varphi}) \vec{H}_0 \cdot \vec{i}_z dS \right], \quad (7-35)$$

$$I_0 \cong i_\pi = \frac{p_\pi v_\pi}{-j\omega\mu}$$

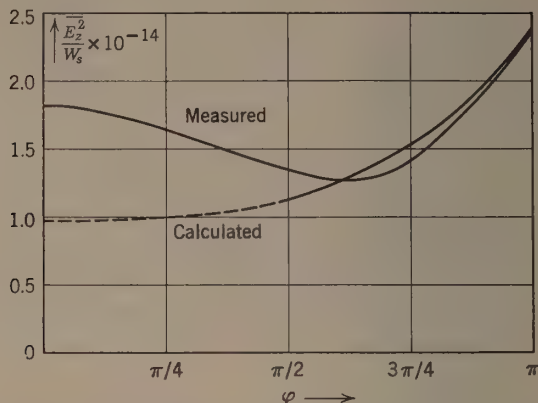


Fig. 7-19. $\overline{E_z^2}/W_s$, with $\overline{E_z^2}$ averaged on the axis across the cavity, for a cloverleaf chain. The calculated curve was obtained from Eq. 7-35 according to the remarks following that equation.

and v_g is the group velocity calculated from Eq. 7-34. The final equation in Appendix C yields a curve of

$$\int_0^L \frac{|E_z(0)|^2 dz}{W_s}$$

based upon the values of p_s , M_{cs} , and m_{cs} for the ω - φ curve which is plotted in Fig. 7-19. The π point has been fitted to the measured value because of the difficulty of normalizing the mode patterns to the cavity volume. Agreement is good down to $\varphi \cong 2\pi/3$, below which $E_z(0)$ rises as the field takes on the character of the “straight-through” mode at $\varphi = 0$.

We believe, on the basis of this discussion and subsequent computations, that the coupled-mode analysis of the cloverleaf structure is good enough to use in formulating the equations for the interaction between such a cavity chain and an electron beam. The beam interacts only with the $\bar{e}_z$ field of the “cavity” π mode at frequencies near the π cutoff so that the interaction is completely determined by the “cold” circuit dispersion relation and by the precise $\bar{e}_{\pi z}(0)$ distribution of the π mode pattern along the axis. In Section 8-6 we elaborate on this point.

Problems

1. Derive Eq. 7-3 from Maxwell's equations with tangential magnetic field on the boundaries of the slot cell.
2. In the resonant-slot coupled chain estimate the amplitude V_2 of the second resonant cavity mode, with $P_2 a = 5.52$, and show it is small with respect to V_1 at frequencies in the first “cavity” passband.

3. Estimate E^2/W_s at frequencies in the "cavity" passband by the first two short-circuit "cavity" modes, with I_1 given by Eqs. 7-1 and 7-8 and I_2 given by an analogous expression. The estimated curve, normalized to the value for the first cavity mode alone, predicts the decrease in the upper curve of Fig. 7-5 over the range $0 < \beta_0 L < \pi/2$.
4. Show that the irrotational modes of Eq. 7-14 satisfy the defining Eqs. 5-13 and 5-17.
5. Verify that the corrections to the lower dispersion curve of the rotated-cavity chain due to the second solenoidal cavity mode and the $\sin(3\pi x/\ell)$ -component of slot voltage are both negligible.
6. Show that any other choice but Eq. 7-22 does not yield a proper dispersion relation for the resonant-slot coupled chain.
7. How are the five parameters of Eq. 7-26 interrelated?
8. Study the $\bar{e}$ pattern on the U-loop coupling surface in Fig. 7-9. Verify the fact that it will raise the π point on Fig. 7-10 according to Table 4-1 on page 103. This implies that the variationally derived dispersion relation for the slot-coupled chain also serves to describe the U-loop coupled chain.
9. Derive Eq. 7-30a.
10. Does Eq. 7-31 imply the frequency-squared sum rule for the first two passbands of a centipede chain?
11. If two ω - φ curves did cross at a point in the range $0 < \varphi < \pi$, as they occasionally do in a structure with mirror symmetry, what is implied about the symmetries of their fields? (*Hint*: consider the properties of the group velocity as determined by the extended version of Eq. 5-50).
12. Verify from the cloverleaf symmetry, Fig. 7-17, that all the slots in a plane do cooperate in exciting the fields of the adjacent cavities.

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Chapter VIII

Analyses of Electron Beam Interaction with Various Slow Wave Structures

8-1. Synopsis

This Chapter is primarily devoted to a variational description of longitudinal electron beam interaction with general helices and cavity chains. In Section 8-2 we review H. A. Haus' derivation of the Pierce determinantal equation for the beam interacting with essentially one "cold" circuit mode of a helix. The fields are represented by this circuit mode as well as by the ordinary fast and slow space charge modes, which also represent the beam kinetic voltage and current. By inserting these trial fields and beam excitations into a variational expression for propagation constant Γ ($e^{-\Gamma z}$ dependence of all quantities) the amplitudes are adjusted so as to make Γ stationary. When we make certain reasonable approximations the equation for the incremental propagation constant assumes the Pierce form.

In Section 8-3 we essentially derive the same determinantal equation from a variational expression either for $j\omega$ at fixed Γ or for Γ at fixed ω , which has the merit of accounting more carefully for the boundary conditions in the "hot" circuit. If the beam is interacting with only one circuit mode and its propagation constant is only slightly perturbed, we show that the equation for the incremental propagation constants in the composite system must be of the Pierce form. For more general helices with axial periodicity and more complicated boundary conditions this extended variational expression is more appropriate. For example, we could use it to study beam interaction with two modes of the tape helix, say, at a frequency and beam velocity such that the beam and circuit are

synchronous near a "cold" circuit resonance where the group velocity is zero or nearly so.

The variational expression for $j\omega$, appropriately modified to include the electron beam, also serves to yield a determinantal equation for complex phase shift γ ($\epsilon^{-\gamma}$ phase shift per cavity) of a symmetrical cavity chain. Now we represent the fields of the n th passband by the two open- and short-circuit modes, the amplitudes of which are chosen so as to make ω stationary. Provided the beam is nearly synchronous with the "cold" circuit fields and the "cold" phase shift is not altered too much, we can cast the determinantal equation for the incremental phase shift per cavity into Pierce form. In many structures the modes of the second passband contribute appreciably to the fields at frequencies in the first passband, but if certain weak-beam conditions are met the dispersion equation remains in Pierce form.

In Section 8-5 we extend the analysis of the resonant-slot coupled chain as treated in Section 7-2 to account for the presence of a beam. The determinantal equation for phase shift γ assumes the Pierce form modified by a factor $(P_1/k)^2 \cong 1$ at frequencies in the "cavity" passband. For more general structures, including the more complicated resonant-slot and resonant-loop coupled chains, we formulate an equivalence principle to simplify the beam-circuit interaction problem. According to this principle the interaction and the dispersion are completely determined provided (a) the beam "sees" only the pair of resonant cavity modes of one passband, (b) the $\vec{E}_z$ field pattern in the beam region is completely known, and (c) an accurate "cold" circuit dispersion relation for this passband is known. The ordinary resonant-slot coupled chain obeys all these criteria at frequencies in the "cavity" passband where the beam does not "see" the slot fields which are very weak in the beam region.

8-2. Variational Treatment of the Interaction between One Helix Mode and the Beam

We first consider the interaction of one circuit mode-harmonic propagating on an arbitrary helix or other slow wave structure as $\epsilon^{-j\beta_c z}$ in the absence of the beam with the fast and slow space charge modes which propagate as $\epsilon^{-j(\beta_c \pm \beta_0)z}$ on the uncoupled beam. Here β_0 is the reduced plasma propagation constant. The analysis follows that of Haus [1] with some changes in his notation and more details of the development.

We first construct a variational expression for propagation constant Γ of the *coupled* beam-helix system. Periodicity along the axial direction is neglected, so that Maxwell's equations with the curl operator split into

transverse and longitudinal portions are

$$\nabla_T \times \hat{E}_+ - \Gamma \hat{i}_z \times \hat{E}_+ = -j\omega\mu\hat{H}_+ \quad (8-1a)$$

$$\nabla_T \times \hat{H}_+ - \Gamma \hat{i}_z \times \hat{H}_+ = j\omega\varepsilon\hat{E}_+ + \hat{J}_+ \quad (8-1b)$$

Subscript plus identifies one complete set of field and beam excitations which propagate as $e^{-\Gamma z}$. The circuit is coupled to the beam through $\hat{J}_+$, and the beam equations of Appendix D take the form

$$\begin{aligned} (j\beta_e - \Gamma)\hat{V}_{k+} &= \hat{E}_{z+} \\ (j\beta_e - \Gamma)\hat{J}_+ &= jY_0\beta_q\hat{V}_{k+} \end{aligned} \quad Y_0 = Z_0^{-1} = \frac{1}{2} \frac{|J_0|}{U_0} \frac{\omega}{\omega_q} R \quad (8-2)$$

Throughout this chapter the beam is considered to flow only along the z axis. To form the expression for Γ with the derivation of Eq. 6-45 as a guide we dot-multiply Eq. 8-1a by an independent (adjoint) magnetic field $\hat{H}_-^*$ and Eq. 8-1b, by $-\hat{E}_-^*$. Here the asterisk denotes a complex conjugate. Subscript minus denotes these independent fields, the nature of which will be specified in order to guarantee the variational property of the Γ expression. We also multiply Eq. 8-2a by $\hat{J}_-^*$ and Eq. 8-2b by $\hat{V}_{k-}^*$. Addition of these four equations yields the variational expression for Γ :

$$\begin{aligned} \Gamma \int_A [(\hat{E}_+ \times \hat{H}_-^* + \hat{E}_-^* \times \hat{H}_+) \cdot \hat{i}_z + \hat{V}_{k+} \cdot \hat{J}_-^* + \hat{V}_{k-}^* \cdot \hat{J}_+] dA \\ = \int_A [(\nabla_T \times \hat{E}_+ + j\omega\mu\hat{H}_+) \cdot \hat{H}_-^* - (\nabla_T \times \hat{H}_+ - j\omega\varepsilon\hat{E}_+ - \hat{J}_+) \cdot \hat{E}_-^* \\ + (j\beta_e\hat{V}_{k+} - \hat{E}_+) \cdot \hat{J}_-^* + (j\beta_e\hat{J}_+ - jY_0\beta_q\hat{V}_{k+}) \cdot \hat{V}_{k-}^*] dA \quad (8-3) \end{aligned}$$

We may prove the variational nature of Eq. 8-3 by demonstrating either (1) that small errors in the correct fields, $\delta\hat{E}_+$, $\delta\hat{H}_-$, and the others generate an error in Γ , $\delta\Gamma$, which is proportional to $(\delta\hat{E}_+)^2$, $(\delta\hat{H}_-)^2$, and the others, or (2) that if $\delta\Gamma$ is set equal to zero for all small variations $\delta\hat{E}_+$, $\delta\hat{H}_-$, and the others, with neglect of the quadratic errors, then the trial fields and beam excitations must satisfy Eqs. 8-1 and 8-2. The latter procedure is adopted in practice. From the remarks in Section 6-3A it is obvious that if $\delta\Gamma$ is zero relative to all small variations of the minus fields and beam excitations, the plus quantities must satisfy Eqs. 8-1 and 8-2 with neglect of quadratic terms. Now let us set $\delta\Gamma = 0$ relative to $\delta\hat{E}_+$, say. When we invert the $\nabla_T \times \delta\hat{E}_+$ term we obtain

$$\int_A \nabla_T \times \delta\hat{E}_+ \cdot \hat{H}_-^* dA = \int_A \delta\hat{E}_+ \cdot \nabla_T \times \hat{H}_-^* dA + \oint_C \delta\hat{E}_+ \times \hat{H}_-^* \cdot d\bar{S}$$

C being the contour of intersection of the cross-sectional plane A with metal surfaces. We assume the trial fields $\hat{E}_\pm$ are properly normal to C so

that this contour integral vanishes. Then Eq. 8-3 leaves us with

$$\begin{aligned} \Gamma \int_A \delta \hat{E}_+ \cdot \hat{H}_-^* \times \hat{i}_z dA \\ = \int_A (\delta \hat{E}_+ \cdot \nabla_T \times \hat{H}_-^* + j\omega\epsilon \delta \hat{E}_+ \cdot \hat{E}_-^* - \delta \hat{E}_+ \cdot \hat{J}_-^*) dA \end{aligned} \quad (8-4a)$$

and this rearranges into

$$0 = \int_A (\nabla_T \times \hat{H}_-^* + \Gamma \hat{i}_z \times \hat{H}_-^* + j\omega\epsilon \hat{E}_-^* - \hat{J}_-^*) \cdot \delta \hat{E}_+ dA \quad (8-4b)$$

Since this is true for *arbitrary* $\delta \hat{E}_+$ within the cross section we see that the adjoint quantities obey the equation

$$\nabla_T \times \hat{H}_- + \Gamma^* \hat{i}_z \times \hat{H}_- = j\omega\epsilon \hat{E}_- + \hat{J}_- \quad (8-5)$$

that is, the same equation as the plus quantities except that $-\Gamma$ is replaced by $+\Gamma^*$. The adjoint excitations are *defined* as propagating as $\epsilon^{+\Gamma^*z}$. Analogously we may prove that if $\delta\Gamma = 0$ relative to $\delta\hat{H}_+$, $\delta\hat{V}_{k+}$, and $\delta\hat{J}_+$, then the remainder of Eqs. 8-1 and 8-2 are satisfied by the adjoint excitations provided $+\Gamma^*$ replaces $-\Gamma$.

In a lossless system with no energy sources $\Gamma = j\beta$ and $-\Gamma = +\Gamma^*$. But with ohmic loss present (which we neglect) or with a beam as an energy source, $\Gamma = \alpha + j\beta$, whereupon it follows that if the plus excitations propagating as $\epsilon^{(-\alpha-j\beta)z}$ represent a growing wave the minus excitations form a decaying wave. Physically these two waves are equally significant in a system with no ohmic loss, so we always have a growing-decaying wave pair in a circuit coupled to a beam.

Now that we are assured of the variational nature of Eq. 8-3 we can proceed to derive a determinantal equation for Γ . The most practical case covered by this analysis is the one in which the d.c. beam is synchronous or nearly so with the predominant $n = 0$ mode of a tape helix, at point *A*, for example, in Fig. 8-1. This point is sufficiently below resonance point *D* so that the "cold" circuit is truly propagating with a phase velocity $k/\beta \cong \tan \psi$. Our development to follow will also be appropriate for interaction with the $n = +1$ mode at point *B*. It will only be appropriate, however, for interaction near points *C* and *D* with appropriate modification of the trial functions. In the former case *both* the $n = 0$ and $+1$ circuit modes would be required because the $n = +1$ mode with propagation constant $\beta_{+1} \cong k$ primarily determines the "cold" circuit dispersion according to the remarks of Section 6-3C. In the latter case of interaction near point *D*, the excitation of the $n = -1$ mode is comparable to that of the $n = 0$ mode for the same reason.

To represent the fields at the operating frequency and beam velocity

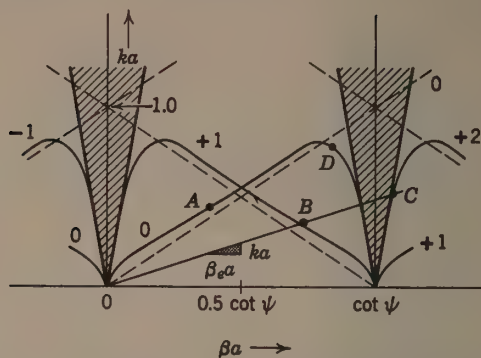


Fig. 8-1. A portion of the ω - β diagram (Fig. 2-4) for the single-tape helix, showing an electron beam of relative velocity $k/\beta_0 = u_0/c$ in synchronism with circuit waves at points B and C .

such that the beam is synchronous or nearly so at point A we employ the “cold” circuit mode of field patterns $\bar{E}_c(x, y)$ and $\bar{H}_c(x, y)$, as well as the slow and fast beam modes of patterns $\bar{E}_1(x, y)$, $\bar{H}_1(x, y)$, $\bar{V}_{k1}(x, y)$, and $\bar{J}_1(x, y)$ and $\bar{E}_2, \dots, \bar{J}_2$, respectively. We assign them amplitudes a_c , a_1 , and a_2 and write the trial excitations as

$$\hat{E}_+(x, y) = a_c \bar{E}_c + a_1 \bar{E}_1 + a_2 \bar{E}_2, \quad \hat{V}_{k+}(x, y) = a_1 \bar{V}_{k1} + a_2 \bar{V}_{k2} \quad (8-6a, c)$$

$$\hat{H}_+(x, y) = a_0 \bar{H}_0 + a_1 \bar{H}_1 + a_2 \bar{H}_2, \quad \hat{J}_+(x, y) = a_1 \bar{J}_1 + a_2 \bar{J}_2 \quad (8-6b, d)$$

The "cold" circuit patterns $\bar{E}_c$ and $\bar{H}_c$ are understood to satisfy the relations

$$\nabla_T \times \bar{E}_c - j\beta_c \bar{i}_z \times \bar{E}_c = -j\omega\mu\bar{H}_c, \quad (8-7a)$$

$$\nabla_T \times \bar{H}_c - j\beta_c \bar{i}_z \times \bar{H}_c = j\omega\epsilon \bar{E}_c \quad (8-7b)$$

while the “uncoupled” slow beam mode patterns satisfy

$$\nabla_T \times \bar{E}_1 - j\beta_g \bar{i}_z \times \bar{E}_1 = -j\omega\mu\bar{H}_1, \quad -j\beta_g \bar{V}_{k1} = \bar{E}_{z1} \quad (8-8a,c)$$

$$\nabla_T \times \bar{H}_1 - j\beta_g \bar{i}_z \times \bar{H}_1 = j\omega\epsilon\bar{E}_1 + \bar{J}_1, \quad -j\beta_g \bar{J}_1 = jY_0\beta_g \bar{V}_{k1} \quad (8-8b,d)$$

with $\beta_s = \beta_e + \beta_y$

This mode has a negative admittance $J_1/\mathcal{V}_{k1} = -Y_0$, so that the a.c. power flow per unit amplitude a_1 is $-\frac{1}{2}Y_0 |\mathcal{V}_{k1}|^2$ per unit area. The patterns of the fast beam mode are defined by Eqs. 8-8 but in terms of E_2 , H_2 , $\mathcal{V}_{k2}$, and J_2 and with $\beta_f = \beta_e - \beta_a$ in place of β_s . We may represent the adjoint wave excitations of subscript minus by exactly the same patterns but with independent amplitudes b_c , b_1 , and b_2 in Eqs. 8-6. All these trial fields and excitations now go into variational Eq. 8-3, which simplifies somewhat when we use Eqs. 8-7 and 8-8. Two of the $a_1 b_2^*$ and $a_2 b_1^*$ terms also

cancel if we specify *pattern* $\bar{J}_1$ as real and equal to $\bar{J}_2$, whereupon the complex *amplitudes* will specify the precise excitation of each beam mode. It follows from the last of Eqs. 8-8 for the slow mode and the relation $+j\beta_c \bar{J}_2 = jY_0 \beta_q \bar{V}_{k2}$ for the fast mode that $\bar{V}_{k1} = -\bar{V}_{k2}$.

If we define these "cold" circuit and uncoupled beam mode power flows, per unit amplitude of a_c , a_1 , and a_2 ,

$$\begin{aligned}
 P_{av,c} &= \frac{1}{4} \int_A (\bar{E}_c \times \bar{H}_c^* + \bar{E}_c^* \times \bar{H}_c) \cdot \bar{i}_z dA, & \text{circuit mode power} \\
 P_{av,c1} &= \frac{1}{4} \int_A (\bar{E}_c \times \bar{H}_1^* + \bar{E}_1^* \times \bar{H}_c) \cdot \bar{i}_z dA, & \text{a complex crosspower} \\
 P_{av,c2} &= \frac{1}{4} \int_A (\bar{E}_c \times \bar{H}_2^* + \bar{E}_2^* \times \bar{H}_c) \cdot \bar{i}_z dA, & \text{another crosspower} \\
 P_{av,1} &= \frac{1}{4} \int_A [(\bar{E}_1 \times \bar{H}_1^* + \bar{E}_1^* \times \bar{H}_1) \cdot \bar{i}_z & \text{beam} \\
 & \quad + \bar{V}_{k1} \cdot \bar{J}_1^* + \bar{V}_{k1}^* \cdot \bar{J}_1] dA & \text{mode} \\
 P_{av,2} &= \frac{1}{4} \int_A [(\bar{E}_2 \times \bar{H}_2^* + \bar{E}_2^* \times \bar{H}_2) \cdot \bar{i}_z & \text{powers} \\
 & \quad + \bar{V}_{k2} \cdot \bar{J}_2^* + \bar{V}_{k2}^* \cdot \bar{J}_2] dA
 \end{aligned} \tag{8-9}$$

and set $\delta\Gamma = 0$ relative to δb_c , δb_1 , and δb_2 successively, we obtain the following three equations for amplitudes a_c , a_1 , and a_2 :

$$\begin{aligned}
 0 &= a_c j\beta_c (4P_{av,c}) + a_1 j\beta_s (4P_{av,c1}^*) + a_2 j\beta_f (4P_{av,c2}^*) \\
 & \quad - \Gamma [a_c (4P_{av,c}) + a_1 (4P_{av,c1}^*) + a_2 (4P_{av,c2}^*)] \\
 0 &= a_c \left[j\beta_c (4P_{av,c1}) - \int_A \bar{E}_c \cdot \bar{J}_1^* dA \right] + a_1 j\beta_s (4P_{av,1}) \\
 & \quad + a_2 j\beta_f \int_A (\bar{E}_2 \times \bar{H}_1^* + \bar{E}_1^* \times \bar{H}_2) \cdot \bar{i}_z dA \\
 & \quad - \Gamma \left[a_c (4P_{av,c1}) + a_1 (4P_{av,1}) + a_2 \int_A (\bar{E}_2 \times \bar{H}_1^* + \bar{E}_1^* \times \bar{H}_2) \cdot \bar{i}_z dA \right] \\
 0 &= a_c \left[j\beta_c (4P_{av,c2}) - \int_A \bar{E}_c \cdot \bar{J}_2^* dA \right] \\
 & \quad + a_1 j\beta_s \int_A (\bar{E}_1 \times \bar{H}_2^* + \bar{E}_2^* \times \bar{H}_1) \cdot \bar{i}_z dA + a_2 j\beta_f (4P_{av,2}) \\
 & \quad - \Gamma \left[a_c (4P_{av,c2}) + a_1 \int_A (\bar{E}_1 \times \bar{H}_2^* + \bar{E}_2^* \times \bar{H}_1) \cdot \bar{i}_z dA + a_2 (4P_{av,2}) \right]
 \end{aligned} \tag{8-10}$$

The determinant of this matrix of coefficients set equal to zero yields the allowed values of Γ , one for each of the three independent waves of the composite system.

We can eliminate four terms above by showing

$$\int_A (\bar{E}_2 \times \bar{H}_1^* + \bar{E}_1^* \times \bar{H}_2) \cdot \bar{i}_z dA \equiv 0$$

from an orthogonality relation obtained by dot-multiplying Eq. 8-8a conjugated by $-\bar{H}_2$, Eq. 8-8b conjugated by $\bar{E}_2$, and the corresponding equations for the fast beam mode by $-\bar{H}_1^*$ and $\bar{E}_1^*$, respectively. By addition we obtain, with the aid of Eq. 8-8c,d,

$$\begin{aligned} -2j\beta_q \int_A (\bar{E}_2 \times \bar{H}_1^* + \bar{E}_1^* \times \bar{H}_2) \cdot \bar{i}_z dA \\ = \int_A (J_1^* \cdot \bar{E}_2 + J_2 \cdot \bar{E}_1^*) dA = +j\beta_q \cdot \\ \int_A (\bar{V}_{k1}^* \cdot \bar{J}_2 + \bar{V}_{k2} \cdot \bar{J}_1^*) dA \sim \int_A (\bar{V}_{k1} \cdot \bar{J}_1 - \bar{V}_{k1} \cdot \bar{J}_1) dA = 0 \quad (8-11) \end{aligned}$$

Equation 8-10 therefore yields this determinantal equation for Γ :

$$\begin{vmatrix} (j\beta_c - \Gamma)4P_{av,c} & (j\beta_s - \Gamma)4P_{av,c1}^* & (j\beta_f - \Gamma)4P_{av,c2}^* \\ (j\beta_c - \Gamma)4P_{av,c1} - \int_A \bar{E}_c \cdot \bar{J}_1 dA & (j\beta_s - \Gamma)4P_{av,1} & 0 \\ (j\beta_c - \Gamma)4P_{av,c2} - \int_A \bar{E}_c \cdot \bar{J}_1 dA & 0 & (j\beta_f - \Gamma)4P_{av,2} \end{vmatrix} = 0 \quad (8-12)$$

The 12 (row 1, column 2) and 13 terms are converted into forms resembling the 21 and 31 terms as follows. From Eqs. 8-7 and 8-8 we can derive

$$(j\beta_{s,f} - j\beta_c) \int_A (\bar{E}_{1,2} \times \bar{H}_c^* + \bar{E}_c^* \times \bar{H}_{1,2}) \cdot \bar{i}_z dA = \int_A \bar{E}_c^* \cdot \bar{J}_1 dA \quad (8-13a)$$

by which the 12 and 13 terms can be written

$$\begin{aligned} (j\beta_c - \Gamma)4P_{av,c1}^* + \int_A \bar{E}_c^* \cdot \bar{J}_1 dA \quad (J_1 \equiv J_2) \\ (j\beta_c - \Gamma)4P_{av,c2}^* + \int_A \bar{E}_c^* \cdot \bar{J}_1 dA \quad \text{respectively} \end{aligned} \quad (8-13b)$$

To simplify these equations we introduce the Pierce parameters [2]. The synchronism parameter b_P (subscript P for Pierce) relates the "cold" circuit propagation constant to that of the d.c. beam as

$$\beta_c = \beta_e(1 + C_P b_P), \quad C_P \text{ to be defined} \quad (8-14a)$$

and incremental propagation constant δ_P is defined by

$$\Gamma = j\beta_e - \beta_e C_P \delta_P \quad (8-14b)$$

We now make the usual assumption that the crosspower terms in Eq. 8-13 are small with respect to

$$\int_A \bar{E}_c^* \cdot J_1 dA$$

Then Eq. 8-12 is expanded and simplified into the well-known form

$$(j\delta_P - b_P) \left[\delta_P^2 + \left(\frac{\beta_q}{\beta_e C_P} \right)^2 \right] = 1 \quad (8-15a)$$

provided we define C_P as

$$C_P^3 = \frac{\left| \int_A \bar{E}_c \cdot J_1 dA \right|^2 \beta_q}{8\beta_e^3 P_{av,c} (-P_{av,1})}, \quad P_{av,1} < 0 \quad (8-15b)$$

For a thin beam, $\bar{V}_{k1}$ and J_1 are nearly uniform in the cross section, so $P_{av,1} = +\frac{1}{2} \bar{V}_{k1} J_1 A_b$, A_b being the beam area. With $J_1 / \bar{V}_{k1} = -\frac{1}{2} |J_0| \beta_e / U_0 \beta_q$ we obtain

$$C_P^3 (\text{thin beam}) = \frac{|E_c|^2}{2\beta_e^2 P_{av,c}} \frac{|I_0|}{4U_0} \quad (8-16)$$

Here C_P is the gain parameter because propagation of each independent wave, in particular the growing wave, is as $e^{-\Gamma z} = e^{-j\beta_e z} e^{\beta_e C_P \delta_P z}$. Many solutions of Eq. 8-15 are available [3,4] over a wide range of values of C_P , b_P , and space charge parameter $\beta_q / \beta_e C_P$, so commonly denoted by $\sqrt{4(QC)_P}$.

Suppose we treat briefly the case in which only one of the beam modes is nearly in synchronism with the "cold" circuit. Then two of the composite system waves will have propagation constants which are nearly equal to the "cold" circuit propagation constant; the third wave is essentially the uncoupled beam mode. To find the incremental propagation constants of the first two waves we can reduce Eq. 8-12 to the 2×2 form which involves the "cold" circuit mode and the nearly synchronous beam mode. The resultant expression for δ shows that the two new waves are a growing-decaying pair if the power flows of the uncoupled modes have opposite signs as, for example, a circuit mode with group velocity in the direction of the beam flow and the slow beam mode. The two new waves are a constant-amplitude pair if these power flows have the same signs.

8-3. Extended Variational Treatment of the Helix-Beam Interaction

Variational Eq. 8-3 is easy enough to use when we account for axial periodicity over a period L ; we simply write the trial fields in the form

$$\hat{E}(x, y, z) = \hat{E}_p(x, y, z)e^{-\Gamma z}, \quad \text{where} \quad \hat{E}_p(x, y, z + L) = \hat{E}_p(x, y, z)$$

and integrate over a period L . However, it suffers from a more fundamental defect. It *assumes* that the “cold” circuit fields are known to the extent they satisfy the proper boundary conditions on metal surfaces and open surfaces between the inner and outer helix regions. But the “cold” circuit fields are known exactly only for the idealistic sheath helix. In fact, the whole of Section 6-3 treated the problem of determining the relative amplitudes of sheath helix modes comprising a tape helix wave, and the dispersion relation. Therefore we now analyze the problem so as to account not only for more than one axial harmonic in the trial fields but also the boundary conditions in the “hot” circuit. To modify variational Eq. 8-3 we simply change Γ to $-\partial/\partial z$ and add to the left side the W_i and W_o integrals over the wire (or other metal surfaces) and the D integrals which appear on the right side of Eq. 6-45. The argument of the n th Bessel functions I_n and K_n is now understood to be the complex γ_n . Alternatively, we could obtain the same variational expression by modifying Eq. 6-45 as follows. Assuming the beam travels in the inner A_i region of the helix, the changes are:

- (a) The first bracketed expression is augmented by $-\hat{J}_+$
- (b) These integrals are added to the right side:

$$\begin{aligned} & - \int_{A_i} \hat{J}_-^* \cdot \left[\left(j\beta_o + \frac{\partial}{\partial z} \right) \hat{V}_{k+} - \hat{e}_+ \right] dA \\ & \quad - \int_{A_i} \hat{V}_{k-}^* \cdot \left[\left(j\beta_o + \frac{\partial}{\partial z} \right) \hat{J}_+ - jY_0\beta_o \hat{V}_{k+} \right] dA \end{aligned}$$

Then it is easy to prove that if $\delta(j\omega) = 0$ for arbitrary $\delta V_{k+}, \dots, \delta \hat{J}_-$, not only are Maxwell's equations satisfied by the plus and minus excitations, but also the beam equations

$$\left(j\beta_o + \frac{\partial}{\partial z} \right) \hat{V}_{k\pm} = \hat{e}_{z\pm} = (\hat{e}_{z\pm})_{\text{cav}} + jZ_0\beta_o \hat{J}_{\pm} \quad (8-17a)$$

$$\left(j\beta_o + \frac{\partial}{\partial z} \right) \hat{J}_{\pm} = jY_0\beta_o \hat{V}_{k\pm} \quad (8-17b)$$

are satisfied. The proof is developed with the work following Eq. 6-45 as a guide. The modified Eq. 8-3 in terms of Γ is equivalent to the modified Eq. 6-45 in terms of $j\omega$, with the curl operator equal to $\nabla_T \times + \hat{i}_z(\partial/\partial z) \times$, since Γ is stationary at fixed ω when ω is stationary at fixed Γ .

We now merely outline the derivation of determinantal Eq. 8-15 from the new variational expression rather than from Eq. 8-3. The beam is again assumed to interact with the single $n = 0$ "cold" circuit mode at the frequency corresponding to point *A* of Fig. 8-1. We assume $e^{-\Gamma z}$ propagation, with $\text{Im } \Gamma \cong j\beta_0$, and neglect the periodic nature of the weaker components of the "cold" circuit wave to which the $n = 0$ mode belongs. The circuit contribution to the trial fields is now furnished by the $n = 0$ sheath mode components of Eqs. 6-51 and 6-52 with $n = 0$, Γ replacing $j\beta_0$, and radial propagation constant $\gamma = j\sqrt{\Gamma^2 + k^2}$ understood to be complex. These fields replace $a_c \bar{E}_c$ and $a_c \bar{H}_c$ of Eq. 8-6a,b so that $\hat{E}_+$, for example, reads

$$\begin{aligned} \hat{E}_+ = \hat{i}_z B_3 I_0(\gamma r) + \hat{i}_r B_3 \frac{\Gamma}{\gamma} I_1(\gamma r) \\ - \hat{i}_\theta B_1 \frac{j\omega\mu}{\gamma} I_1 + a_1 \bar{E}_1 + a_2 \bar{E}_2, \quad r \leq a \quad (8-18a) \end{aligned}$$

The first three terms with variational amplitudes B_3 and B_1 satisfy Maxwell's equations, without the beam terms but with γ complex, exactly. Here $a_1 \bar{E}_1$ and $a_2 \bar{E}_2$ again represent the beam mode contributions to the field. In the outer helix region, $r \geq a$, we have from Eq. 6-52

$$\hat{E}_+ = \hat{i}_z B_4 K_0(\gamma r) - \hat{i}_r B_4 \frac{\Gamma}{\gamma} K_1(\gamma r) + \hat{i}_\theta B_2 \frac{j\omega\mu}{\gamma} K_1, \quad r \geq a \quad (8-18b)$$

in which B_4 and B_2 are variational amplitudes. We have analogous expressions for $\hat{H}_+$ in both regions, and the adjoint fields are identical in form but for independent amplitudes $C_1, \dots, C_4$.

With these new trial fields in the new variational equation, we obtain a dispersion relation for $\omega(\Gamma)$ by varying each adjoint amplitude in turn, including those adjoint to b_1 and b_2 . We choose to specify $\hat{e}_\parallel = -\hat{E}_\parallel$ explicitly just as in Section 6-3D, whereby one of the sheath mode amplitudes, say B_4 , is expressed in terms of the other three. Three circuit equations arise, in terms of B_1 , B_2 , and B_3 , which are exactly the same as in the "cold" circuit except for the complex γ and $\Gamma = j\beta$ and the new beam coupling terms

$$a_1 \int_A I_0^*(\gamma r) J_{1z} dA + a_2 \int_A I_0^*(\gamma r) J_{2z} dA$$

in the equation obtained by varying C_3^* . Variation of the adjoint beam

mode amplitudes yields essentially Eq. 8-10*b,c* except that the circuit terms proportional to a_c are replaced by

$$-B_3 \int I_0(\gamma r) \bar{J}_{1z}^* dA \quad \text{and} \quad -B_3 \int I_0(\gamma r) \bar{J}_{2z}^* dA$$

respectively. We again neglect the crosspowers defined in Eq. 8-9*b,c*. Then the functional *form* of the dispersion relation is

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	0	0
$\mathcal{D}$	$\mathcal{E}$	$\mathcal{F}$	0	0
$\mathcal{G}$	$\mathcal{H}$	$\mathcal{K}$	$\int I_0^* \bar{J}_{1z}$	$\int I_0^* \bar{J}_{2z}$
0	0	$-\int I_0 \bar{J}_{1z}^*$	$(j\beta_s - \Gamma)4P_{av,1}$	0
0	0	$-\int I_0 \bar{J}_{2z}^*$	0	$(j\beta_f - \Gamma)4P_{av,2}$

$$= 0 \quad (8-19)$$

$$\text{with } \beta_s = \beta_e + \beta_a, \quad \beta_f = \beta_e - \beta_a$$

The determinant of the upper 3×3 matrix set equal to zero would give us the “cold” helix dispersion relation; the determinant of the lower right-hand 2×2 portion set equal to zero yields $\Gamma^0 = j(\beta_e \pm \beta_a)$ of the uncoupled beam.

We can expand the determinant of Eq. 8-19, introduce the Pierce notation, and proceed to derive Eq. 8-15 for the incremental propagation constant, where now

$$\int_{A_i} I_0(\gamma r) \bar{J}_{1z} dA \quad \text{replaces} \quad \int_{A_i} \bar{E}_c \cdot \bar{J}_1 dA$$

in the numerator of C_P^3 . In this derivation we do not obtain the term $P_{av,c}$, “cold” circuit power flow per unit amplitude B_3 , directly. Rather we obtain a term of the form $D(\gamma)$, which has the form of the “cold” circuit dispersion relation. But this term can be approximated as

$$D(\gamma) = D(\gamma_0) + \left. \frac{\partial D}{\partial \gamma} \frac{\partial \gamma}{\partial \Gamma} \right|_{\gamma=\gamma_0} \cdot (\Gamma - j\beta_0), \quad D(\gamma_0) = 0 \quad (8-20)$$

with γ a function of propagation constant Γ at constant k . Now it is merely a laborious task to prove that $(\partial D / \partial \gamma)(\partial \gamma / \partial \Gamma)_{\gamma=\gamma_0}$ is proportional to $P_{av,c}$ —and that Pierce’s equation obtains—if we neglect the small terms proportional to the relative tape width t defined in Eq. 6-53. But it is impractical to demonstrate this point if we retain all the terms which arise in Eq. 8-19.

Fortunately we can resort to a general proof based directly on the variational properties to show that any determinantal equation for Γ obtained like Eq. 8-19 and formulated in terms of one propagating mode inside and outside the helix *must lead to a Pierce determinantal equation in the form of Eq. 8-15* provided the beam is weak so that

$$|\Gamma - j\beta_e| = |-\beta_e C_P \delta_P - j\beta_e C_P b_P| \ll \beta_e \quad (8-21)$$

The proof is in Appendix A Section A.2.

At this time we must level a rather serious criticism of any numerical calculation of C_P based on just a few trial fields determined variationally. The variational method gives the expected formula for the incremental propagation constant, but it does not guarantee an accurate numerical evaluation. Now, we have direct evidence that the calculated impedance parameter $Z = E_z^2/(2\beta_e^2 P_{av})$ in C_P is often erroneous by a factor of 2 or more; see Fig. 6-5 for example. Obviously the gain parameter should not be calculated from such a crude estimate of the impedance. This result may seem strange in view of the fact we are using a variational expression for $\Gamma(\omega)$ which should tend to make the complex Γ quite accurate compared to the trial fields. However, consider the following point. A variationally calculated "cold" circuit dispersion curve will be approximately parallel to the correct curve. The slope k/β is usually about 0.1 or so for a slow wave, however, which means the error $\Delta k/\Delta\beta$ is about 0.1. Therefore, although the error Δk at fixed β might be small, the error $\Delta\beta$ at fixed k can be perhaps ten times larger. Thus, the incremental propagation constants in the "hot" circuit can have large errors if C_P depends upon the *calculated* rather than the measured impedance parameter.

To avoid error in the impedance parameter we can either perform a more accurate calculation of the fields with more trial functions or resort to measurement of $E_z^2/2\beta_e^2 P_{av}$, since the variational method has obviously justified the form of the correct dispersion relation. Various methods of measuring this impedance parameter are described in Appendix B. The apparent disadvantage of extending the variational calculation is the labor of evaluating a larger number of relative amplitudes of the trial field modes so as to be consistent with the "cold" circuit boundary conditions.

We now modify the preceding discussion in order to describe the interaction between a beam and the predominant $n = +1$ "backward" mode with synchronism near point B of Fig. 8-1. This mode has $e_z \sim I_1(\gamma r) \cong 0$ near the axis so we shall assume a hollow beam flowing in the inner region close to the tape at $r = a$. The analysis is precisely that for the $n = 0$ mode interaction except that we represent the circuit fields of $\epsilon^{+j\theta}$ variation by the $n = +1$ mode components of Eqs. 6-51 and 6-52.

Although we could regard the beam as composed of a number of filaments, each one "driven" by the local $\hat{e}_z$ field, it is more accurate to represent the space charge field by the $n = +1$ space charge modes. These modes propagate on the "free" beam within a drift tube of radius equal to the helix radius and have the components $\hat{e}_r$, $\hat{e}_z$, $\hat{h}_\theta$, $\hat{V}_{kz}$, and $\hat{J}_z$ if the axial d.c. magnetic field is high. Each propagates as $e^{-j(\beta_e \pm \beta_{q1})z}$, where β_{q1} must be ascertained from a solution of the beam and field equations within the drift tube.¹

The total $\hat{E}_+$ trial field of the $n = +1$ mode in the inner helix region, $r \leq a$, is

$$\begin{aligned} \hat{E}_+ = & \hat{i}_z (B_3)_1 I_1(\gamma r) + \hat{i}_r \left[(B_3)_1 \frac{\Gamma}{\gamma} I_1' - (B_1)_1 \frac{\omega \mu}{\gamma^2 r} I_1 \right] \\ & - \hat{i}_\theta \left[-(B_3)_1 \frac{j\Gamma}{\gamma^2 r} I_1 + (B_1)_1 \frac{j\omega \mu}{\gamma} I_1' \right] + a_1 \bar{E}_1 + a_2 \bar{E}_2, \quad r \leq a, \\ & \text{with } \gamma^2 = -\Gamma^2 - k^2 \quad \text{and } ' = \frac{\partial}{\partial(\gamma r)} \quad (8-22) \end{aligned}$$

and this varies as $e^{j\theta - \Gamma z}$. If we again explicitly specify continuous $\hat{e}_\perp = \hat{E}_\perp$ and $\hat{h}_\parallel = \hat{H}_\parallel$ (as in Section 6-3C), on the surface $r = a$ and set $\delta\{j\omega\} = 0$ with respect to variations of two of the adjoint circuit amplitudes as well as two adjoint beam amplitudes, we obtain an equation like 8-19. As we observed several paragraphs ago, this can be converted into the Pierce form. The numerator of C_P^3 now contains the coupling integral

$$\left| \int_{A_i} I_1(\gamma r) J_{1z} dA \right|^2$$

as well as β_{q1} of the $n = 1$ space charge mode. Here $P_{av,c}$ is the circuit power flow per unit amplitude of the inner $\bar{e}_z$ field and is negative because the group velocity at point *B* of Fig. 8-1 is negative.

By way of summary we can say that the preceding analysis based on a variational expression which accounts more carefully for the boundary conditions is consistent with the usual Pierce formulation. When only one circuit mode interacts with the beam the incremental propagation constant depends upon the coupling parameter

$$\frac{\left| \int_A \bar{E}_c \cdot J_1^* dA \right|^2}{2\beta_e^2 P_{av,c}}$$

which either must be calculated more carefully than the "cold" circuit

¹ See W. H. Louisell [5]. His Appendix B treats the solid cylindrical beam in a drift tube.

dispersion curve or else measured directly. The full power of the variational method is available for those situations in which two or more circuit modes predominate in the fields. Such is the case when the beam interacts with the helix at point *C* or *D* of Fig. 8-1.

8-4. Interaction between the Beam and Certain Cavity Chains

In this section we discuss only those cavity chains in which the fields of both the first and second passbands are characterized by resonant "cavity" modes the amplitudes of which phase shift abruptly across the coupling surfaces. The resonant-slot and resonant-loop coupled structures, in which the fields of one of the passbands are characterized by a pair of "slot" or "loop" modes, the amplitudes of which tend to phase shift rapidly across the *cavity midplanes*, are relegated to the next section. Then our variational expression is Eq. 6-77, with these additions to the numerator of the right side:

(a) $-\hat{E}_-^* \cdot \hat{J}_+$ supplements the terms within the very first pair of parentheses

(b) The terms

$$\begin{aligned}
 & - \int_V \hat{J}_-^* \cdot \left[\left(j\beta_e + \frac{\partial}{\partial z} \right) \hat{V}_{k+} - \hat{E}_+ \right] dV \\
 & - \int_V \hat{V}_{k-}^* \cdot \left[\left(j\beta_e + \frac{\partial}{\partial z} \right) \hat{J}_+ - jY_0\beta_a \hat{V}_{k+} \right] dV \quad (8-23)
 \end{aligned}$$

are added

These integrals are the ones which converted Eq. 6-45 into a variational expression for the beam-helix system except that now we integrate over a periodic volume. The demonstration of the variational properties is similar.

A. Derivation of the Single Mode-Pair Pierce Equation

Let us set up the equation for complex phase shift/cavity,² γ , when the beam interacts with the cavity fields at a frequency within the *n*th passband. These fields are described in first approximation by the one short-circuit

² This phase shift γ should not be confused with the radial propagation constant of a helix mode.

and open-circuit mode of that passband, defined in Section 5-3A as

$$\begin{aligned}\nabla \times \bar{E}_n &= P_n \bar{H}_n & \nabla \times \bar{e}_n &= p_n \bar{h}_n \\ \nabla \times \bar{H}_n &= P_n \bar{E}_n & \nabla \times \bar{h}_n &= p_n \bar{e}_n\end{aligned}\quad (8-24)$$

$$\int_V \bar{E}_n^2 dV = \int_V \bar{H}_n^2 dV = \int_V \bar{e}_n^2 dV = \int_V \bar{h}_n^2 dV = \tau$$

The normalized short-circuit mode resonant frequency, $P_n = \omega_n/c$, could be either the zero or π phase shift cutoff frequency, whereupon p_n of the open-circuit mode is the other cutoff frequency. In variational Eq. 6-77, modified by the terms of Eq. 8-23, let us represent the trial fields as

$$\begin{aligned}\hat{E}_- &= v_n \bar{e}_n & \hat{E}_+ &= V_n \bar{E}_n \\ \hat{H}_- &= i_n \bar{h}_n & \hat{H}_+ &= I_n \bar{H}_n\end{aligned}\quad (8-25)$$

where the minus trial field is presumed to phase shift by $\epsilon^{\nu*}$ and the plus field, by $\epsilon^{-\nu}$ from cavity (m) to cavity ($m+1$) for the same reason that the fields propagated as $\epsilon^{\Gamma*z}$ and $\epsilon^{-\Gamma z}$ respectively, in the beam-helix system.

When we set $\delta\{j\omega\} = 0$ with respect to δi_n^* and δv_n^* in the variational equation and eliminate I_n , we obtain this circuit equation within cavity (0) of reference phase zero:

$$\begin{aligned}[(P_n^2 - k^2)T_{nn} - P_n M_{nn}(1 \mp \cosh \gamma)]V_n &= -\frac{j\omega\mu}{\tau} \int_{\text{cav}(0)} \mathcal{J} \cdot \bar{e}_n dV \\ M_{nn} &= \frac{1}{\tau} \int_{M-} \bar{e}_n^{(0)} \times \bar{H}_n^{(0)} \cdot \bar{i}_z dS \\ T_{nn} &= \frac{1}{\tau} \int_V \bar{E}_n \cdot \bar{e}_n dV \quad (\cong 1 \text{ for a narrow passband})\end{aligned}\quad (8-26)$$

In the "cold" circuit $\mathcal{J}$ is zero and $\cos \varphi$ replaces $\cosh \gamma$. Therefore the upper sign applies if the short-circuit mode resonates at $k = P_n$ when $\varphi = 0$; otherwise, the lower or plus sign is selected. The term $|M_{nn}|$ measures very nearly the passband width $|P_n - p_n|$, and its sign determines the slope of the passband (see Table 4-1). The two amplitude equations for V_n and I_n may be cast into equivalent circuit form with the definitions of equivalent circuit quantities in Eq. 4-43, whereupon a current generator proportional to

$$\int_{\text{cav}} \mathcal{J} \cdot \bar{e}_n dV$$

appears in shunt with capacitor C_{nc} in the circuits of Fig. 4-3a,b. And if we had interchanged $\hat{E}_+$ and $\hat{E}_-$, $\hat{H}_+$ and $\hat{H}_-$, and defined $i = UI$ along

with $V = Tv$, we would have obtained two amplitude equations for v_n and i_n which, with the definitions of Eq. 4-45, would be represented by the circuits of Fig. 4-4a,b with a current generator again in shunt with C_{nc} .

If $\delta\{j\omega\} = 0$ relative to $\delta\hat{J}_-^*$ and $\delta\hat{V}_{k-}^*$ in Eq. 8-23 the beam equations must be true on an integral basis. The beam is regarded as composed of a number of filaments, each one driven by the circuit field which it experiences according to Eq. 8-17. Description of the beam by filaments, in each of which the space charge electric field is $jZ_0\beta_0\hat{J}$, is equivalent to a description by fast and slow space charge modes of propagation constants $\beta_e \pm \beta_q$ in a thin beam.

Equations 8-26 and the beam equations as they stand will determine allowed values of phase shift γ provided we realize that $\bar{V}_k(z + L) = V_k(z) e^{-\gamma}$ and $\hat{J}(z + L) = \hat{J}(z) e^{-\gamma}$. And it is important to realize that the accuracy in the calculated γ is *not affected by a large gain per cavity*. This follows from the fact that the errors in our fields are measured by the extent to which modes of neighboring passbands are excited, not upon the gain. These neighboring modes are weakly excited if their resonant frequencies are far from the operating frequency. But if the adjacent passbands lie close to the one of interest so that their modes contribute significantly to the fields at the operating frequency, they will do so for *both* large and small gain.

If, however, the gain per cavity, $\text{Re } \gamma$, is much less than 1, we can obtain a Pierce determinantal equation for γ as follows [6]. First, we define some incremental parameters: δ_P , the incremental phase shift, as

$$\gamma = j\theta_e - \theta_e C_P \delta_P, \quad \theta_e = \beta_e L \quad (C_P \text{ to be defined}) \quad (8-27)$$

and b_P , the Pierce synchronism parameter, by

$$\varphi^0, \text{ "cold" circuit phase shift, } = \theta_e(1 + C_P b_P) \quad (8-28)$$

φ^0 lies on the ω - φ curve at the operating frequency, k . Now, $\cosh \gamma$ in terms of b_P and δ_P is

$$\cosh \gamma \cong \cos \varphi^0 + \sin \varphi^0 (\theta_e C_P b_P - j\theta_e C_P \delta_P) \quad (8-29)$$

provided

$$|\theta_e C_P b_P| \ll 1 \quad \text{and} \quad |\theta_e C_P \delta_P| \ll 1$$

In order to obtain the Pierce determinantal equation for δ_P we must add the necessary proviso that the resonant cavity modes of neighboring passbands are negligibly excited.

When we substitute Eq. 8-29 into 8-26 and realize that, by definition of φ^0 ,

$$(P_n^2 - k^2)T_{nn} - P_n M_{nn}(1 \mp \cos \varphi^0) \equiv 0 \quad (8-30)$$

we can write Eq. 8-26 as

$$\mp \frac{P_n M_{nn} \tau \sin \varphi^0}{j \omega \mu} \theta_e C_P (b_P - j \delta_P) V_n = \int_{\text{cav}(0)} \mathcal{J} \cdot \bar{e}_n dV \quad (8-31)$$

But the expression for "cold" circuit power flow, $P_{av,c}$, would be derived in Section 4-4 as

$$P_{av,c} = \mp \frac{1}{4} \frac{P_n M_{nn} \tau}{\omega \mu} \sin \varphi^0, \quad P_n \cong p_n \quad (8-32)$$

per unit of circuit field amplitude V_n . Now if we make the convenient definitions

$$\mathcal{J} = J(z) \epsilon^{-j\beta_e z}, \quad \hat{V}_k = V_k(z) \epsilon^{-j\beta_e z} \quad (8-33)$$

in which V_k and J are very slowly varying across the cavity according to the small-gain approximation, we can write the circuit Eq. 8-31 as

$$j(4P_{av,c}) \theta_e C_P (j\delta_P - b_P) V_n = + \int_A J(0) \int_{\text{cav}(0)} \epsilon^{-j\beta_e z} \bar{e}_{nz} dz dA \quad (8-34)$$

Here $J(0)$ is $J(z=0)$ entering reference cavity (0).

In Eq. 8-17a, $(\partial/\partial z) \hat{V}_k = [(\partial/\partial z) V_k - j\beta_e V_k] \epsilon^{-j\beta_e z}$ so that when we move the exponential factor to the right side and integrate across the cavity from $z=0$ to $z=L$ we obtain

$$\begin{aligned} V_k(L) - V_k(0) &= V_k(0) (\epsilon^{\theta_e C_P \delta_P} - 1) \cong V_k(0) \theta_e C_P \delta_P \\ &= V_n \int_0^L \epsilon^{j\beta_e z} \bar{E}_{nz} dz + jZ_0 \theta_q J(0) \end{aligned} \quad (8-35)$$

The integrated form of Eq. 8-17b is

$$J(0) \theta_e C_P \delta_P = jY_0 \theta_q V_k(0) \quad (8-36)$$

Now, when we put these last three equations together, the usual determinantal equation for δ_P obtains as

$$(j\delta_P - b_P) \left[\delta_P^2 + \left(\frac{\theta_q}{\theta_e C_P} \right)^2 \right] = 1 \quad (8-37a)$$

provided C_P^3 is defined as

$$C_P^3 = \frac{\int_A \left(\int_0^L \epsilon^{-j\beta_e z} \bar{e}_{nz} dz \int_0^L \epsilon^{j\beta_e z} \bar{E}_{nz} dz \right) dA Y_0 \theta_q}{4\theta_e^3 P_{av,c}} \quad (8-37b)$$

A thin beam would have uniform excitation over each cross section, so if we define effective electric field as

$$E_{\text{eff}}^2 = \frac{1}{L^2} \int_0^L \epsilon^{-j\beta_e z} \bar{e}_{nz} dz \int_0^L \epsilon^{j\beta_e z} \bar{E}_{nz} dz \quad (8-38a)$$

then C_P^3 reduces to the familiar

$$C_P^3 \text{ (thin beam)} = \frac{E_{\text{eff}}^2}{2\beta_e^2 P_{\text{av},c}} \frac{|I_0|}{4U_0} \quad (8-38b)$$

Notice that both the open- and short-circuit mode patterns appear in E_{eff}^2 . The calculated E_{eff}^2 is subject to the same kind of error inherent in the numerator of the impedance parameter of the helix tube, but the percentage error is smaller for smaller holes through which the beam passes. Thus, at frequencies near the open-circuit mode resonance in the n th passband E_{eff}^2 should be measured by $\bar{e}_{nz}$; at frequencies near the short-circuit mode resonance, by $\bar{E}_{nz}$ alone. At intermediate frequencies E_{eff}^2 as measured by Eq. 8-38 will usually be sufficiently accurate.

B. Correction for the Modes of the Second Passband

Suppose, now, that the operating frequency lies in the first passband of a cavity chain and we wish to account for the modes of the second passband. We assume that the fields at frequencies in the second passband tend to phase shift rapidly across the same coupling surfaces as do the fields at frequencies in the first passband. Denote the short-circuit mode patterns of the second passband by $\bar{E}_2$ and $\bar{H}_2$ and the open-circuit mode patterns by $\bar{e}_2$ and $\bar{h}_2$. Then our trial fields of Eq. 8-25 can be augmented as follows:

$$\begin{aligned} \hat{E}_- &= v_1 \bar{e}_1 + v_2 \bar{e}_2 & \hat{E}_+ &= V_1 \bar{E}_1 + V_2 \bar{E}_2 \\ \hat{H}_- &= i_1 \bar{h}_1 + i_2 \bar{h}_2 & \hat{H}_+ &= I_1 \bar{H}_1 + I_2 \bar{H}_2 \end{aligned} \quad (8-39)$$

When we set $\delta\{j\omega\} = 0$ in the variational expression with respect to δv_1^* and δi_1^* we now obtain

$$\begin{aligned} [(P_1^2 - k^2)T_{11} - P_1 M_{11}(1 \mp \cosh \gamma)]V_1 + \mathcal{A}V_2 &= -\frac{j\omega\mu}{\tau} \int_{\text{cav}(0)} \hat{J} \cdot \bar{e}_1 dV \\ \mathcal{A} &= (P_1^2 - k^2)T_{12} - P_1 M_{12}(1 \mp \cosh \gamma) \text{ or } \mp \sinh \gamma P_1 M_{12} \end{aligned} \quad (8-40)$$

The two choices of coefficient $\mathcal{A}$ correspond to a second passband of the same character (mode symmetry) as the first passband, or of the opposite character, respectively. The discussion of Section 5-5 will clarify this matter. Thus, if the modes of both passbands have even symmetry, both $-\cosh \gamma$ terms are selected; if they both have odd symmetry, both $+\cosh \gamma$ terms are apropos. If the first passband has even symmetry and the second one odd symmetry, the $-\cosh \gamma$ term of the V_1 coefficient and

the $-\sinh \gamma$ term of the V_2 coefficient are taken. Lastly, if these symmetries are reversed we take the $+\cosh \gamma$ term and the $+\sinh \gamma$ term, respectively.

Just as we obtained Eq. 8-34 we now convert Eq. 8-40 into

$$\begin{aligned} j(4P_{av,1})\theta_e C_P(j\delta_P - b_P)V_1 + \mathcal{A}\left(-\frac{\tau}{j\omega\mu}\right)V_2 \\ = \int_A J(0) \int_{\text{cav}(0)} \epsilon^{-j\beta_e z} \bar{e}_{1z} dz dA \quad (8-41) \end{aligned}$$

By taking variations δv_2^* and δi_2^* we obtain the circuit equation for mode 2 as

$$\begin{aligned} \mathcal{B}V_1 + [(P_2^2 - k^2)T_{22} - P_2 M_{22}(1 \mp \cosh \gamma)]V_2 \\ = -\frac{j\omega\mu}{\tau} \int_A J(0) \cdot \int_{\text{cav}(0)} \epsilon^{-j\beta_e z} \bar{e}_{2z} dz dA \quad (8-42) \end{aligned}$$

$$\mathcal{B} = (P_2^2 - k^2)T_{21} - P_2 M_{21}(1 \mp \cosh \gamma) \quad \text{or} \quad \pm \sinh \gamma P_2 M_{21}$$

The sign of this $\sinh \gamma$ term is taken in the opposite sense described before. This equation reduces to the form of "cold" circuit Eq. 5-49 when $J(0) = 0$ and $\gamma = j\varphi^0$.

The beam equations are Eq. 8-35, 8-36 but with the additional circuit excitation of $V_2 \bar{E}_{2z}$:

$$Y_k(0)\theta_e C_P \delta_P = V_1 \int_0^L \epsilon^{j\beta_e z} \bar{E}_{1z} dz + V_2 \int_0^L \epsilon^{j\beta_e z} \bar{E}_{2z} dz + jZ_0 \theta_q J(0) \quad (8-43a)$$

$$J(0)\theta_e C_P \delta_P = jY_0 \theta_q V_k(0) \quad (8-43b)$$

These last four equations in terms of V_1 , V_2 , $Y_k(0)$, and $J(0)$ enable us to form an equation for periodic phase shift γ . If the beam is weak in the sense defined by

$$\left| \frac{Y_0 \theta_q}{(\theta_e C_P \delta_P)^2 + \theta_q^2} \int_A \int_0^L \epsilon^{j\beta_e z} \bar{E}_{2z} dz \int_0^L \epsilon^{-j\beta_e z} \bar{e}_{1z} dz dA \right| \ll |\mathcal{A}| \frac{\tau}{\omega\mu} \quad (8-44a)$$

and

$$\left| \frac{Y_0 \theta_q}{(\theta_e C_P \delta_P)^2 + \theta_q^2} \int_A \int_0^L \epsilon^{j\beta_e z} \bar{E}_{1z} dz \int_0^L \epsilon^{-j\beta_e z} \bar{e}_{2z} dz dA \right| \ll |\mathcal{B}| \frac{\tau}{\omega\mu} \quad (8-44b)$$

then we can simplify the equation for γ . Here P_2 is the short-circuit mode resonant frequency of the second passband while k lies in the first passband, so it certainly follows that $(P_2^2 - k^2)T_{22}(\tau/\omega\mu)$ is much greater than the

left side of Eq. 8-44b written in terms of $\bar{E}_{2z}$ and $\bar{e}_{2z}$. All these approximations enable us to write the determinantal equation for γ as

$$\left[(P_1^2 - k^2)T_{11} - P_1 M_{11}(1 \mp \cosh \gamma) - \frac{\omega \mu}{\tau} \frac{Y_0 \theta_q L^2 \int E_{\text{eff}}^2 dA}{(\theta_e C_P \delta_P)^2 + \theta_q^2} \right] \cdot [(P_2^2 - k^2)T_{22} - P_2 M_{22}(1 \mp \cosh \gamma)] - \mathcal{A} \mathcal{B} = 0 \quad (8-45)$$

The first portion of this equation is just the earlier expression. If we expand $\sinh \gamma$ with the inequalities of Eq. 8-29 and cancel the zero-order terms which satisfy the "cold" circuit dispersion relation, we obtain the form of Eq. 8-37 (with $n = 1$ now) but $P_{\text{av},c}$ of Eq. 8-32 is written a little differently. In case *both* passbands have either the even symmetry or the odd symmetry, passband parameter M_{11} is replaced by $M_{11} - M_{12}T_{21}$. In the alternative case in which the first passband has the even symmetry (upper sign) or the odd symmetry (lower sign) and the second passband has the *other* symmetry we must make this change in $P_{\text{av},c}$:

$$\mp M_{11} \rightarrow \mp M_{11} + \frac{2P_2 M_{12} M_{21}}{(P_2^2 - k^2)T_{22}} j \cos \varphi^0$$

These changes may be important if the width of the stopband between the first and second passbands is much less than the width of the first passband.

8-5. Interaction of a Resonant-Slot Coupled Structure with the Beam

We briefly treat these structures discussed in Section 7-2 in order to show that the Pierce equation describes the propagation. The C_P^3 parameter is given by the Eq. 8-37 expression multiplied by the factor $(k/P_1)^2$, where $k/P_1 \cong 1$ if the operating frequency is near the short-circuit mode resonant frequency P_1 .

Cavity electric field amplitude V_1 of the short-circuit mode of the "cavity" passband is given by Eq. 7-1 modified to include the beam term as it appears in Eq. 8-26:

$$(P_1^2 - k^2)V_1 = -\frac{P_1}{\tau} \int_{S_{\pm}} \hat{e} \times \bar{H}_1^{(0)} \cdot \bar{n} dS - \frac{j\omega\mu}{\tau} \int_{\text{cav}(0)} \bar{J} \cdot \bar{E}_{1z} dV \quad (8-46)$$

Here we recall the previous discussion to the effect that $\hat{e}$ on the slots ($S_{\pm}$) (neglecting the small centerholes now) is given by a solution of the TEM transmission line equation for the slot. Now it is an excellent

assumption to say that the slot fields do not interact with the beam; they are very weak in the beam region. Therefore our expression for the preceding surface integral on the ($S-$) slot is again Eq. 7-8 but with $j\varphi$ replaced by γ . Let us again write $\hat{J} = J \epsilon^{-j\beta_e z}$, where J is essentially constant across a single cavity. With $P_1 I_1 = j\omega \epsilon V_1$ we can now write Eq. 8-46 as

$$I_1 = \frac{-j\omega \epsilon}{(P_1^2 - k^2)\tau} \left[j\omega L_0 I_1 H_{1x}^2 (1 - \epsilon^\gamma - \epsilon^{-\gamma} + 1) k^{-3} \left(2 \tan \frac{k\ell}{2} - k\ell \right) S \right] \\ + \frac{j\omega \epsilon}{P_1} \frac{-j\omega \mu}{\tau(P_1^2 - k^2)} \int_A J(0) \int_0^L \epsilon^{-j\beta_e z} \bar{E}_{1z} dz dA \quad (8-47)$$

From Eqs. 8-35 and 8-36 we have $J(0)$ in terms of V_1 and of I_1 as

$$J(0) = \frac{P_1 I_1}{j\omega \epsilon} \frac{jY_0 \theta_q}{(\theta_e C_P \delta_P)^2 + \theta_q^2} \int_0^L \epsilon^{j\beta_e z} \bar{E}_{1z} dz \quad (8-48)$$

Substitution of this latter equation into Eq. 8-47 enables us to cancel amplitude I_1 and form this determinantal equation for incremental phase shift δ_P defined in Eq. 8-27:

$$\left[-\left(\frac{P_1}{k} \right)^3 \omega \epsilon \tau \frac{1}{2} \alpha_\pi \sin \varphi^0 \left(\tan \frac{k\ell}{2} - \frac{k\ell}{2} \right) \right] (j\delta_P - b_P) \cdot \left[\delta_P^2 + \left(\frac{\theta_q}{\theta_e C_P} \right)^2 \right] \\ = \frac{\int_A \left| \int_0^L \epsilon^{-j\beta_e z} \bar{E}_{1z} dz \right|^2 dA Y_0 \theta_q}{\theta_e^3 C_P^3} \quad (8-49)$$

The term α_π was defined in Eq. 7-10.

If the first bracketed entity were $4P_{av,c}$ of the "cold" circuit we would have the usual Pierce Eq. 8-37. It is in fact $4P_{av,c}$ but for the additional factor $(P_1/k)^2$, as the following calculation shows. From Eq. 7-8 we have

$$P_{av} = -\frac{1}{2} S \operatorname{Re} \int_{S-} \hat{e} \times \hat{H}_1^{(0)*} \cdot \bar{n} dS \\ = -\operatorname{Re} \frac{1}{2} j\omega L_0 |I_1|^2 H_{1x}^2 S \cdot (1 - \epsilon^{j\varphi}) 2k^{-3} \left(\tan \frac{k\ell}{2} - \frac{k\ell}{2} \right) \quad (8-50)$$

With α_π of Eq. 7-10 and $I_1 = j\omega \epsilon V_1 / P_1$ the power flow per unit amplitude V_1 is

$$4P_{av,c} = -\frac{P_1}{k} \omega \epsilon \tau \frac{1}{2} \alpha_\pi \sin \varphi^0 \left(\tan \frac{k\ell}{2} - \frac{k\ell}{2} \right) \quad (8-51)$$

However, $P_1/k \cong 1$ at frequencies near P_1 so the quantity within brackets in Eq. 8-49 is essentially $4P_{av,c}$.

8-6. An Equivalence Principle for All Types of Cavity Chains

When we come to consider the interaction between a beam and a more complicated slot- or loop-coupled chain from the variational point of view we find the theoretical impedance parameter too ambiguous to be useful. Two reasons make this so: the ambiguity in $E_z(0)^2$ along the axis, except at frequencies close to one cutoff frequency or the other, and the errors in power flow implied by the variational treatment. Recall that the work in Section 7-5 led to expressions for the power flow which ignored the crosspower between the "slot" (or "loop") modes of one passband and the "cavity" modes of the other passband. Rather than develop a satisfactory expression for $E_z^2(0)/2\beta_e^2 P_{av,c}$ in terms of the relative amplitudes of these modes it is preferable to develop the "hot" circuit dispersion equation somewhat empirically so as to apply to a larger group of structures. The ensuing discussion will point out the conditions under which we can derive a meaningful dispersion relation.

To generalize the discussion to include the irrotational "cavity" modes as well as the solenoidal ones, consider an expansion of the fields in the (presumably complete) set of open-circuit modes as defined and used in Section 5-3D. These modes include only irrotational electric field as excited by the tangential magnetic field on the cavity coupling surfaces and by the beam; the magnetic field remains purely solenoidal and may be represented by solenoidal short-circuit modes alone (See Eq. 5-54). From Eq. 5-24 on page 131 and Eq. 5-44 on page 137 the equation for the solenoidal mode amplitude v_1 of the predominant electric field component in our passband of interest can be written in the form

$$(p_1^2 - k^2)U_{11}v_1 + \sum_{i=1}^{\infty} K_i(k, \gamma)v_i = -\frac{j\omega\mu}{\tau} \int_{\text{cav}} \mathbf{j} \cdot \bar{\mathbf{e}}_{1z} dV$$

$$U_{11} = \frac{1}{\tau} \int_V \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{h}}_1 dV \quad (8-52)$$

The irrotational modes for electric field do not furnish magnetic field drive for the first open-circuit cavity mode of interest. *By supposition* the beam couples only to the *first* solenoidal mode so that the v_i amplitudes ($i \neq 1$) of the other modes may be implicitly related back to v_1 alone through a series of equations of the form of Eq. 5-24. Therefore we can write Eq. 8-52 in the form

$$[(p_1^2 - k^2)U_{11} + K_{11}(k, \gamma)]v_1 = -\frac{j\omega\mu}{\tau} \int_{\text{cav}} \mathbf{j} \cdot \bar{\mathbf{e}}_{1z} dV \quad (8-53)$$

Now if the beam is sufficiently weak so that $\gamma \cong j\varphi^0$, we can expand K_{11} about the "cold" circuit phase shift φ^0 as

$$K_{11}(k, \gamma) \cong K_{11}(k, j\varphi^0) + \left(\frac{\partial K_{11}}{\partial j\varphi} \right)_{\varphi^0} \cdot (\gamma - j\varphi^0) \quad (8-54)$$

Since $(p_1^2 - k^2)U_{11} + K_{11}(k, j\varphi^0) \equiv 0$ and we can differentiate this expression along the $k - \varphi$ curve with respect to φ ,

$$-2kU_{11} \frac{\partial k}{\partial \varphi} + \frac{\partial K_{11}}{\partial \varphi} + \frac{\partial K_{11}}{\partial k} \frac{\partial k}{\partial \varphi} = 0 \quad \text{at } \varphi = \varphi^0 \quad (8-55)$$

we can put Eq. 8-53 into the form

$$\frac{\partial k}{\partial \varphi} \left(2kU_{11} - \frac{\partial K_{11}(k, \varphi^0)}{\partial k} \right) (-j\gamma - \varphi^0)v_1 = -\frac{j\omega\mu}{\tau} \int_{\text{cav}} \mathbf{J} \cdot \bar{\mathbf{e}}_{1z} dV \quad (8-56)$$

Now let us think of $K_{11}(k, \varphi^0)$ in terms of an effective coupling coefficient $M_{\text{eff}}(k)$ since the "cold" circuit phase shift is implicitly a function of frequency k . We write $K_{11} = p_1 M_{\text{eff}}(k) \cdot (1 \pm \cos \varphi^0)$ so that Eq. 8-53 will assume a modified form of the single-passband Eq. 8-26:

$$\frac{\partial k}{\partial \varphi} \left[2kU_{11} - p_1 \frac{\partial M_{\text{eff}}}{\partial k} (1 \pm \cos \varphi^0) \right] (-j\gamma - \varphi^0)v_1 = -\frac{j\omega\mu}{\tau} \int_{\text{cav}} \mathbf{J} \cdot \bar{\mathbf{e}}_{1z} dV \quad (8-57)$$

which, with definitions in Eqs. 8-27, and 8-28 becomes

$$\begin{aligned} -\frac{\tau}{j\omega\mu} \frac{\partial k}{\partial \varphi} 2kU_{11} \left[1 - \frac{p_1}{2kU_{11}} \frac{\partial M_{\text{eff}}}{\partial k} (1 \pm \cos \varphi^0) \right] \theta_e C_P (j\delta_P - b_P)v_1 \\ = \int_{\text{cav}} \mathbf{J} \cdot \bar{\mathbf{e}}_{1z} dV \end{aligned} \quad (8-58)$$

The upper sign refers to a passband with even $\bar{\mathbf{e}}_z$ symmetry.

This equation so closely resembles Eq. 8-34 that we can see Eqs. 8-58, 8-35, and 8-36 will yield the Pierce determinantal Eq. 8-37 except that $P_{\text{av},c}$ in the denominator of C_P^3 is replaced by

$$\begin{aligned} P'_{\text{av},c} &= \frac{W'_s}{L} \frac{\partial \omega}{\partial \beta_0} \left[1 - \frac{p_1}{2kU_{11}} \frac{\partial M_{\text{eff}}(k)}{\partial k} (1 \pm \cos \varphi^0) \right] \\ W'_s &= \frac{1}{2} \varepsilon \int_V \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{h}}_1 dV = \frac{1}{2} \varepsilon U_{11} \tau = \frac{1}{2} \varepsilon \int_V \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{e}}_1 dV \frac{P_1}{P_1} \end{aligned} \quad (8-59)$$

(This last equivalence comes from Green's theorem.) Here $P'_{\text{av},c}$ is almost the energy stored per unit length in the first-passband modes alone, times the group velocity; and W'_s is almost twice the time-average electric energy stored by these modes. The $M_{\text{eff}}(k)$ term accounts for the presence

of the other solenoidal modes with respect to which we have assumed the beam uncoupled.

The equivalence principle states that all cavity chains in which the (weak) beam interacts with the E_z field of the modes of the first passband alone are equivalent if (a) they have the same $U_{11}\tau/L$ ratio, with U_{11} defined in Eq. 8-52; (b) the same group velocities and $(p_1/kU_{11})\partial M_{\text{eff}}/\partial k$, where group velocity is related to $\partial M_{\text{eff}}/\partial k$ in Eq. 8-55; and (c) the same effective axial electric field

$$\frac{1}{L} \int_0^L \epsilon^{-j\beta_e z} \bar{e}_{1z} dz$$

Now let us discuss the practical implications of these formulas. The assumption that the beam interacts only with the first cavity solenoidal mode is a good one in resonant-slot coupled structures in which the “slot” modes of the second passband have weak axial electric field patterns in the beam region. It is not quite so good for resonant-loop coupled chains because the short-circuit “loop” mode (see Fig. 7-12c) has a strong component of E_z field along the axis. Since, this pattern has the “straight-through” character across each cavity its cumulative effect upon the beam may not be small when the “cold” circuit phase shift lies between zero and $\pi/2$ radians at lower frequencies in the cavity passband. The weak-beam assumption is, of course, often fulfilled. Most important of all, it is clear that if we know the parameters U_{11} and $M_{\text{eff}}(k)$ in the “cold” circuit dispersion equation

$$(p_1^2 - k^2)U_{11} + p_1 M_{\text{eff}}(k)(1 \pm \cos \varphi^0) = 0 \quad (8-60)$$

no matter how $M_{\text{eff}}(k)$ is derived, we can describe the waves in the “hot” structure. The parameter U_{11} usually lies in the range $0.80 \leq U_{11} < 1.0$ and is closer to 1.0 if the passband is narrower. Unfortunately there is no way to measure U_{11} alone in the “cold” chain because the associated T_{11} is also unknown. However, the fractional error in U_{11} tends to be small in any case; it is more important to obtain an $M_{\text{eff}}(k)$ which describes the first passband accurately and to measure $\bar{e}_{1z}(z)$ at each point within the cavity in the beam region.

Problems

1. Prove Eq. 8-3 is variational by setting $\delta\Gamma = 0$ for a variation $\delta V_{k\pm}$, for example.
2. Show that the off-diagonal terms in Eq. 8-12 would be proportional to

$$\int \bar{E}_c \cdot J_1 dA$$

with no crosspower terms if the fields of the beam modes were neglected. But then $P_{av,1}$ and $P_{av,2}$ would represent only the beam kinetic power in these modes.

3. This problem investigates the interaction of a beam with a slow wave structure at the π cutoff frequency, where the d.c. beam and "cold" circuit have equal phase velocities. Modify the work leading to Eq. 8-12 by including a backward circuit harmonic propagating as $e^{-j(2\pi/L - \beta_0)z}$, β_0 being the propagation constant of the forward harmonic. These two harmonics are orthogonal in the sense that

$$\int (\hat{E}_+ \times \hat{H}_-^* + \hat{E}_-^* \times \hat{H}_+) \cdot \hat{i}_z dS = 0$$

over the cross section. Show that the Pierce determinantal equation yields only imaginary values of δ_P when β_0 approaches π/L . Since gain does occur, the interaction of the beam with the other circuit harmonics must be important.

4. Verify the remarks of the paragraph following Eq. 8-16.
5. With respect to the remarks following Eq. 8-17 prove that in the modified Eq. 8-3 Γ is stationary with respect to field variations at fixed ω and also that ω is stationary with respect to fixed Γ .
6. Derive the dispersion relation for complex Γ assuming the slow space charge mode is in synchronism with the circuit at point D of Fig. 8-1. Represent the circuit field with the $n = 0$ and -1 sheath helix modes, the latter of which is predominant according to discussion in Section 2-3.
7. Modify Eq. 7-21 so as to make it variational with an electron beam present. The beam equations are Eq. 8-17. Then use Eq. 7-22 for the circuit fields (the slot mode does not interact with the beam) and derive a dispersion relation for incremental phase shift δ_P to replace Eq. 8-49. Equations 8-35 and 8-36 may be used to eliminate the beam amplitudes.
8. Modify Eq. 8-49 to include the presence of centerhole coupling. Equations 7-11 and 7-12 are helpful. Is the new quantity within the first brackets nearly the new $4P_{av,c}$?
9. Show that circuit Eq. 8-58 for the cavity chain under general conditions reduces to Eq. 8-34 when only one circuit mode is effective and $\partial M_{ett}/\partial k = 0$.
10. Study the behavior of the coupling parameter $M_{ett}(k)$ in the cloverleaf structure by applying Eq. 8-60 to the curve of Fig. 7-18. Take $p_1 = \omega_\pi/c$ as known and plot M_{eff}/U_{11} vs. k .

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Chapter IX

The Cavity-Chain Beam Tube

9-1. Synopsis¹

In this chapter we investigate many of the properties of the cavity chain which interacts with an electron beam at a frequency near the π cutoff frequency of the first (narrow) passband. The velocity of the beam is adjusted so that it is synchronous with the "cold" circuit at the $3\pi/4$, π , or $7\pi/4$ point on the dispersion curve; the operating frequency is then chosen to correspond to this "synchronous" frequency; and the properties of the waves are studied as a function of beam voltage about the synchronous value, space charge, beam-circuit coupling coefficient, and passband width. Our analysis is strictly small-signal and is formulated in terms of the one pair of open- and short-circuit modes of the first passband and in terms of the fast and slow space charge modes. We do not report on the nature of the waves with the beam and circuit synchronous, or nearly so, at the $\pi/2$ point where the tube behaves more like a helix traveling wave tube than at any other operating frequency.

After we determine the nature of the waves as a function of various parameters we use them to study the excitation of the growing wave in the first cavity of the amplifier tube, the reexcitation of the growing wave just beyond a detuned cavity inserted into the center of the chain, and the reflection of the backward wave in the last cavity coupled to a load. We also study the effect of the impedance walls upon the gain of the growing

¹ The author thanks the Board of Regents of the University of California for free time granted on the IBM 704 computer at Berkeley and also Mr. R. J. Lanstein, currently a programmer with The Mineral Technology Department, University of California, Berkeley, who programmed many of the equations.

wave. The conclusions from these studies will be found in the last paragraph of this synopsis.

The reference structure for our analysis is the centerhole-coupled chain of Fig. 9-1, with positive group velocity over the interval $0 \leq \beta_0 L \leq \pi$, fractional passband width of $-M/p = 0.10$, and uniform $\bar{e}_z$ electric field pattern exciting the thin beam.

In Section 9-2 we outline the derivation of the determinantal equation for phase shift γ in the infinitely long tube from the beam and circuit equations. In this equation a beam-circuit coupling coefficient C_B appears naturally; it is proportional to Pierce's C_P parameter (subscript P for Pierce) in the helix tubes. However, our C_B parameter remains well behaved as the frequency approaches the cutoff frequency. Space charge and synchronism parameters are also appropriately defined for the cavity chain.

Section 9-3 presents graphs of the incremental phase shifts of the independent waves in the centerhole-coupled cavity-chain beam tube for uniform $\bar{e}_z$ field pattern and operation near the $3\pi/4$ synchronous point. That is, the operating frequency was chosen to accord with the "cold" circuit phase shift of $3\pi/4$, and the phase shifts were computed by the IBM 704 computer for a range of voltages about the synchronous voltage and for various values of C_B and space charge. Many of the growing waves have the property that the direction of the circuit power flow changes from backward to forward as the beam voltage increases through a certain transition voltage.

The gain per cavity is very nearly maximum for low values of space charge at the synchronous voltage. Since the gain is also nearly

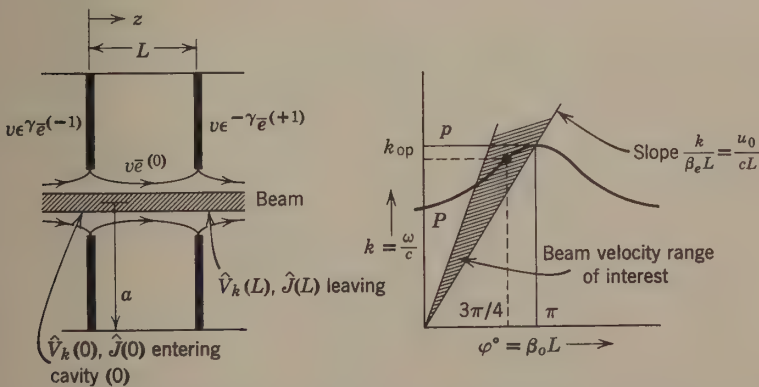


Fig. 9-1. A cavity-chain beam tube of small fractional passband width, $-M/p$. Operating frequency k_{op} corresponds to a "cold" circuit phase shift φ^0 of $3\pi/4$ radians. γ is the complex periodic phase shift.

proportional to C_B in the range $C_B \leq 0.12$, this beam-circuit coupling coefficient has the additional significance of a gain parameter. For operation near the π synchronous point this is not quite so, however, as discussed in Section 9-6.

The preceding resumé concerned basically the centerhole-coupled chain with wide cavities and nearly uniform $\bar{e}_z$ field, operated near the $3\pi/4$ synchronous point. It is instructive to consider a tube with positive- M coupling beyond the centerholes such that the group velocity is positive over the interval $\pi \leq \beta_0 L \leq 2\pi$. However, examination of the growing waves which result for synchronism at the $7\pi/4$ point reveals that this tube is very inefficient. That is, on the basis of comparable C_B and space charge this tube has half or less the gain in decibels of the tube operated near the $3\pi/4$ synchronous point. Section 9-4 delves into this matter.

In Section 9-5 we verify the consistency of our formalism with respect to the power flow across a coupling surface if any linear combination of the waves is present.

We then examine in Section 9-6 the nature of the waves for uniform e_z cavity field pattern but synchronism of beam and circuit near or at the π cutoff point. The operating frequency is now the π cutoff frequency. Each growing wave carries less circuit power per unit amplitude than for comparable values of C_B and $4(QC)_B$ at operation near the $3\pi/4$ synchronous point. But the rate of growth per cavity tends to be higher, although it is no longer linear with C_B for small C_B .

Section 9-7 is devoted to a brief study of the growing wave when the cavities are relatively narrow so that the axial electric field varies across each one nearly as $\sin(\pi z/L)$ instead of being uniform. The operating frequency is presumed to be close to the π resonant frequency of the open-circuit mode with this sinusoidally varying $\bar{e}_z$ pattern. On the basis of comparable C_B and space charge the optimum growing wave gain for operation near the $3\pi/4$ point is slightly less with the sinusoidal $\bar{e}_z$ field. But the optimum growing wave gain for operation near the π point is greater with the sinusoidal field. Optimum gain is somewhat arbitrary and means a gain which is a compromise between high growth rate and low circuit power per unit amplitude.

Returning to the cavity-chain tube with uniform $\bar{e}_z$ field, knowledge of the properties of the independent waves enables us to determine linear combinations of them which satisfy the boundary conditions at both ends of the finite tube. In particular, we determine in Section 9-8 the relative amount of growing wave excited by the generator along with the decaying and perturbed fast beam waves in the first cavity of an amplifier. The growing wave component of the field is usually 0.5 or more of the total cavity field for a wide range of parameters.

Because these high-power tubes tend to reflect some backward wave from the last cavity in spite of careful load adjustment, a detuned (lossless) cavity may be inserted about half way down the tube to reduce the tendency to oscillate. The discussion of Section 9-9 points out that, under many conditions of beam-circuit coupling, space charge, and beam velocity adjusted for optimum growing wave gain, the amplitude of growing wave reexcited beyond the detuned cavity will be about equal to *or greater than* the incident amplitude. For low values of space charge the beam drift angle through the detuned cavity should be about 60° in all cases studied, but for higher values of space charge and for all cases of operation near the π synchronous point the reexcitation of the growing wave is virtually independent of this drift angle!

Next, we examine the matching of the output cavity to the growing wave through loop coupling to the load. For each case studied a matched load exists which is more critical for amplifier operation near the π synchronous point than for operation near the $3\pi/4$ point. The load for maximum power output differs somewhat from the matched load which produces no backward wave due to the fact that the latter wave, if present, can increase both the negative beam power flow out of the last cavity and the positive power flow into the load.

We also answered the question of whether a match exists for the input cavity, such that an incoming backward wave will reexcite no new growing wave. Typical operating conditions for beam-circuit synchronism near the $3\pi/4$ and π phase shift points indicate that such a match *does* generally exist.

Finally, in Section 9-11 we study the effect on the growing wave of finite cavity wall impedance. Tangential currents flowing through this impedance establish electric fields which in turn excite new components of electric and magnetic field within each cavity. Computations reveal that the reduction in gain due to this impedance is considerably less if the beam is present. In fact, for reasonable values of space charge the impedance walls actually *enhance* the gain. This is the so-called resistive wall amplifier effect (or, more properly, the impedance wall effect) which is more pronounced for operation near the π synchronous point than near the $3\pi/4$ point.

The important conclusions from our study are the following. For operation at the $3\pi/4$ "cold" circuit phase shift frequency gain per unit length is comparable to gain in the helix traveling wave tube. For operation at the π cutoff frequency, the gain per cavity is higher although there is less circuit power flow per unit amplitude of growing wave. It definitely pays to operate an amplifier at the $3\pi/4$ or π synchronous points, that is, within the region $0 \leq \beta_0 L \leq \pi$ of the ω - q curve rather than in the region

$\pi \leq \beta_0 L \leq 2\pi$ where the gain is much lower. The optimum gain is quite independent of whether the $\hat{e}_z$ cavity field is uniform or varies as $\sin(\pi z/L)$ across the cavity, where the maximum value of the sinusoidal field equals the magnitude of the uniform field. The growing wave can be easily excited to a relatively large magnitude by the generator in the first cavity of an amplifier. If a detuned cavity is inserted into the chain to prevent backward wave oscillation the magnitude of the reestablished growing wave is usually greater than the magnitude of the incident growing wave. In addition, the output cavity can be matched to the incident growing wave just as the input cavity can be matched, in general, to the incoming backward wave if one is reflected from the last cavity. The only real disadvantage of the cavity-chain traveling wave tube with respect to the helix traveling wave tube is its vastly reduced frequency bandwidth, which is less than the fractional passband width of the "cold" structure.

9-2. Description of the Structure and Definitions of Terms

We first study the properties of the four independent waves which exist when a longitudinally confined beam interacts with the electric field of an infinitely long cavity chain, such as shown in Figs. 9-1 and 9-2. We are primarily concerned with amplifier operation at frequencies within the first passband. We assume the first passband is narrow compared to the stopband width between the first and second passbands so we can use the open- and short-circuit modes of the first passband to represent the cavity solenoidal fields. At frequencies near the π cutoff frequency the electric field is better represented by the $\bar{e}$ field pattern of the open-circuit mode, shown in Fig. 9-1. There γ is the complex phase shift per cavity of the electric field amplitude v . We examine the waves for a range of beam velocities about that velocity corresponding to synchronism of the d.c. beam and the "cold" circuit, both at the $3\pi/4$ point and at the π point of the "cold" circuit ω - φ curve. We also examine briefly the growing wave for synchronism at the $7\pi/4$ point of Fig. 9-2. The operating frequency will always correspond to the "cold" circuit phase shift at the synchronous point. The four waves which exist in the "hot" structure may be regarded as perturbations of the fast and slow space charge modes which propagate as $e^{j(\beta_0 \pm \beta_0)z}$ in the absence of the fins and circuit field, and of the forward and backward "cold" circuit waves with phase shifts given by the ω - φ curve. Generally we cannot identify a "hot" wave as a perturbation of one particular beam mode or "cold" circuit wave, nor is it necessary to do so.

We make essentially four assumptions about the total electric field

exciting the beam in each cavity. The first assumption that the operating frequency is very near the resonant frequencies of the modes of the first passband and far from the resonances of the modes of the other passbands has been noted. We therefore represent the solenoidal electric field within reference cavity (0) of Figs. 9-1 and 9-2 by $v\bar{e}_z$ of the first open-circuit mode which resonates at $k = p$ in the "cold" chain (see the discussion in Section 4-6).

We also represent the irrotational portion of the axial electric field by $jZ_0\beta_q\hat{J}$, Z_0 being the beam impedance per unit area as defined by Eq. 8-2, β_q the reduced plasma propagation constant, and $\hat{J}$ the a.c. beam current density. Our third assumption is that of a wide cavity such that the length L is at least twice the centerhole diameter. In such a cavity the $\bar{e}_z$ pattern will be nearly uniform in the beam region across the cavity, as shown in Figs. 9-1 and 9-2. We simply assume that the $\bar{e}_z$ pattern is completely uniform across each cavity, with an abrupt phase shift of electric field amplitude v (but not of the beam excitations) across each centerhole. Our fourth assumption is that of uniform $\bar{e}_z$ pattern throughout every cross section of the beam. These assumptions are better for a thin, hollow beam

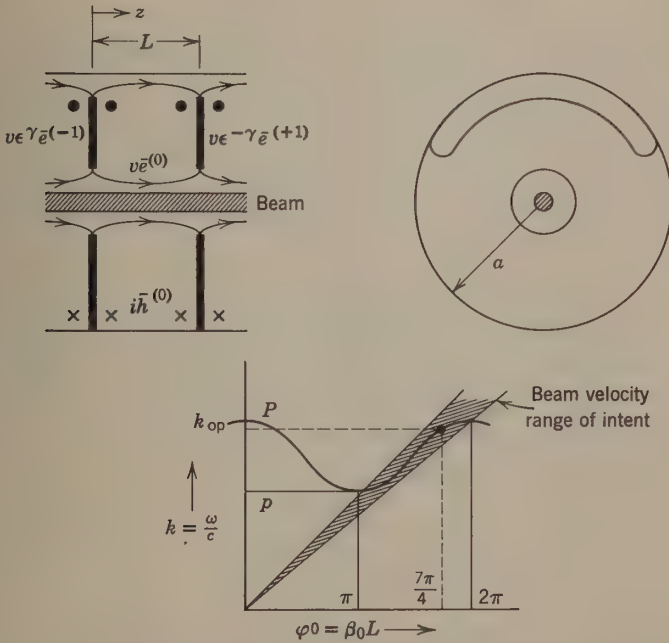


Fig. 9-2. Another cavity-chain beam tube of small fractional passband width, M/p . The operating frequency, k_{op} , corresponds to near-synchronism of d.c. beam and "cold" circuit at $\varphi_0 = 7\pi/4$ radians. γ is the complex periodic phase shift.

which almost grazes the centerhole edges than for a thin, solid beam traveling on the axis. In Section 9-5 we prove that, in spite of the discontinuities in field amplitudes across the centerholes, this model of the interaction implies continuity of electromagnetic power flow across each hole, where continuity of beam power will be explicitly evident.

To obtain a determinantal equation for $\epsilon^{-\gamma}$ or γ we proceed just as we did in Section 8-4 to obtain one equation for v in terms of kinetic voltage $\hat{V}_k(0)$ and beam current density $\hat{J}(0)$ entering the cavity and two more expressions in terms of $\hat{V}_k(0)$ and $\hat{J}(0)$ which are excited by $v\bar{e}_z$ across the cavity. The three basic equations for the interaction were developed previously as (Eq. 8-26)

$$[p^2 - k^2 + pM(1 + \cosh \gamma)]v = -\frac{j\omega\mu}{\tau} \iint_{\text{cav}} \mathbf{J}_z \cdot \bar{e}_z dA dz$$

for the circuit² and

$$\begin{aligned} \left(j\beta_e + \frac{\partial}{\partial z}\right) \hat{V}_{kz}(z) &= v\bar{e}_z + jZ_0\beta_q \hat{J}_z(z) \\ \left(j\beta_e + \frac{\partial}{\partial z}\right) \hat{J}_z(z) &= jY_0\beta_q \hat{V}_{kz} \end{aligned}$$

for the beam.³ Since uniform $\bar{e}_z$ is known very accurately within the cavity these equations will determine three homogeneous equations in terms of v , $\hat{V}_k(0)$, and $\hat{J}(0)$, provided we utilize the periodicity relations $\hat{V}_k(L) = \hat{V}_k(0) \epsilon^{-\gamma}$ and $\hat{J}(L) = \hat{J}(0) \epsilon^{-\gamma}$. The determinantal equation for γ results from setting the determinant of the matrix of these coefficients equal to zero. In Appendix E, Section E-1, we derive this relation in full, with neglect of small terms by the assumption of $\beta_q/\beta_e = \omega_q/\omega < \frac{1}{3}$ so that $(\beta_q/\beta_e)^2 \ll 1$.

Before we discuss the nature of the waves according to the values of γ obtained with the IBM 704 computer let us introduce some notation to correspond with the usual notation for the helix type of traveling wave tube.⁴ According to our earlier Eqs. 8-37b, 8-38, and 8-32, the Pierce gain parameter C_P (subscript P always denotes a Pierce symbol) for the cavity chain at frequencies near the π cutoff frequency is

$$C_P^3 = \frac{E_{\text{eff}}^2}{2\beta_e^2 P_{\text{av}}} \frac{|I_0|}{4U_0} \quad (9-1)$$

² The bracketed expression obtains by substituting Eq. 4-12 into Eq. 8-26, with $T_1 \cong 1$ and $P_1 \cong p_1$.

³ The relativistic corrections included multiplying J_0 , which appears in Z_0 and Y_0 , by the R factor of Eq. D-12 and also correcting the reduction factor ω_q/ω_p as seen in the stationary reference frame.

⁴ J. R. Pierce [1, Chapters 7-9].

where

$$E_{\text{eff}}^2 = \left| \frac{1}{L} \int_0^L \bar{e}_z e^{-j\beta_e z} dz \right|^2, \quad \beta_e L \cong \varphi^0$$

$$P_{\text{av}} = \frac{p(-M)\tau}{4\omega\mu} \sin \varphi^0,$$

($M < 0$ in the Fig. 9-1 structure). Here φ^0 is the "cold" circuit phase shift at the operating frequency $k_{\text{op}} = \omega_0 \sqrt{\mu\epsilon}$ in Figs. 9-1 and 9-2. Since $\bar{e}_z$ is uniform in this chain, E_{eff}^2 is just

$$E_{\text{eff}}^2 = e_z^2 \left[\frac{\sin(\theta_e/2)}{\theta_e/2} \right]^2, \quad \theta_e = \beta_e L = \frac{\omega}{u_0} L$$

and we can write the Pierce parameter as

$$C_P^3 = \frac{2\omega\mu e_z^2}{\beta_e^2 p(-M)\tau \sin \varphi^0} \left[\frac{\sin(\theta_e/2)}{\theta_e/2} \right]^2 \frac{|I_0|}{4U_0}, \quad \theta_e \cong \varphi^0 \quad (9-2)$$

Obviously C_P is not well behaved as k approaches p and φ^0 approaches π , but we can define a beam-circuit coupling coefficient C_B which appears naturally in determinantal Eq. E-8 of Appendix E and which is well behaved throughout the passband:

$$C_B^2 = \frac{\omega\mu e_z^2 A_b Y_0}{p^2 \beta_e^2 \tau} = \frac{\omega\mu e_z^2}{p^2 \beta_e^2 \tau} \frac{1}{2} \frac{|I_0|}{U_0} \frac{\theta_e}{\theta_q} \quad (9-3)$$

Here A_b is the beam area, τ is the cavity volume, and p is the cutoff frequency at the π point. Comparison of Eqs. 9-2 and 9-3 shows that C_B^2 is related to C_P^3 at beam voltages near synchronism ($\theta_{e0} = \varphi^0$) as

$$C_B^2 = \left(-\frac{M}{p} \right) \frac{\theta_{e0}}{\theta_q} \left[\frac{\theta_{e0}/2}{\sin(\theta_{e0}/2)} \right]^2 \sin \varphi^0 C_P^3, \quad \theta_{e0} \equiv \varphi^0 \quad (9-4)$$

This is our first relation between the parameters of the cavity-chain tube and the Pierce parameters of the helix tube.

The physical parameters that naturally arise in the analysis of the cavity-chain tube are C_B and θ_q/θ_e . But let us define a space charge parameter $4(QC)_B$ analogous to $4(QC)_P$ for the helix tube as

$$\frac{\theta_q}{\theta_{e0}} = C_B \sqrt{4(QC)_B} \quad (9-5)$$

By itself Q has no meaning. Since the Pierce definition of θ_q/θ_{e0} is

$$\frac{\theta_q}{\theta_{e0}} = C_P \sqrt{4(QC)_P} \quad (9-6)$$

we obtain the second relation among the parameters of the two tubes, namely

$$C_B \sqrt{4(QC)_B} = C_P \sqrt{4(QC)_P} \quad (9-7)$$

Lastly, we define the synchronism parameter b_B as a measure of the phase velocity of the d.c. beam relative to the "cold" circuit wave as

$$\varphi^0 = \theta_{e0} = \theta_e(1 + C_B b_B) \quad (9-8)$$

Positive b_B means the d.c. beam travels faster than the "cold" circuit phase velocity. For the helix tube Pierce has defined

$$\beta_e = \beta_e(1 + C_P b_P), \quad \beta_e L = \varphi^0, \quad \beta_e L = \theta_e \quad (9-9)$$

So the relation between the two b 's is

$$C_B b_B = C_P b_P \quad (9-10)$$

In terms of the definition of θ_q/θ_{e0} in Eq. 9-5 we can express the relation between C_B and C_P of Eq. 9-4 alternatively as

$$C_B = \left(-\frac{M \sin \varphi^0}{p} \right)^{1/4} [4(QC)_B]^{-1/4} \left[\frac{\theta_{e0}/2}{\sin(\theta_{e0}/2)} \right]^{3/4} C_P \quad (9-11)$$

This completes our definitions of the cavity-chain parameters as related to the Pierce parameters.

Table 9-1 Relation between C_B and C_P in the Cavity Chain with Uniform $\bar{e}_z$ Field Pattern. In Eq. 9-11 $-M/p = 0.10$.

At $\varphi^0 = \pi/2$		At $\varphi^0 = 3\pi/4$	
$4(QC)_B$	C_B	$4(QC)_B$	C_B
1	$0.498 C_P$	1	$0.486 C_P$
4	$0.395 C_P$	4	$0.386 C_P$
10	$0.339 C_P$	10	$0.331 C_P$
25	$0.291 C_P$	25	$0.284 C_P$

Typical values of C_B for cavity-chain tubes are lower than the values of C_P defined by Eq. 9-2 by factors of 2 or 3. Table 9-1 compares them in a chain with a fractional passband width $-M/p = 0.10$, for "cold" circuit phase shift φ^0 of $\pi/2$ and $3\pi/4$ and for various values of space charge. As the frequency approaches the "cold" circuit π cutoff frequency, C_B becomes a small fraction of C_P because the latter approaches infinity (see Eq. 9-2 as φ^0 approaches π).

We now estimate a typical value of C_B for a cavity chain with its first passband centered at 3 kMc. A narrow passband implies small coupling holes and normalized open-circuit mode field patterns which are nearly those of the short-circuit mode patterns given by Eq. 3-17:

$$\begin{aligned} \bar{e}(\bar{r}) &\cong \bar{i}_z \sqrt{3.71} J_0(Pr) \\ \bar{h}(\bar{r}) &\cong \bar{i}_\theta \sqrt{3.71} J_1(Pr) \end{aligned} \quad P \cong k \cong p \text{ in Fig. 9-1}$$

Then e_z^2 for Eq. 9-3 is about 3.7. Since $Pa = 2.405$ in order that $e_z(r = a)$ be zero on the cavity wall we have $a = 3.84 \text{ cm} = 1.51 \text{ in.}$ Suppose the cavity length L is 0.5 in. and we wish the d.c. beam transit angle $\beta_e L = \omega L/u_0$ to be $3\pi/4$ radians. This makes $u_0 \cong 10^8 \text{ m/sec.}$ about a third the speed of light, so that $U_0 = 32 \text{ kv}$ (including relativistic effects). If the beam has a perveance $K = I_0/U_0^{3/2} = 10^{-6}$, the d.c. current I_0 is 5.7 amp. We have now specified all the parameters for C_B^2 of Eq. 9-3 except θ_q/θ_e ; if this is estimated to be $\frac{1}{4}$, then C_B^2 is 3.8×10^{-3} . If, now, the cavity were longer by a factor of $\frac{4}{3}$, the beam voltage U_0 would be 60 kv instead of 32 kv so as to maintain $\beta_e L = 3\pi/4$. Then β_e^2 would decrease by a factor of $(\frac{3}{4})^2$, τ would increase by a factor of $\frac{4}{3}$ and I_0/U_0 would increase by a factor of $(\frac{60}{32})^{1/2} \cong 1.4$ for the same perveance and θ_q/θ_e ratio. Now $C_B^2 \cong 7.1 \times 10^{-3}$. We consider a large value of C_B^2 to be 50×10^{-3} .

We can also estimate the effect on coupling coefficient C_B of Eq. 9-3 of increasing the operating frequency. Suppose we modify the chain so as to operate at 10 kMc and maintain the d.c. beam transit angle at $3\pi/4$. Now ω is larger by a factor of $\frac{10}{3}$ and since we cannot increase u_0 by this same factor we must also decrease L . Suppose we raise the operating voltage from 32 to 60 kv and multiply u_0 by the factor $\frac{4}{3}$, as before. Then L must be reduced to two-fifths of its former value at 3 kMc. This requires smaller centerholes and beams in accord with our requirement of uniform cavity field across each cavity. The cavity diameter decreases by the same factor as $k \cong P$ increases, so that the cavity volume τ is now multiplied by the factor $\frac{2}{5}(\frac{3}{10})^2$. If we maintain the perveance, I_0/U_0 is multiplied by $(\frac{60}{32})^{1/2}$. Lastly, we should account for the change in $\theta_e/\theta_q = \omega/\omega_q$, where ω_q is proportional to $\rho_0^{1/2} = (I_0/A_b u_0)^{1/2}$. All these effects change C_B^2 by an overall factor of

$$\frac{\frac{10}{3}}{(\frac{10}{3})^2 (\frac{5}{2})^2 (\frac{3}{10})^2 (\frac{2}{5})} \left(\frac{60}{32}\right)^{1/2} \frac{\frac{10}{3}}{[(\frac{60}{32})^{3/2} (\frac{10}{3})^2 (\frac{3}{4})]^{1/2}} \cong 1.3$$

Thus beam-circuit coupling coefficient C_B need not change very much when we are redesigning a tube for operation at a higher frequency, provided the beam perveance is maintained.

We may choose to avoid the large beam voltages required for synchronism near $3\pi/4$ and π in the negative- M structure of Fig. 9-1 and choose instead to operate the positive- M structure of Fig. 9-2 at beam transit angles near $7\pi/4$. We can modify the Fig. 9-1 structure into that of Fig. 9-2 by punching coupling slots along the outer radius so as to change the sign of the coupling coefficient (refer to the discussion of example 2 in Table 4-1) and by either increasing the cavity length L or decreasing the beam voltage or both. Unfortunately, when we increase L so as to lengthen the beam transit angle we reduce C_B^2 by the same factor,

and if we reduce the beam voltage instead we reduce C_B^2 even more. A reduction in C_B^2 always implies less growth per cavity in the growing wave. Furthermore, a comparison of the growing wave in this structure and in the Fig. 9-1 structure at the $3\pi/4$ synchronism points on the basis of the same C_B and $4(QC)_B$ (see Section 9-4) reveals that the rate of growth is less in the Fig. 9-2 structure *by a factor of 2 or more in decibels*. Amplifier operation of this structure at $7\pi/4$ synchronism is considerably less efficient than is the higher-voltage operation of the Fig. 9-1 structure.

9-3. Nature of the Waves for Uniform Cavity Electric Field

The d.c. beam is synchronous with the "cold" circuit of Fig. 9-1 at the $3\pi/4$ "cold" circuit phase shift frequency. Figures 9-3 through 9-6 illustrate the incremental phase shifts of the waves for various values of space charge, measured by $4(QC)_B$, and various values of the beam-circuit coupling coefficient C_B . With the phase shift factor $\epsilon^{-\gamma}$ given by [excitation in cavity ($n + 1$)]/[excitation in cavity (n) a period away], the incremental phase shift $X + jY$ is defined as

$$X + jY = -\gamma + j\theta_e \quad (9-12)$$

Positive X in the growing wave, identified by subscript 1, is the gain per cavity in nepers *provided* the circuit power flow per unit amplitude of v is in the forward or $+z$ direction. We also encounter growing waves in which the *backward* circuit power flow represents power fed into the beam from the collector end of the tube; that is, accelerator-type growing waves. A positive value of Y denotes a wave which has a lesser magnitude of phase lag from cavity (n) to cavity ($n + 1$) than does the d.c. beam and therefore a greater phase velocity than u_0 (since $v_p = \omega/\beta \propto \omega/\theta$). Subscript 2 denotes the decaying wave, for which $X_2 = -X_1$ (X_2 is not shown on any of the figures) and $Y_2 = Y_1$. Incremental parameter Y_3 belongs to the faster constant-amplitude wave, and Y_4 to the slower one or the perturbed "cold" circuit backward wave. An arbitrary distribution of field and beam excitations established from either or both ends of a finite tube can be represented by a linear combination of these four waves.

Figure 9-3 shows the behavior of the waves as a function of beam velocity measured by b_B of Eq. 9-8 for fixed C_B , for various values of the space charge parameter $4(QC)_B$ defined in Eq. 9-5. The fractional pass-band width is $-M/p = 0.10$ for the structure of Fig. 9-1. The three waves shown are perturbations of the forward "cold" circuit wave and the fast and slow beam modes. The fourth wave (not shown) is, for practical purposes, the backward "cold" circuit wave essentially unperturbed in the

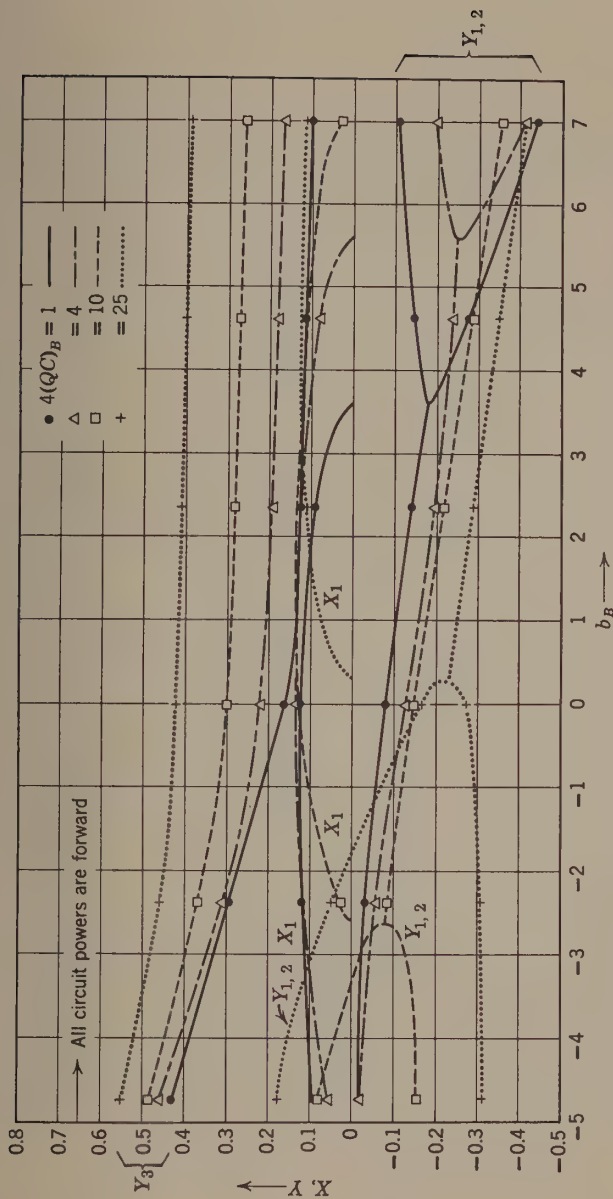


Fig. 9-3. Incremental periodic phase shift $X + jY$ for the waves in the tube with uniform e_s field in each cavity. Synchronism at $3\pi/4$. $C_B^2 = 0.001$. $-M/p = 0.10$. $4(QC)_B$ is the parameter. The perturbed backward "cold" circuit wave is not represented.

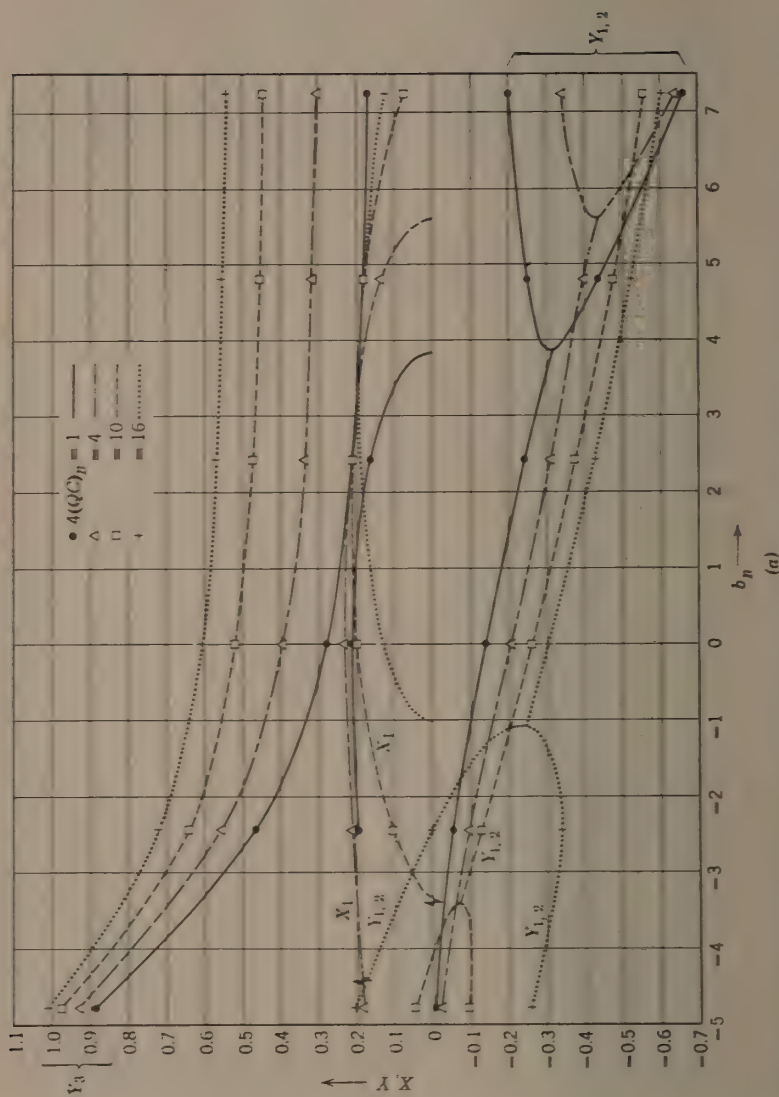


Fig. 9-4. Incremental periodic phase shift for the waves with uniform e field. Synchronism at $3\pi/4$. $C_B^3 = 0.003$. $4(QC)_n$ is the parameter. (a) $-M/p = 0.10$, and (b) $-M/p = 0.15$. The perturbed backward "cold" circuit wave is not represented.

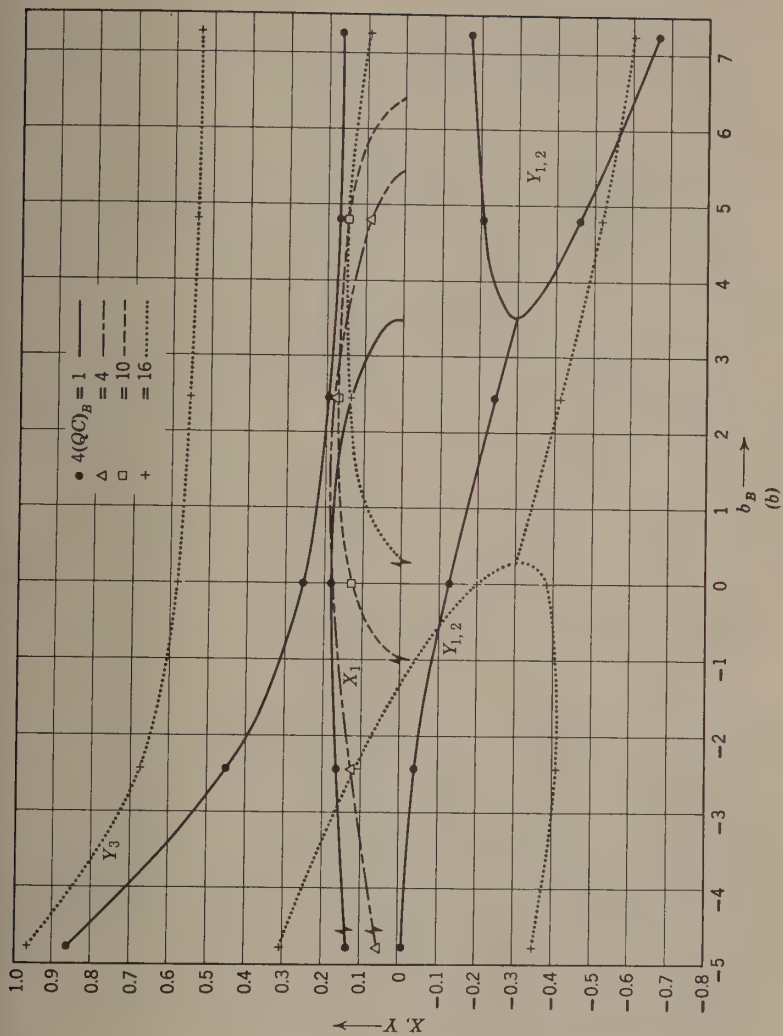


Fig. 9-4 (continued)

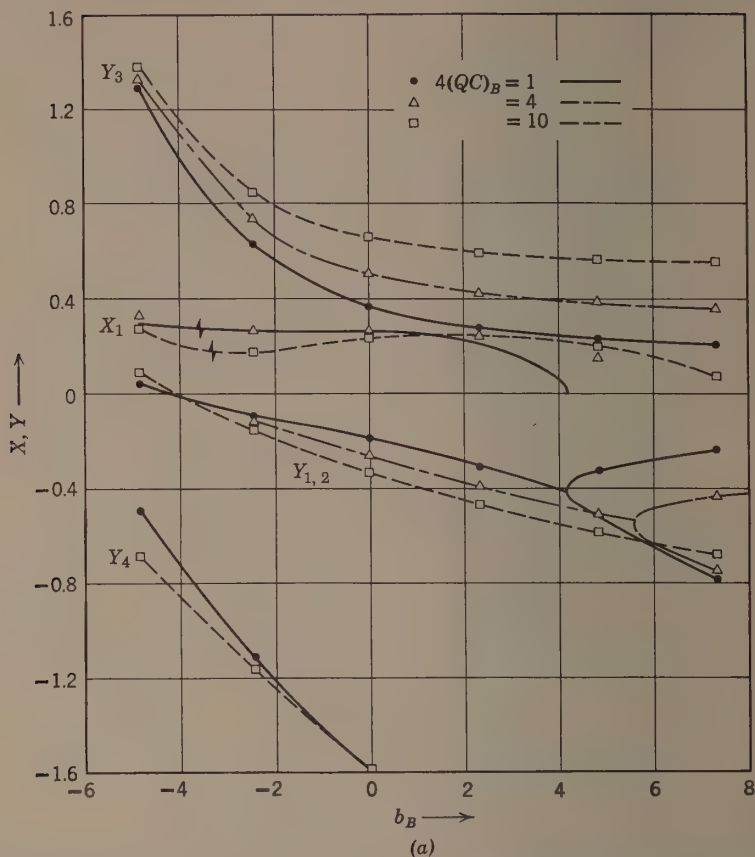


Fig. 9-5. Incremental periodic phase shift for the waves with uniform e_z field. Synchronism at $3\pi/4$. $C_B^2 = 0.005$. $4(QC)_B$ is the parameter. (a) $-M/p = 0.10$.

range of beam velocities shown. The magnitude $C_B^2 = 0.001$ represents weak beam-circuit coupling, in spite of which the maximum gain per cavity lies in the range 0.12–0.14 neper for the various values of space charge. The large range of b_B in which useful gain occurs illustrates the fact that a cavity chain operated for synchronism near π “cold” circuit phase shift tends to have a much larger *voltage bandwidth* than does a helix tube. For example, in Fig. 9-3 useful gain occurs for $4(QC)_B = 1$ in the range $2.8 > \theta_e > 2.1$ and for $4(QC)_B = 10$, in the range $2.55 > \theta_e > 1.9$.

The curves of Fig. 9-3 show that, as space charge increases, the maximum gain remains nearly constant and the voltage range for gain shifts toward higher voltages ($b_B > 0$). We expect this because the beam tends to deliver

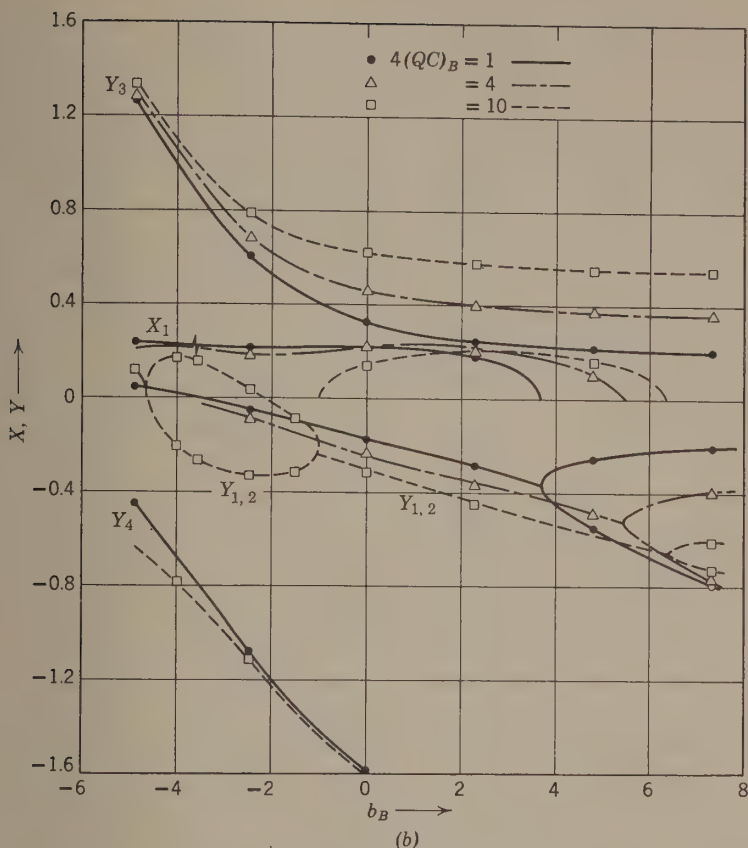


Fig. 9-5 (continued). (b) $-M/p = 0.15$.

the most power to the circuit when the slow beam mode is nearly synchronous with the "cold" circuit, so that as β_q increases the slow beam mode phase velocity $\omega/(\beta_e + \beta_q)$ is kept constant by increasing the d.c. beam velocity. A space charge of $4(QC)_B = 25$ in Fig. 9-3 implies a maximum space charge factor of $\theta_q/\theta_e = 0.19$ at the high-voltage end of the figure; this does not violate our assumption of $(\theta_q/\theta_e)^2 \ll 1$. These and succeeding curves have been calculated on the basis of constant C_B and $4(QC)_B$ with changes in beam voltage about the synchronous value ($b_B = 0$), whereas the physical parameters change slightly according to Eqs. 9-3 and 9-5.

If we increase the fractional passband width from 0.10 to 0.15 but continue to operate around the $3\pi/4$ synchronous point we obtain a very similar set of curves (not shown). The maximum gain for each value of space charge is now about 20% lower except for the case of $4(QC)_B = 25$ for

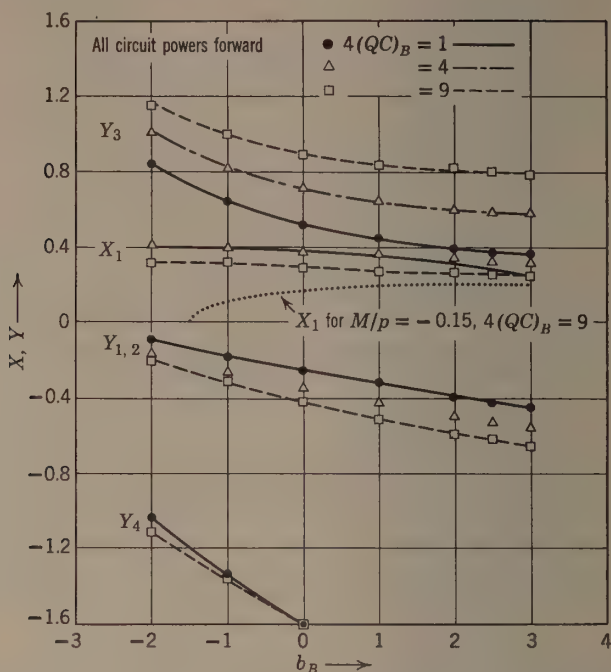


Fig. 9-6. Incremental periodic phase shift for the waves with uniform e_z field. Synchronism at $3\pi/4$. $C_B^2 = 0.010$. $4(QC)_B$ is the parameter. $-M/p = 0.10$, except for one case.

which the gain is nearly the same. The larger passband width also reduces somewhat the voltage range of gain operation for each value of $4(QC)_B$.

Figure 9-4 repeats the information of Fig. 9-3 but for an increased coupling coefficient $C_B^2 = 0.003$. In Fig. 9-4a we see that the maximum gain for each value of $4(QC)_B$ is about double the corresponding gain in Fig. 9-3 and is again nearly independent of space charge for low values of space charge. The voltage bandwidths are about the same as for $C_B^2 = 0.001$. Figure 9-4b shows the effect on the growing wave gain of increasing the fractional passband width to 0.15 but maintaining synchronism at $3\pi/4$. The maximum gain for each value of $4(QC)_B$ drops somewhat, but not much, and the voltage bandwidths decrease noticeably.

Another interesting aspect of these curves is just beginning to manifest itself on the $4(QC)_B = 1$ and 4 curves of Fig. 9-4a, near $b_B = -4.5$, at the points indicated by the pips. As the beam voltage increases through these points the phase lag per cavity $\text{Im } \gamma = \theta_e - Y$ of both the growing and decaying waves passes through π radians. In Section 9-5 we show that the circuit electromagnetic power flow in any wave is proportional to $(-M/p) \sin(\text{Im } \gamma)$. Thus, for $b_B \ll -4.5$ the circuit power flow in the

growing wave is *negative*, which means it flows in the backward or $-z$ direction, from the circuit to the beam. The *positive* beam power in this situation increases exponentially in the $+z$ direction. The associated decaying wave represents a transfer of initial positive beam power (if any is present) to the circuit and out the gun end of the circuit. For beam voltages such that $b_B \gtrsim -4.5$ in Fig. 9-4a we have positive circuit power in the growing wave increasing exponentially in the forward direction at the expense of negative beam power; that is, the ordinary useful traveling wave tube growing wave.

By this reasoning we have at one point near $b_B \simeq -4.5$ on each of the $4(QC)_B = 1$ and 4 curves a growing-decaying wave pair neither of which carries forward or backward circuit power; that is, we have a *reactive growing and decaying wave*. In each of these curious waves the real beam power is also zero down the tube. At the transition voltage where these reactive waves exist the growth or decay rate X is a smooth function of beam voltage as $\theta_e - Y$ goes through π radians. These waves are not responsible for experimentally observed oscillations for the simple reason that the power lost into the circuit loads could not be replenished by the beam. Our remarks imply the need for operating an amplifier away from the transition voltage where the circuit power flow per unit amplitude of circuit field is small even though the rate of growth is usually substantial.

In Fig. 9-5a we repeat the information of Fig. 9-4a but for $C_B^2 = 0.005$ and three values of the space charge parameter $4(QC)_B$. We now include Y_4 for the perturbed "cold" circuit backward wave. The maximum gains are somewhat higher than for Fig. 9-4a, as we expect. Notice how the pips which mark the transition from backward-circuit to forward-circuit power flow in the growing wave have moved to the right. The effect of increasing the fractional passband width $-M/p$ from 0.10 to 0.15 is now rather striking, as Fig. 9-5b shows. Here we see, for $4(QC)_B = 10$, the transformation of the growing wave from one carrying backward circuit power ($b_B < -4.6$) to a constant amplitude wave at $b_B = -3.05$ where the fast beam mode is synchronous with the "cold" circuit ($\theta_e - \theta_q = 3\pi/4$) and then to a growing wave of forward circuit power flow in the range $b_B > -1.0$, in which the slow beam mode is more nearly in synchronism at $3\pi/4$. When the fast beam mode is synchronous with the "cold" circuit at $3\pi/4$, $b_B = -3.05$, no growth results because the power flows in the two waves have the same signs (see discussion immediately following Eq. 8-16). The slow beam mode is synchronous with the "cold" circuit at $3\pi/4$, for $4(QC)_B = 10$, at $b_B = +4.1$, where the growth rate is about optimum.

The incremental phase shifts of the waves for $C_B^2 = 0.007$ (not shown) closely resemble those for $C_B^2 = 0.005$ in Fig. 9-5, except that for the former the maximum gain is somewhat higher by nearly the ratio of the

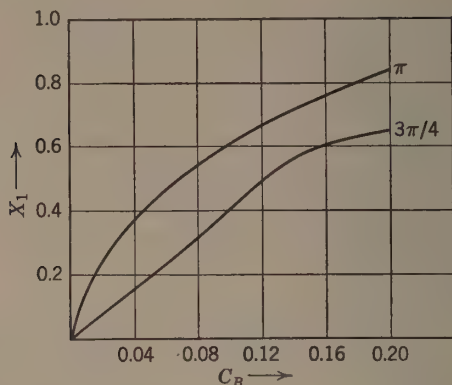


Fig. 9-7. Optimum gain per cavity, $X_1 = \text{Re}(-\gamma_1)$ vs. C_B for synchronism at $3\pi/4$ and also near-synchronism at π . $-M/p = 0.10$. The optimum X_1 is nearly independent of space charge for low values of space charge.

two values of C_B if $4(QC)_B = 1$ or 4 but is no higher for $4(QC)_B = 10$. Moreover, the useful voltage bandwidth within which the circuit power flow is in the forward direction is somewhat lower for $C_B^2 = 0.007$; this is true for both $-M/p = 0.10$ and 0.15. Incidentally, for $4(QC)_B = 10$ and $b_B = 7.4$ on Fig. 9-5 $(\theta_q/\theta_e)^2$ has the largest value of 0.115, which lies within our assumption of $(\theta_q/\theta_e)^2 \ll 1$.

In Fig. 9-6 we show the incremental phase shifts of the waves for the rather strong coupling of $C_B^2 = 0.010$, with $-M/p = 0.10$. If $-M/p$ is changed to 0.15 the rate of growth X for $4(QC)_B = 1$ and 4 is slightly lowered and also the voltage bandwidth for $4(QC)_B = 9$ is considerably shrunk. In Fig. 9-6, $4(QC)_B = 9$, $b_B = 3$ corresponds to $(\theta_q/\theta_e)_{\max}^2 = 0.15$, which somewhat violates our assumption about this parameter.

The gain per cavity X is very nearly maximum for low values of $4(QC)_B$ at the synchronous voltage: $b_B = 0$. It is also nearly independent of the space charge for low values of $4(QC)_B$. In Fig. 9-7 we plot X vs. C_B as given by the preceding figures with $-M/p = 0.10$. The gain is very nearly proportional to C_B in the range $C_B \leq 0.12$; therefore C_B has the additional significance of a *gain parameter*. The curve for near-synchronism at π will be discussed in Section 9-6.

At this point in our discussion it is instructive to replot the curves of Fig. 9-3 for $4(QC)_B = 10$ with respect to Pierce coordinates in order to show the similarity with the corresponding curves for the helix traveling wave tube. With $4(QC)_B = 10$, Table 9-1 indicates that C_P for this cavity chain is $C_B/0.331 = 0.095$, and $\sqrt{4(QC)_P} = 3.16/3 = 1.053$ by Eq. 9-7. Then $(QC)_P = 0.28$, and since C_P is quite small we can compare

the replotted curves in Fig. 9-8 with Pierce's curves⁵ of x, y vs. b_P for $(QC)_P = 0.25$. Our x and y are obtained logically from Fig. 9-3 as

$$x, y = \frac{X, Y}{\theta_{e0} C_P}, \quad \theta_{e0} = \frac{3\pi}{4} \quad (9-13)$$

and b_B is converted to $b_P = b_B/3$ through Eq. 9-10. In this way we obtain x, y vs. b_P for the three waves (neglecting the slightly perturbed "cold" circuit backward wave) shown in Fig. 9-8. The agreement with Pierce's curves is quite good even though the condition $|\theta_e C_P \delta| \ll 1$ is not well satisfied for the higher values of $\delta_P = x + jy$. We also include on the figure the curves of Fig. 9-4a for $4(QC)_B = 10$ plotted in Pierce coordinates. The Pierce parameters for this second set of curves are $C_P = 0.165$, $(QC)_P = 0.27$. These curves show poorer agreement because C_P is higher.

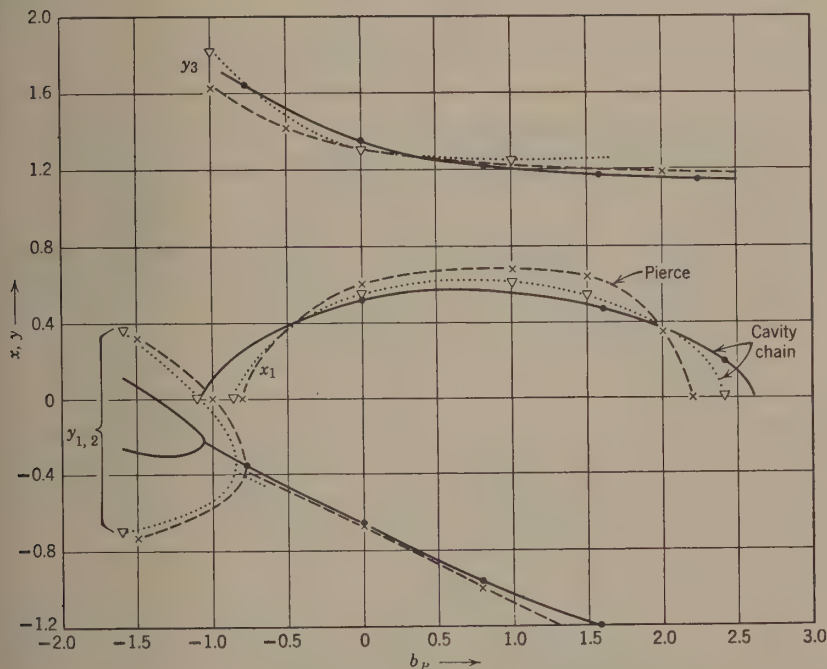


Fig. 9-8. Incremental propagation constants in the cavity-chain beam tube as compared to those in the helix tube. The Pierce coordinates x and y are related to X and Y as: $x, y = X, Y/\theta_{e0} C_P$. (Data from Pierce [1]). From Pierce, p. 177, $(QC)_P = 0.25$ (dashed lines). From Fig. 9-4a, $C_B = 0.055$, $4(QC)_B = 10$, $(QC)_P = 0.27$, and $C_P = 0.165$ (solid lines). From Fig. 9-3, $C_B = 0.032$, $4(QC)_B = 10$, $(QC)_P = 0.28$, and $C_P = 0.095$ (dotted lines).

⁵ J. R. Pierce [1, p. 177].

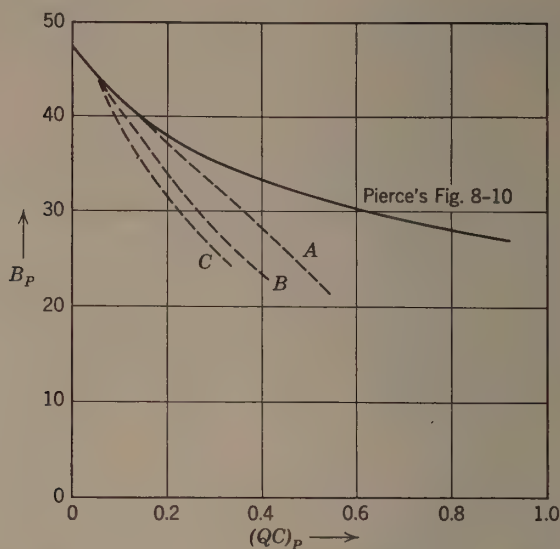


Fig. 9-9. Gain factor B_P vs. $(QC)_P$ in Pierce coordinates for the cavity-chain tube of uniform e_z field. $-M/p = 0.10$. (Data from Pierce [1]). For curve A, $C_B^2 = 0.001$, Fig. 9-3. For curve B, $C_B^2 = 0.003$, Fig. 9-4a. For curve C, $C_B^2 = 0.005$, Fig. 9-5a.

We have also calculated and plotted the Pierce relative gain parameter B_P which is defined in connection with the expression for maximum power gain of the cavity-chain tube:

$$P_{\max} = 10 \log_{10} \epsilon^{2X_{\max}N_c} \quad \text{db} \quad (9-14a)$$

where N_c is the total number of cavities. With the aid of the relation $\theta_{e0}N_c = (2\pi/\lambda_e)LN_c = 2\pi N$, N being the number of electronic wavelengths, we have

$$P_{\max} = 20(2\pi)(\log_{10} \epsilon)x_{\max}C_PN = B_PC_PN \quad \text{db} \quad (9-14b)$$

which serves to define the relative gain parameter B_P as

$$B_P = 54.5x_{\max}$$

where $x_{\max}$ is obtained from Eq. 9-13. From the data of Figs. 9-3, 9-4a, and 9-5a we determined the values of C_P and $4(QC)_P$ for each value of $x_{\max}$ from Table 9-1 and Eq. 9-7 and then found B_P vs. $(QC)_P$ by Eq. 9-13. The results appear in Fig. 9-9 with respect to Pierce's curve for the helix traveling wave tube.⁶ Our curves are deceptive, because the *net* gain per cavity, $X_{\max}$, remains very nearly constant along each of the descending curves of Fig. 9-9. Since $X_{\max} = \theta_{e0}C_Px_{\max}$ by definition of x and we know

⁶ J. R. Pierce [1, p. 126].

from Table 9-1 that C_P increases steadily (C_B is constant now) as $(QC)_P$ increases, it follows that $x_{\max}$ steadily decreases. This explains why B_P decreases with increasing space charge.

9-4. The Growing Wave for Operation near the $7\pi/4$ Synchronous Point

We commented earlier on the possibility of modifying the structure of Fig. 9-1 into that of Fig. 9-2 so as to operate around the synchronous $7\pi/4$ point. Let us see how C_B^2 varies with perveance, beam voltage U_0 , and L so that we can change the operating point from $3\pi/4$ to $7\pi/4$ with the least reduction in C_B^2 . From Eq. 9-3 C_B^2 is proportional to

$$C_B^2 \propto \frac{1}{\beta_e^2 L U_0} \frac{1}{\omega_q}, \quad \text{with } \omega_q \propto \left(\frac{I_0}{u_0} \right)^{1/2}$$

Since $\beta_e = \omega/u_0$ and $I_0 = KU_0^{3/2}$ this becomes

$$C_B^2 \propto \frac{u_0^{3/2}}{U_0^{1/4}} K^{1/2} \frac{u_0}{L} \quad (9-15)$$

This expression tells us that if we wish to increase $\theta_e = \omega(L/u_0)$ from $3\pi/4$ to $7\pi/4$ we can increase L by a factor of $\frac{7}{3}$ but we must also increase K by $(\frac{7}{3})^2$ in order to maintain C_B . If we choose to increase L and also decrease u_0 to obtain the same change in θ_e , we must increase K even more.

We have examined the waves in the structure of Fig. 9-2 for $M/p = 0.10$, $C_B^2 = 0.001, 0.005$, and 0.010 , and various values of $4(QC)_B$ so as to compare the growing wave with that of Figs. 9-3, 9-5a, and 9-6. The range of b_B in which useful gain occurs that is, the voltage bandwidth, is far less in $7\pi/4$ synchronous operation than in previous cases. The maximum rate of growth in the Fig. 9-2 structure occurs for synchronism of the slow beam mode with the "cold" circuit: $\theta_e + \theta_q = 7\pi/4$ and decreases very rapidly to zero as the beam voltage changes. The significant point, however, is the tremendous reduction in gain as compared to the gain for $3\pi/4$ synchronism. For the *same* values of C_B and $4(QC)_B$ as in the negative- M structure the maximum rate of growth in *neper*s is one-half or less the corresponding growth in the negative- M structure. This is plausible when we observe that the effectiveness of beam-circuit coupling is measured by $[\sin(\theta_e/2)/\theta_e/2]^2$ when the cavity field is uniform and this parameter decreases from 0.625 at $\theta_e = 3\pi/4$ to 0.019 at $\theta_e = 7\pi/4$.

We can safely conclude that amplifier efficiency is vastly reduced by changing the beam "cold" circuit synchronous point from the $3\pi/4$ region of the Fig. 9-1 structure to the $7\pi/4$ region of the Fig. 9-2 structure.

9-5. Expressions for the Electromagnetic Power Flow across a Coupling Surface

In Section 9-3 we made the statement that the electromagnetic power flow crossing a coupling surface is, in the narrow-passband approximation, proportional to $\sin(\text{Im } \gamma)$, where the wave phase shifts by $\epsilon^{-\gamma}$ across this surface. We now wish to obtain the precise power flow expression. Then we obtain various equivalent expressions for the total circuit power flow when more than one independent wave is present. Our expressions will imply continuity of power flow across the coupling surfaces as, of course, they must.

Suppose we wish to measure the circuit power flow in one wave at the left coupling surface in Fig. 9-1 or 9-2. One way to obtain this power is by averaging both the electric and the magnetic fields on the surface (denoted hereafter by M), as given by the single open- and short-circuit modes, respectively, of the passband. We write

$$P_{av} = \frac{1}{2} \text{Re} \int_M \frac{1}{2} v (1 \mp \epsilon^\gamma) \bar{e}^{(0)} \times \frac{1}{2} I^* (1 \pm \epsilon^{\gamma*}) \bar{H}^{(0)} \cdot \bar{i}_z dS \quad (9-16)$$

where the upper signs are taken for a passband of even symmetry such as we have in Figs. 9-1 and 9-2 (see Eq. 4-25 for this derivation). Since one of our amplitude relations for the open-circuit mode is, from Eq. 4-34,

$$pv = -j\omega\mu i \quad \text{with} \quad i \cong I \text{ for a narrow passband} \quad (9-17)$$

and we only require the real part of Eq. 9-16, that equation effectively is

$$P_{av} = \frac{1}{8} |v|^2 \frac{pM\tau}{\omega\mu} \text{Im}(\pm \epsilon^{\gamma*} \mp \epsilon^\gamma) = \mp \frac{1}{4} |v|^2 \frac{pM\tau}{\omega\mu} \text{Im}(\epsilon^\gamma) \quad (9-18)$$

where

$$M = \frac{1}{\tau} \int_{M-} \bar{e}^{(0)} \times \bar{H}^{(0)} \cdot \bar{i}_z dS, \quad \begin{array}{l} \text{very nearly the passband} \\ \text{width in units of } k \end{array}$$

As a check we see that in the "cold" circuit $\gamma = j\varphi$ and so

$$P_{av} = \mp \frac{1}{4} |v|^2 \frac{pM\tau}{\omega\mu} \sin \varphi^0$$

in agreement with Eq. 4-27 for a narrow passband such that $P \cong p \cong k$.

We may easily verify that the following expressions for the power flow in a wave are all equivalent to Eq. 9-18:

$$\begin{aligned} P_{av} &= \frac{1}{2} \text{Re} \int_M \hat{e}^{(-1)} \times \bar{\bar{H}}^* \cdot \bar{i}_z dS = \frac{1}{2} \text{Re} \int_M \hat{e}^{(0)} \times \bar{\bar{H}}^* \cdot \bar{i}_z dS \\ &= \frac{1}{2} \text{Re} \int_M \bar{\bar{e}} \times \hat{H}^{(-1)*} \cdot \bar{i}_z dS = \frac{1}{2} \text{Re} \int_M \bar{\bar{e}} \times \hat{H}^{(0)*} \cdot \bar{i}_z dS \end{aligned} \quad (9-19a)$$

in which

$$\vec{H} \triangleq \frac{1}{2}(\hat{H}^{(-1)} + \hat{H}^{(0)}), \quad \vec{e} = \frac{1}{2}(\hat{e}^{(-1)} + \hat{e}^{(0)}). \quad (9-19b)$$

Because the wave under discussion is represented by a resonant cavity mode of which $\hat{e}^{(-1)} \times \hat{H}^{(-1)*}$ in cavity (-1) and $\hat{e}^{(0)} \times \hat{H}^{(0)*}$ in cavity (0) are purely imaginary we can also abbreviate Eq. 9-19a to

$$P_{av} = \frac{1}{4} \text{Re} \int_M \hat{e}^{(-1)} \times \hat{H}^{(0)*} \cdot \bar{i}_z dS = \frac{1}{4} \text{Re} \int_M \hat{e}^{(0)} \times \hat{H}^{(-1)*} \cdot \bar{i}_z dS \quad (9-19c)$$

These relations are true for either choice of signs in Eq. 9-16.

We now wish to outline the proof that Eqs. 9-19 are all valid even though the $\hat{e}$ and $\hat{H}$ fields are composed of a linear combination of the four independent waves with a beam present. This fact will imply continuity of circuit power flow across each coupling surface in the general case. Consider again Eq. 9-16 written as

$$P_{av, \text{tot}} = \frac{1}{2} \text{Re} \int_M \frac{1}{2} \sum_{\alpha=1}^4 v_{\alpha} (1 \mp \epsilon^{\gamma_{\alpha}}) \bar{e}^{(0)} \times \frac{1}{2} \sum_{\beta=1}^4 I_{\beta}^* (1 \pm \epsilon^{\gamma_{\beta}*}) \bar{H}^{(0)} \cdot \bar{i}_z dS \quad (9-20)$$

where subscripts α and β denote the α th and β th waves, all of which are represented by the same mode patterns $\bar{e}$ and $\bar{H}$. With $p v_{\alpha} = -j\omega\mu i_{\alpha}$ and $i_{\alpha} \cong I_{\alpha}$ we can reduce this equation to

$$P_{av, \text{tot}} = \frac{pM\tau}{8\omega\mu} \text{Im} \sum_{\alpha\beta} v_{\alpha} v_{\beta}^* (1 \mp \epsilon^{\gamma_{\alpha}}) (1 \pm \epsilon^{\gamma_{\beta}*}) \quad (9-21)$$

In the double summation a typical term $v_1 v_2^* (1 - \epsilon^{\gamma_1 + \gamma_2*})$ has its imaginary part canceled by the associated term $v_2 v_1^* (1 - \epsilon^{\gamma_2 + \gamma_1*})$ so we obtain

$$\begin{aligned} P_{av, \text{tot}} &= \frac{pM\tau}{8\omega\mu} \text{Im} \sum_{\alpha\beta} v_{\alpha} v_{\beta}^* (\pm \epsilon^{\gamma_{\beta}*} \mp \epsilon^{\gamma_{\alpha}}) \\ &= \mp \frac{pM\tau}{4\omega\mu} \text{Im} \sum_{\alpha\beta} v_{\alpha} v_{\beta}^* \epsilon^{\gamma_{\alpha}} \end{aligned} \quad (9-22)$$

The first expression in Eq. 9-19a is equivalent to this, for we can transform it as follows:

$$\begin{aligned} P_{av, \text{tot}} &= \frac{1}{2} \text{Re} \int_M \hat{e}^{(-1)} \times \vec{H}^* \cdot \bar{i}_z dS \\ &= \frac{1}{4} \text{Re} \int_M \mp \sum_{\alpha} v_{\alpha} \epsilon^{\gamma_{\alpha}} \bar{e}^{(0)} \times \sum_{\beta} I_{\beta}^* (1 \pm \epsilon^{\gamma_{\beta}*}) \bar{H}^{(0)} \cdot \bar{i}_z dS \\ &= \mp \frac{1}{4} \frac{pM\tau}{\omega\mu} \text{Im} \left[\sum_{\alpha\beta} v_{\alpha} \epsilon^{\gamma_{\alpha}} v_{\beta}^* \cdot (1 \pm \epsilon^{\gamma_{\beta}*}) \right] \\ &= \mp \frac{1}{4} \frac{pM\tau}{\omega\mu} \text{Im} \left(\sum_{\alpha\beta} v_{\alpha} v_{\beta}^* \epsilon^{\gamma_{\alpha}} \right) \end{aligned} \quad (9-23)$$

The last step is justified because $\sum_{\alpha\beta} v_\alpha v_\beta^* \epsilon^{\gamma\alpha+\gamma\beta*}$ is pure real. Analogously we may prove that the remainder of the expressions in Eq. 9-19 are all equivalent to Eq. 9-23.

Equation 9-19 tells us that if no field is present on the gun side of the first cavity hole in Figs. 9-1 and 9-2 no circuit power flows out that hole. This fact is consistent with Bethe's small coupling-hole theory, in which the tangential electric field induced on a hole by tangential magnetic field is nearly 90° out of time phase with the magnetic field and creates very small real-power flow into the radiation fields (see Eq. 3-14 as well as Sections 3-5 and 3-6).

To understand why these expressions imply continuity of circuit power flow from the (-1) to the (0) cavity of Figs. 9-1 and 9-2 we observe that the derivation of the open-circuit mode amplitude relations for v and i by the procedure in Section 4-6 may be carried out alternatively from Maxwell's equations written in cavity (0) as

$$\begin{aligned}\nabla \times \hat{e} &= -j\omega\mu\hat{h} \\ \nabla \times \hat{h} &= \hat{J} + \hat{K}_e + j\omega\epsilon\hat{e}, \quad \text{where} \quad \hat{K}_e = (+\hat{i}_z \times \vec{\hat{H}})_{\text{on } M}\end{aligned}\quad (9-24)$$

whereupon we dot-multiply the first equation by $\hat{h}$, the second by $\hat{e}$, and integrate them both over the cavity volume *without* integrating the left sides by parts. This result may also be obtained from the variational expression referred to in Section 8-4 by taking variations δv and δi in $\hat{E}_+$ and $\hat{H}_+$, with $\hat{E}_-$ and $\hat{H}_-$ expanded in short-circuit modes. The procedure is somewhat lengthy because it includes the irrotational open-circuit modes for the electric field, strictly speaking, and so we do not repeat it here.

As a result of all this, surface electric current $\hat{K}_e$ is equivalent to the coupling surface excitation for the open-circuit modes. If, now, we dot-multiply Eq. 9-24a by the total magnetic field $\hat{h}^*$, the conjugate of Eq. 9-24b by $\hat{e}$, subtract them, and integrate again over the cavity (0) volume, we obtain a Poynting theorem which says that the power flow *into* cavity (0) through this hole is

$$P_{av,\text{tot}} = -\frac{1}{2} \text{Re} \int_M \hat{K}_e^* \cdot \hat{e}^{(0)} dS = \frac{1}{2} \text{Re} \int_M \hat{e}^{(0)} \times \vec{\hat{H}}^* \cdot \hat{i}_z dS \quad (9-25a)$$

in which $\vec{\hat{H}}$ is identified in Eq. 9-19b. By analogous reasoning the power flowing out of cavity (-1) is interpreted as

$$P_{av,\text{tot}} = \frac{1}{2} \text{Re} \int_M \hat{e}^{(-1)} \times \vec{\hat{H}}^* \cdot \hat{i}_z dS \quad (9-25b)$$

Equation 9-19a shows these two expressions to be equal. Therefore the interpretation of power flow across coupling surface M is consistent in this formalism.

The beam power flow in each wave is explicitly continuous at the coupling surfaces because the beam equations were solved so that $\hat{V}_k(L)$ flowing out of a cavity = $\hat{V}_k(0)\epsilon^{-\gamma}$ flowing into the next cavity, and similarly for $\hat{J}(L)$ in each wave.

We must observe that the power flows of the separate waves in an expression like Eq. 9-21 for the circuit power are *not* orthogonal; the crosspowers between the various waves contribute to the total power flow in the circuit and in the beam.⁷

The general relations in terms of *all* the resonant cavity modes for the power flow are found in Appendix A, Section A.1, along with the proof of the power flow-stored energy-group velocity relation for the "cold" chain.

9-6. Nature of the Waves for Uniform Cavity Electric Field, in a Voltage Range about the Synchronous Voltage at π Cutoff

The incremental phase shifts for the four independent waves appear in Figs. 9-10 through 9-12, for $C_B^2 = 0.0025, 0.010$, and 0.0225 , respectively, and for the designated passband width of $-M/p = 0.10$ in Fig. 9-1. At

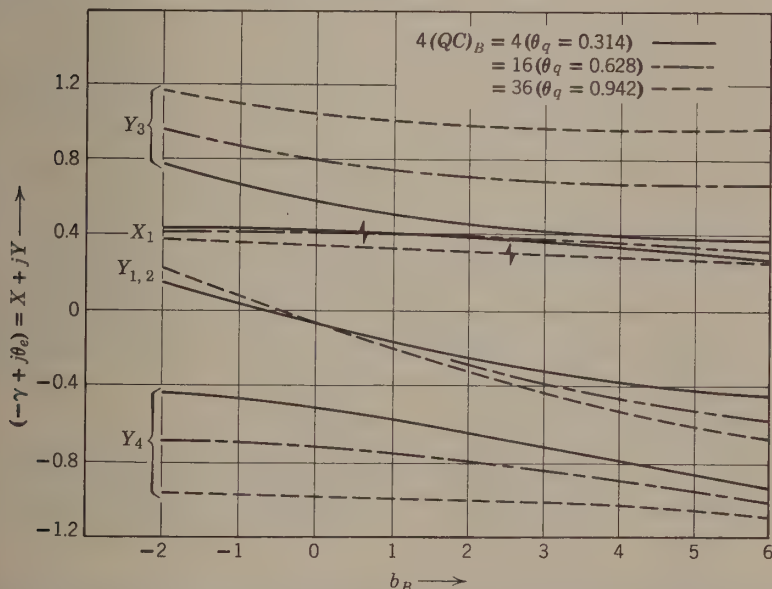


Fig. 9-10. Incremental periodic phase shift $X + jY$ for the waves in the tube with uniform e_z field in each cavity. Synchronism at π . $C_B^2 = 0.0025$. $4(QC)_B$ is the parameter. $-M/p = 0.10$.

⁷ The very last sentence is wrong in Appendix C of Ref. [2].

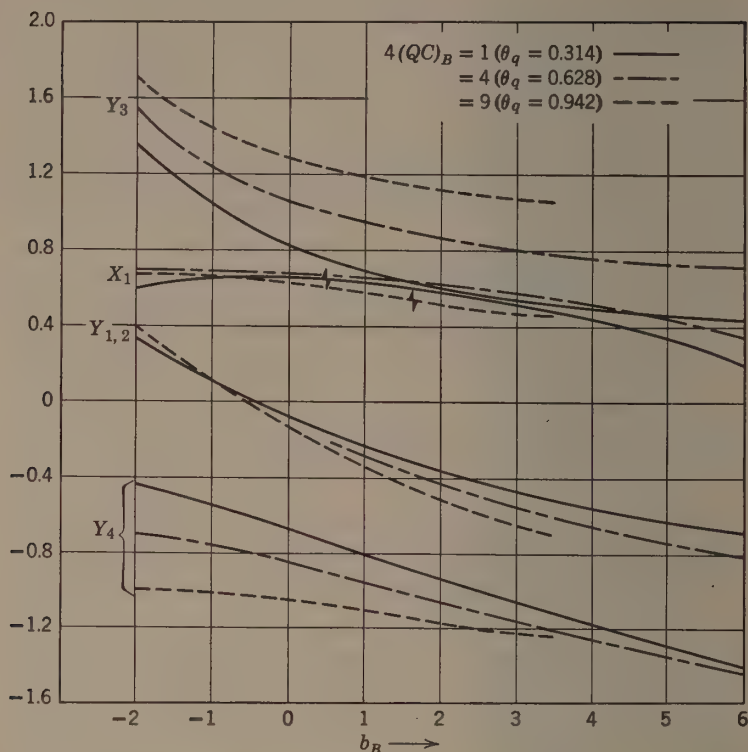


Fig. 9-11. Incremental phase shift for the waves in the tube with uniform e_z field. Synchronism at π . $C_B^2 = 0.010$. $-M/p = 0.10$. $4(QC)_B$ is the parameter.

the cutoff frequency of operation the Pierce parameter C_P of Eq. 9-2 is useless because $\sin \varphi^0 = 0$, so we use the parameters C_B and $4(QC)_B$ exclusively. The incremental phase shifts have the same characteristics as they did for interaction near $3\pi/4$ and pips on the curves indicate the transition from backward-circuit power flow in the growing wave to forward-circuit power flow at the higher beam voltages.

Figure 9-10 shows the insensitivity of the phase shifts to changes in beam voltage in the range $-2 \leq b_B \leq 6$, which corresponds for this value of $C_B = 0.05$, to $3.49 \geq \theta_e \geq 2.42$. The gain per cavity X is high and higher, in fact, than the gain for the same C_B at the $3\pi/4$ synchronous point. We would operate an amplifier well to the right of a transition voltage such that appreciable circuit power grows in the direction of the d.c. beam at the expense of beam power. The effect of increasing the fractional pass-band width to 0.15 is a slight decrease in gain; the transition points essentially do not move.

In Fig. 9-11 we see the incremental phase shifts for the larger coupling of $C_B^2 = 0.010$. The useful gain per cavity to the right of the transition points is noticeably greater than for $C_B^2 = 0.0025$ of the previous figure. But the gain diminishes more rapidly with increasing b_B than it did previously. Maximum $(\theta_q/\theta_e)^2$ is 0.15 and occurs at $b_B = 3$, $4(QC)_B = 9$. The effect on the growing wave of increasing $-M/p$ to 0.15 (the curves are not shown) is to move all the transition points about 1 unit of b_B to the right and to lower the curves, reducing the optimum gain about 0.13 neper. This term "optimum gain" is somewhat ambiguous; it is the gain beyond the transition point at a voltage which represents a compromise between low circuit power per unit amplitude of growing wave and low gain per cavity.

In Fig. 9-12 we see the phase shifts for large beam-circuit coupling of $C_B^2 = 0.0225$. The useful gains to the right of the transition points have increased and they decrease somewhat more rapidly with increasing b_B as in Fig. 9-11. The maximum $(\theta_q/\theta_e)^2$ of 0.19 at $b_B = 3$, $4(QC)_B = 4$ violates our assumption of $(\theta_q/\theta_e)^2 \ll 1$; however, the smooth nature of the curve for this value of $4(QC)_B$ indicates that this gain is not greatly in error. An increase of the fractional passband width to 0.15 causes the optimum gains to drop about 0.08 neper but the transition voltages remain nearly fixed.

We also examined the gain as a function of b_B and $4(QC)_B$ for the high-power coupling of $C_B^2 = 0.040$, but the curves are not included here. The optimum gain is about 0.10 neper above that for $C_B^2 = 0.0225$ at $\theta_q/\theta_e = 0.1$ and 0.2 [or $4(QC)_B = 0.445$ and 1.78 in Fig. 9-12] and somewhat less than this for $\theta_q/\theta_e = 0.3$. An increase of passband width reduces the gains somewhat, as before.

In Fig. 9-7 we show the gain for low values of $4(QC)_B$ as a function of C_B , as measured about one unit of b_B to the right of the transition points. The curve labeled π shows decidedly more gain than was obtainable with operation around the $3\pi/4$ synchronous point. For operation near the π synchronous point the gain is only roughly linear with C_B so that it is a gain parameter in a different sense than at the lower operating point.

9.7. Behavior of the Growing Wave with Sinusoidal e_z Cavity Field; Operation near the $3\pi/4$ and π Synchronous Points⁸

In this section we digress to consider the growing wave in a chain consisting of the relatively narrow cavities of Fig. 9-13. These cavities are narrow compared to those of Fig. 9-1 so that the $\bar{e}_z$ pattern is "bowed"

⁸ In the structure of Fig. 9-13, $-M/p = 0.10$.

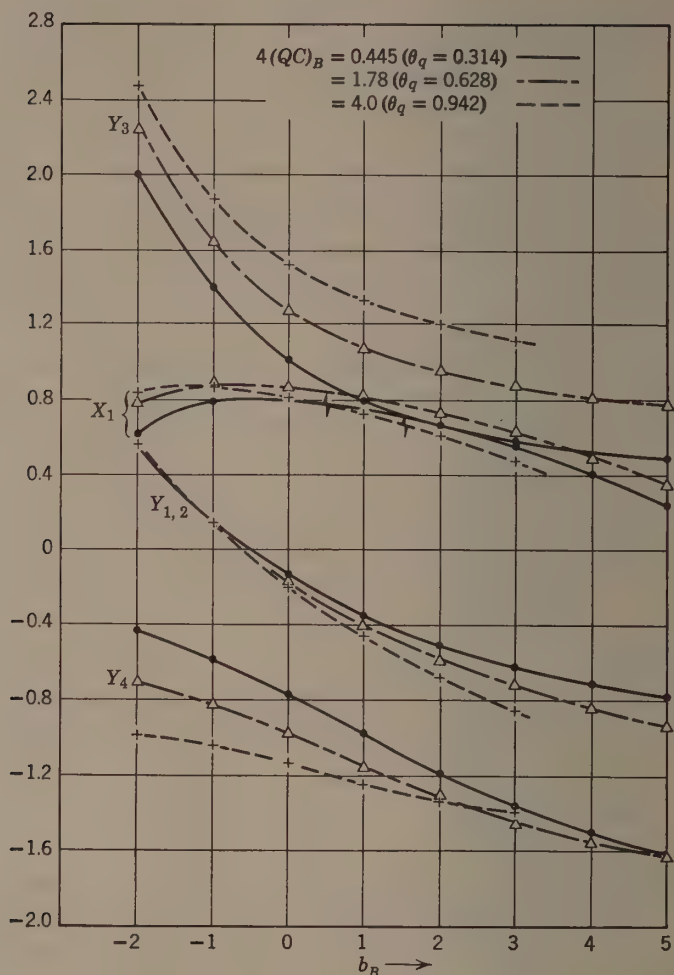


Fig. 9-12. Incremental phase shift for the waves in the tube with uniform e_z field. Synchronism at π . $C_B^2 = 0.0225$. $-M/p = 0.10$. $4(QC)_B$ is the parameter.

across each cavity and $\bar{e}_z \cong e_m \sin(\pi z/L)$ across the cavity. Of the assumptions made early in Section 9-2 the following remarks apply. Since the centerholes of Fig. 9-13 represent more coupling between cavities of smaller volume, the second passband will generally lie closer to the first passband and its modes will contribute more predominantly to the fields. Nevertheless, we can obtain a good description of the growing wave behavior by ignoring them and representing the solenoidal electric field by $v\bar{e}$ of the first open-circuit mode, as before. We should note that $v\bar{e}_z$ only describes

the axial electric field at frequencies near the π cutoff frequency; at frequencies near the other cutoff frequency, P , the electric field would resemble the "straight-through" character of the short-circuit mode. We again take the axial space charge electric field to be $jZ_0\beta_q\hat{J}_z$ and the $\bar{e}_z$ pattern to be uniform in the cross section of the thin beam (hollow or solid), though it varies sinusoidally across the cavity. Our assumption of abrupt phase shift of amplitudes across the centerholes is better for sinusoidal than for uniform $\bar{e}_z$ pattern because the nearly radial field near the centerholes does not accelerate the beam very much.

Our fundamental equations for the interaction are on page 276 but with $\bar{e}_z = e_m \sin(\pi z/L)$. The derivation of the determinantal equation for γ is outlined in Appendix E, Section E-2, and this equation has been solved on the IBM 704 computer for the same values of the parameters on which the previous curves were based. We define the C_B parameter for this chain as

$$C_{Bs}^2 = \frac{\omega \mu e_m^2 A_b Y_0}{p^2 \beta_e^2 \tau} \quad \text{s for sinusoidal field} \quad (9-26)$$

just as in Eq. 9-3, whereupon Eqs. 9-5 and 9-8 still serve as definitions for the space charge parameter $4(QC)_B$ and synchronism parameter b_B . Although we do not work in terms of the Pierce parameters we call attention to the fact that E_{eff}^2 of Eq. 9-1 evaluates as

$$E_{\text{eff}}^2 = \left| \frac{1}{L} \int_0^L e_m \sin \frac{\pi z}{L} e^{-j\beta_e z} dz \right|^2 = e_m^2 \left[\frac{2\pi \cos(\theta_e/2)}{\pi^2 - \theta_e^2} \right]^2 \quad (9-27)$$

which remains well behaved even as θ_e approaches π . This means C_P

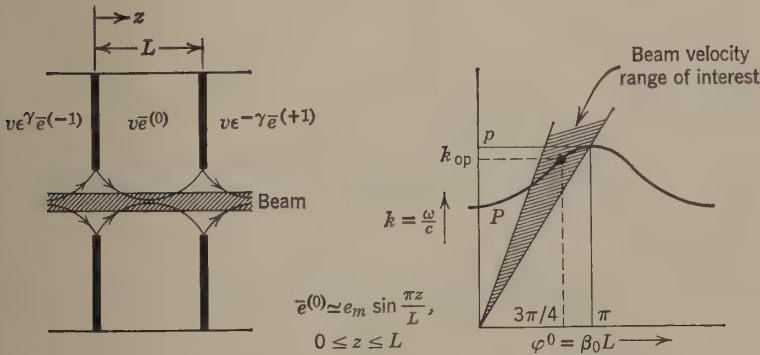


Fig. 9-13. A cavity-chain beam tube of narrow first passband and approximately sinusoidal $\bar{e}_z$ pattern. Operating frequency k_{op} and the synchronous beam velocity both correspond to a "cold" circuit phase shift of $3\pi/4$ in this diagram.

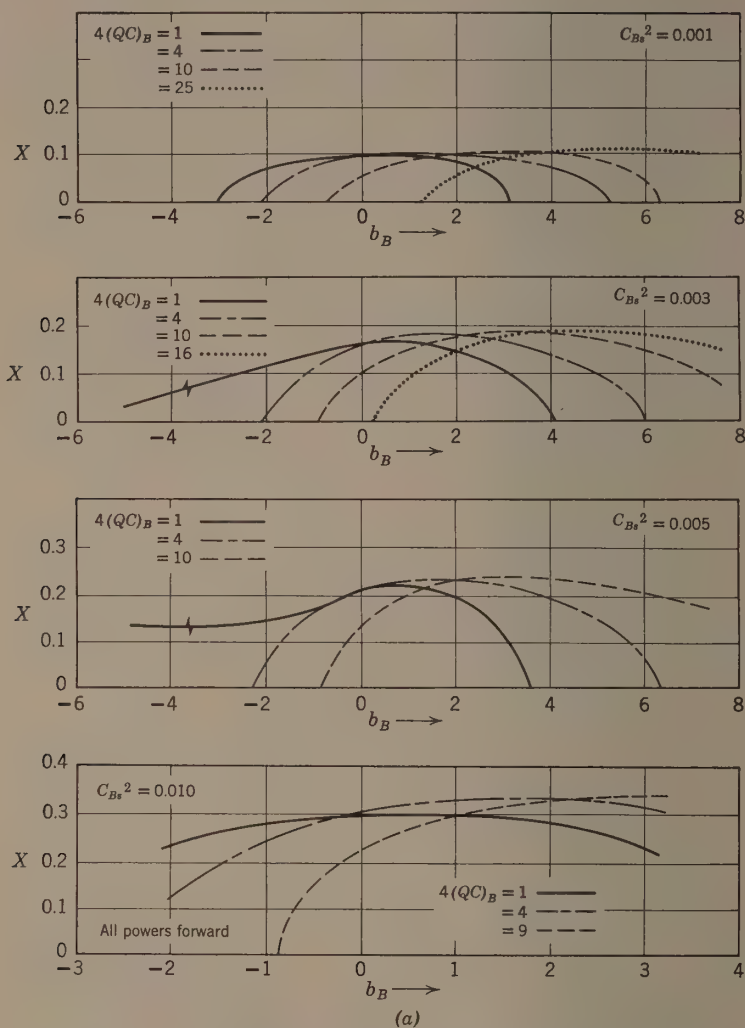


Fig. 9-14. Growth rate X for the cavity chain with sinusoidal e_z field across each cavity. $-M/p = 0.10$. C_{Bs}^2 and $4(QC)_B$ are the parameters. (a) Synchronism at $3\pi/4$.

would be related to C_B as

$$C_B^2 = \left(-\frac{M}{p} \right) \frac{\theta_{e0}}{\theta_a} \left[\frac{\pi^2 - \theta_{e0}^2}{2\pi \cos(\theta_{e0}/2)} \right]^2 \sin \varphi^0 C_P^3, \quad \theta_{e0} \equiv \varphi^0 \quad (9-28)$$

instead of by Eq. 9-4.

Figure 9-14a illustrates the gain per cavity X of the growing wave for the four values of C_B^2 and associated values of $4(QC)_B$ upon which Figs.

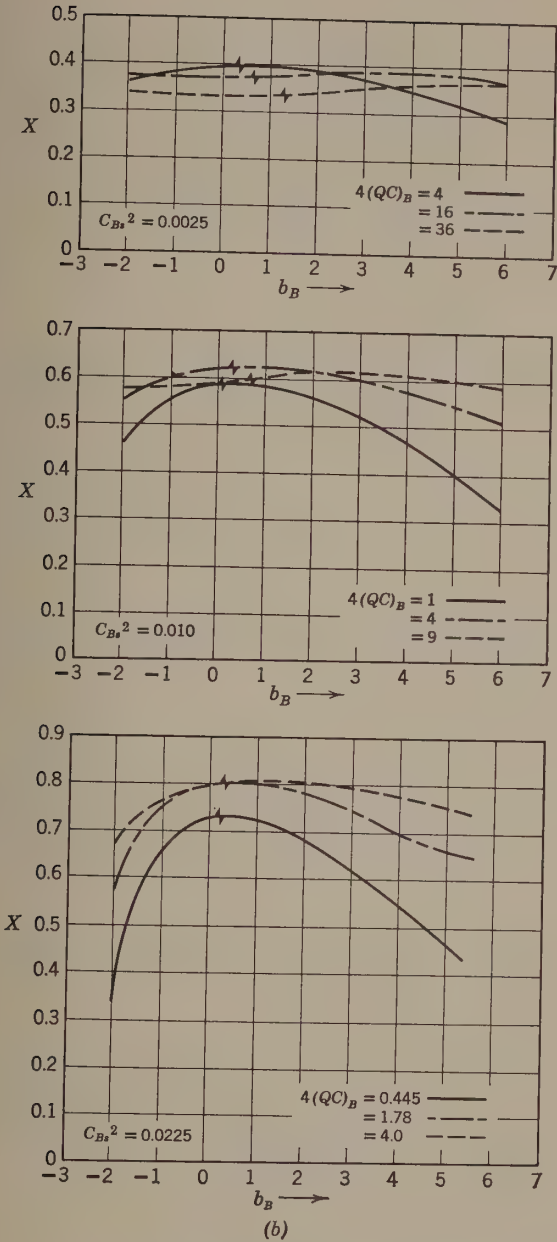


Fig. 9-14 (continued). (b) Synchronism at π .

9-3, 9-4a, 9-5a, and 9-6 are based. If we compare each set of curves in Fig. 9-14a with the corresponding figure for the uniform $\bar{e}_z$ field pattern, we note the following trends. Almost every curve in Fig. 9-14a has a maximum value which is slightly less than the corresponding curve on Figs. 9-3 through 9-6. The single exception is the gain curve for $C_B^2 = 0.010$, $4(QC)_B = 9$, which shows greater maximum gain in Fig. 9-14a than in Fig. 9-6. The close agreement of the gain curves for the two structures of Fig. 9-1 and 9-13 is surprising in view of the fact that we are comparing them on the basis of the same C_B , which means $e_{z, \text{uniform}} \equiv e_{m, \text{max}}$. But the average value of $e_m \sin(\pi z/L)$ across the cavity is only $(2/\pi)e_m \equiv (2/\pi)e_{z, \text{uniform}}$. The curves of Fig. 9-14a also indicate that the voltage bandwidths tend to be narrower in the structure of Fig. 9-13.

In Fig. 9-14b we present the same types of curves for the structure of Fig. 9-13 but interacting with the beam near the π synchronous point, for comparison with Figs. 9-10 through 9-12. Here we observe that the maximum useful gain to the right of the pips in Fig. 9-14b is, in most cases, slightly or else noticeably *greater* than for the comparable structure of uniform $\bar{e}_z$ pattern. The voltage bandwidths are also decidedly wider in Fig. 9-14b.

These facts warrant the statement that, whereas the cavity chain with sinusoidal $\bar{e}_z$ field pattern is not quite as good as the chain with uniform e_z pattern of the same C_B and $4(QC)_B$ for operation near the $3\pi/4$ synchronous point, the former is better with respect to gain near the π synchronous point.

In the remainder of this chapter we study the properties of the cavity-chain beam amplifier with *uniform* $\bar{e}_z$ field pattern in each cavity. We may reasonably expect the tube of sinusoidal pattern to have similar characteristics, but there is no doubt the detailed properties will be different.

9-8. Excitation of the Growing Wave in the First Cavity of the Fig. 9-1 Structure⁹

In this section we discuss the extent to which the growing wave is excited in the first cavity of an amplifier tube fed by the external generator through a coupling loop (see Fig. 9-15). We are not concerned with the reflection of the coaxial waveguide mode due to the mismatch of the cavity at surface S_i but only with the relative amount of growing wave excited by a unit generator current I_g in the loop. We do not consider hole input coupling according to remarks made at the end of Section 9-10.

⁹ $-M/p = 0.10$.

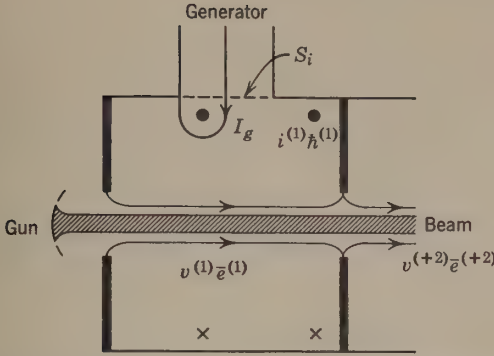


Fig. 9-15. Excitation of the first cavity by the generator. Magnetic field $i^{(1)}\tilde{h}^{(1)}$ threads the loop.

The current I_g is assumed to be uniform along the small loop within the cavity.

The beam enters cavity (1) without initial a.c. modulation, and the problem is to determine the proper linear combination of growing, decaying, and constant-amplitude (perturbed fast beam mode) waves—all with forward-circuit power flow—in order to satisfy the boundary conditions within this cavity. We have solved the problem for operation near both the $3\pi/4$ and the π synchronous points. The magnitude of the growing wave component of electric field, v_g , in the first cavity is plotted vs. b_B in Fig. 9-16 for the various values of C_B and $4(QC)_B$ upon which the curves of Figs. 9-3, 9-5a, and 9-6 are based. The curves are drawn only in the range of b_B in which the growing wave carries forward circuit power; the range of b_B for $C_B^2 = 0.010$ is relatively narrow because C_B is relatively large and so a smaller range of b_B represents about the same fractional change in d.c. beam velocity. The curves tell us that if we adjust the beam voltage somewhat higher than the synchronous value v_g will be 0.5 or more, where $|v_{\text{tot}}| = 1$ for all the curves of Fig. 9-16. The large amount of growing wave excited for $C_B^2 = 0.001$, 0.005, and $4(QC)_B = 10$ at the voltage extremes does not warrant operating at those voltages, where either the rate of growth is going to zero or else the circuit power flow per unit amplitude v_g is approaching zero.

The corresponding curves for operation near the π synchronous point appear in Fig. 9-17, as determined from the properties of the waves in Figs. 9-10 and 9-11 and for the extreme coupling of $C_B^2 = 0.040$. Aside from the fact that the voltage range for useful gain is shifted upward, we see that $|v_g|/|v_{\text{tot}}|$ is typically 0.6; $|v_g|$ for the case $C_B^2 = 0.01$, $4(QC)_B = 1$ unfortunately starts to rise just as the rate of growth starts to diminish.

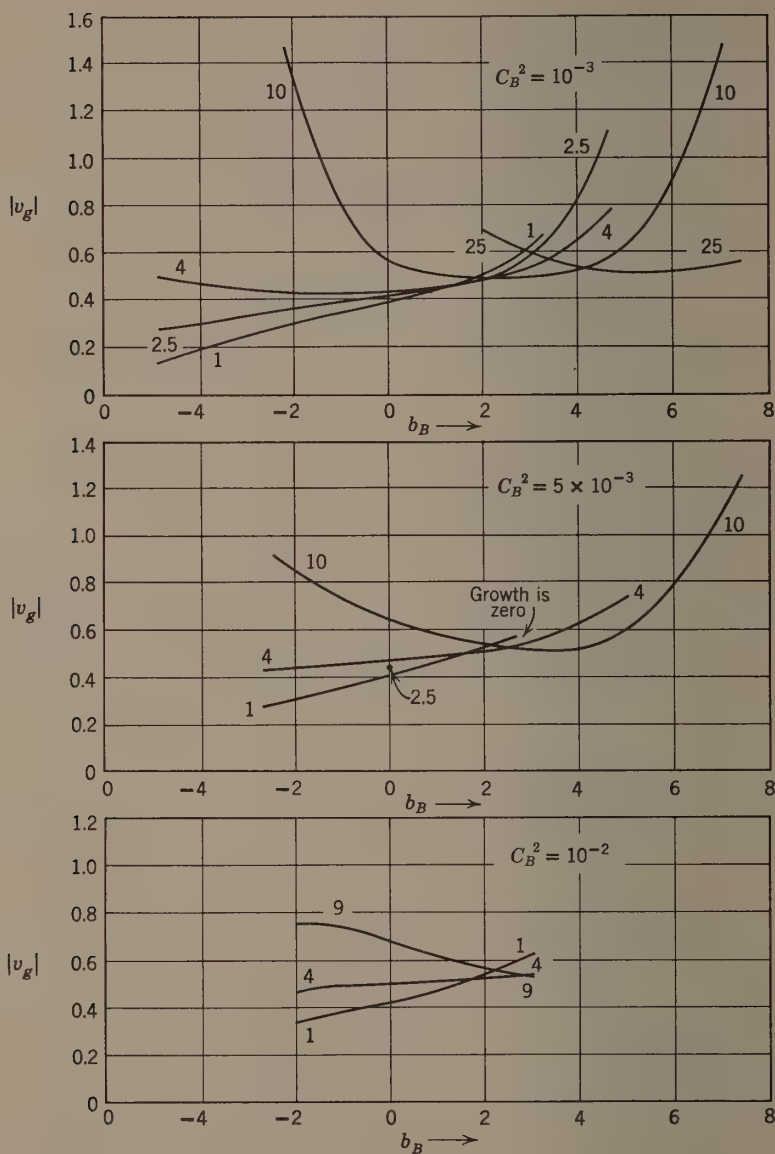


Fig. 9-16. Excitation of the growing wave in the first cavity. Synchronism at $3\pi/4$. $-M/p = 0.10$. C_B^2 and $4(QC)_B$ are the parameters. $|v_{tot}| = 1$ in every case.

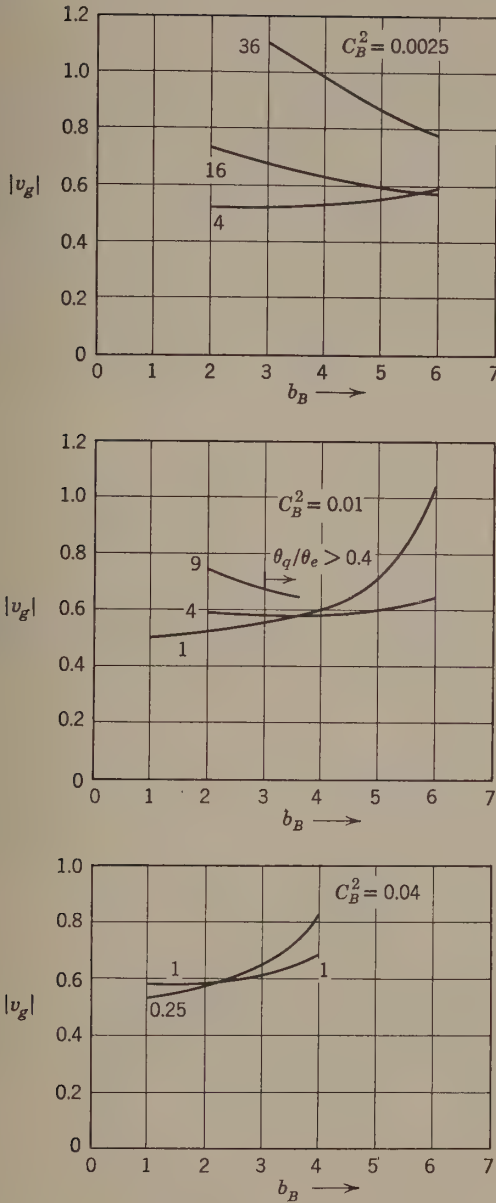


Fig. 9-17. Excitation of the growing wave in the first cavity. Synchronism at π . $-M/p = 0.10$. C_B^2 and $4(QC)_B$ are the parameters. $|v_{tot}| = 1$ in every case.

We should also observe for the case $C_B^2 = 0.0025$, although the growing wave is most effectively generated when $4(QC)_B = 36$, the rate of growth is somewhat less than for lower space charges (see Fig. 9-10).

We calculated the curves of Fig. 9-16 and 9-17 as follows. We first rederived the expressions for open-circuit mode amplitudes v and i , as excited by generator current I_g , the beam, and tangential magnetic field on the right coupling hole of Fig. 9-15. Since no a.c. fields are presumed to exist on the gun side of the first hole, no circuit power flows out of this hole, according to remarks following Eq. 9-23. Following the derivation of Eqs. 4-34 through 4-36, we see that the tangential electric field of the coaxial tube on surface S_i does not excite the cavity field because it is antisymmetric relative to the $\bar{h}$ pattern. Therefore, only I_g of the loop excites the cavity field, and since the loop current is formally equivalent to the beam current we obtain just

$$\begin{aligned} (p^2 - k^2 + pM)v_{\text{tot}}^{(1)} + \frac{1}{2}pMv_{\text{tot}}^{(+2)} + \frac{j\omega\mu}{\tau} \int_V \hat{J}_{\text{tot}} \cdot \bar{e}_z dV \\ = -\frac{j\omega\mu}{\tau} I_g \int_{\text{loop}} \bar{e} \cdot \bar{d}\ell \quad (9-29a) \end{aligned}$$

The last term of this equation will be further justified on a power basis in Section 9-10. We regard $\hat{v}_{\text{tot}}$ and $\hat{J}_{\text{tot}}$ as composed of growing, decaying, and perturbed fast beam waves, as

$$\begin{aligned} \hat{v}_{\text{tot}}^{(1)} = v_g + v_d + v_f, \quad \hat{v}_{\text{tot}}^{(2)} = v_g \epsilon^{-\gamma_g} + v_d \epsilon^{-\gamma_d} + v_f \epsilon^{-\gamma_f} \\ \hat{J}_{\text{tot}}^{(1)} = \hat{J}_g + \hat{J}_d + \hat{J}_f \quad (9-29b) \end{aligned}$$

If we add $\frac{1}{2}pMv_{\text{tot}}^{(0)}$ to both sides of Eq. 9-29a, with $v_{\text{tot}}^{(0)}$ defined as $v_g \epsilon^{+\gamma_g} + \dots + v_f \epsilon^{+\gamma_f}$, the left side will be identically zero. This is so because the growing, decaying, and perturbed fast beam wave portions are separately zero (see Eq. 8-26). Therefore our first equation in terms of v_g , v_d , and v_f is simply

$$\frac{1}{2}pMv_{\text{tot}}^{(0)} = \frac{j\omega\mu}{\tau} I_g \int_{\text{loop}} \bar{e} \cdot \bar{d}\ell \quad (9-29c)$$

We obtain the other two relations by choosing the wave amplitudes so that the kinetic voltage and current which enter the first cavity are matched to zero. In terms of the A and B coefficients of Appendix E, Eq. E-5, these relations are

$$\hat{V}_k(1) = 0 = A_g v_g + A_d v_d + A_f v_f \quad (9-30)$$

$$\hat{J}(1) = 0 = B_g v_g + B_d v_d + B_f v_f \quad (9-31)$$

Then Eqs. 9-29 through 9-31 enable us to compute v_g with the magnitude of I_g chosen so that $|v_{\text{tot}}^{(1)}| = |v_g + v_d + v_f| = 1$ for convenience; this value of $|v_g|$ is plotted in Figs. 9-16 and 9-17.

9.9. Effect of a Reactive Discontinuity (Detuned Cavity)

We now investigate the effect upon the growing wave of inserting a detuned lossless cavity into the center of a long chain in order to reduce the tendency to oscillate. The detuned cavity and circuit field components are depicted in Fig. 9-18. We can understand how a detuned cavity in the center of a long chain will reduce the tendency to oscillate by recourse to the following reasoning. Suppose the chain without the detuned cavity has unit amplitude of growing wave in the first cavity, which grows to $|\epsilon^{-\gamma_o N}| = \epsilon^{N\alpha}$ at the last cavity. If this cavity reflects a fraction R_o in the form of the constant-amplitude backward wave, then $\epsilon^{N\alpha} R_o$ returns to the first cavity. If this cavity reflects a fraction R_i of the field into the growing wave we have a tendency to oscillate if $|R_i R_o| \epsilon^{\alpha N} = 1$. With the detuned cavity present a unit amplitude of growing wave in the first cavity reaches it with amplitude $\epsilon^{N\alpha/2}$. If a fraction R_o' is reflected as a backward wave, the first $N/2$ cavities will tend to oscillate if $|R_o' R_i| \epsilon^{N\alpha/2} = 1$. But if $|R_o'| < |R_o| \epsilon^{N\alpha/2}$ the first half of the tube shows less tendency to oscillate. The latter half of the tube will tend to oscillate by itself if $|R_o R_i'| \epsilon^{N\alpha/2} = 1$, where R_i' is the fraction of backward wave converted into growing wave in the cavity following the detuned one. Therefore, if $|R_i'| < |R_i| \epsilon^{N\alpha/2}$ the last half of the tube shows less tendency to oscillate.

In a twenty-cavity tube of 50-db gain, say, $\epsilon^{\alpha N/2} = 18$, so that even though R_o' and R_i' may well lie in the range 1–3 (greater than 1 as we discover next) it may prove safer to insert the detuned cavity rather than assume the external loads are good matches, particularly since the matched generator and output loads in the “hot” circuit differ from those in the

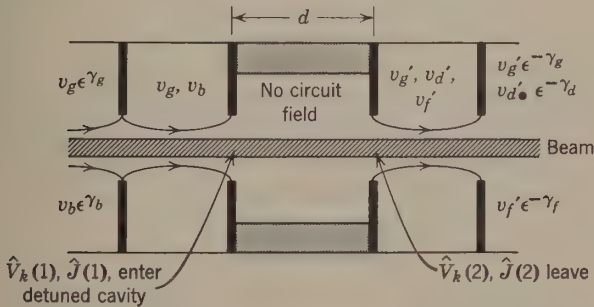


Fig. 9-18. A detuned cavity, with incident growing wave and reflected backward wave on the left, and generated growing, decaying, and perturbed fast waves on the right. The beam drifts under the action of its own space charge fields through the detuned cavity of angle $\theta_d = \omega_c d / u_0$.

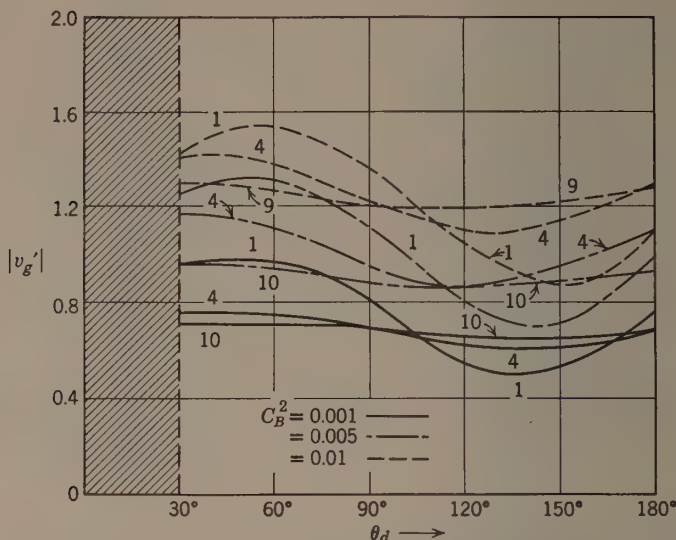


Fig. 9-19. Amplitude of growing wave excited beyond a detuned cavity of drift angle $\theta_d = \omega_d d / u_0$. Synchronism at $3\pi/4$. $-M/p = 0.10$. $4(QC)_B$ and C_B^2 are the parameters. b_B is chosen for optimum gain.

“cold” circuit. In the next section we find that the output cavity load which matches to the growing wave and creates no backward wave is rather critical and can differ considerably from the “cold” match if C_B is reasonably high.

We now discuss the cavity-chain tube of fractional passband width $-M/p = 0.10$, first for operation near the $3\pi/4$ synchronous point in Fig. 9-1. We first selected representative values of C_B and $4(QC)_B$ and chose a beam voltage for each pair of values at which both the circuit power flow in the growing wave and the rate of growth were substantial. We then assumed an incident growing wave of amplitude $v_g = 1$ in the cavity preceding the detuned one. For each pair of values of C_B and $4(QC)_B$ we then computed the magnitude $|v_g'|$ of growing wave generated in the cavity just beyond the detuned one, as a function of the drift angle $\theta_d = \omega_d d / u_0$ of the detuned cavity. These computations gave us the series of curves shown on Fig. 9-19 for $C_B^2 = 0.001, 0.005$, and 0.010 , and $4(QC)_B = 1, 4$, and 9 or 10 . We see that a healthy amount of growing wave is always established beyond the detuned cavity, and more, in some cases, than was incident. This is understandable from the fact that the amplitude of backward wave $|v_b|$ engendered in the incident cavity is greater than 1 in all these cases and increases as $|v_g'|$ increases, so that $|v_g + v_b| = |1 + v_b|$ remains roughly equal to $|v_g'|$. However, $|v_b| < 1.50$

for the parameters of Fig. 9-19 so the reflected backward wave does not become unduly large.

The curves show that, for low values of space charge, $\theta_d = 60^\circ$ is about the best drift angle. For the large space charge of $4(QC)_B = 10$ the drift angle is immaterial, indicating that both the kinetic voltage and current reexcite the growing wave.

Now we change the beam voltage and operating frequency so as to operate near the π synchronous point. Figure 9-20 shows the reexcitation of the growing wave for $C_B^2 = 0.0025$ and 0.010 and for various values of $4(QC)_B$. For each pair of values of C_B and $4(QC)_B$ we choose the beam voltage so as to compromise between a large rate of growth and a large amount of circuit power per unit amplitude v_θ . The growing wave amplitude $|v_g'|$ reexcited in the cavity beyond the detuned one is definitely larger than the total field $|1 + v_b|$ in the cavity preceding the detuned one for the two cases $C_B^2 = 0.0025$, $4(QC)_B = 25, 36$ and $C_B^2 = 0.010$, $4(QC)_B = 6.25, 9$. Although these growing waves are not as practical to utilize because their circuit power flows per unit amplitude v_θ are small and decrease with increasing $4(QC)_B$, they nevertheless deserve comment.

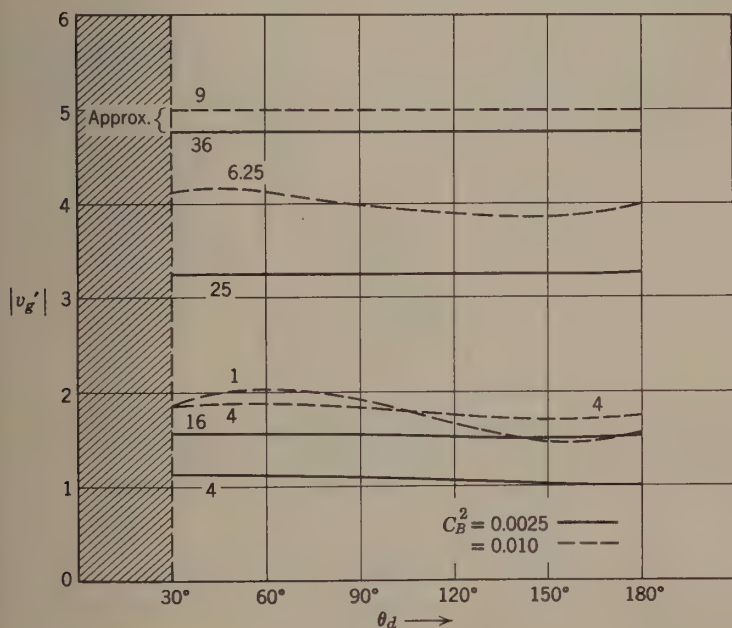


Fig. 9-20. Amplitude of growing wave excited beyond a detuned cavity of drift angle $\theta_d = \omega_d d / u_0$. Synchronism at π . $-M/p = 0.10$. $4(QC)_B$ and C_B^2 are the parameters. b_B is chosen for optimum gain.

In order to understand the large values of $|v_g'|$ in Fig. 9-20 we state a fact which is not obvious from the preceding descriptions of the waves, namely, that the backward wave possesses large components of kinetic voltage and current which, in fact, carry negative beam power. This backward wave, which exists when the tube is operated near the π synchronous point, is distinctly different from the ordinary helix traveling wave tube backward wave, which is essentially only a circuit excitation. The backward wave components of kinetic voltage and current add to those of the incident growing wave and produce large excitations $\hat{V}_k(1)$ and $\hat{J}(1)$ entering the detuned cavity of Fig. 9-18. In fact, the engendered backward wave increases $\hat{V}_k(1)$ and $\hat{J}(1)$ by about the same factor as $|v_g'|$ is greater than 1 for the upper curves of Fig. 9-20. The excitations $\hat{V}_k(1)$ and $\hat{J}(1)$ then transform into large components of $\hat{V}_k(2)$ and $\hat{J}(2)$ leaving the detuned cavity, and these reexcite the growing, decaying, and perturbed fast waves. The growing wave has the smallest components of beam excitation per unit amplitude v_g' , and the perturbed fast wave has the greatest amount of beam excitation. Therefore $|v_g'|$ tends to be larger than either $|v_a'|$ or $|v_f'|$ in the cavity beyond the detuned one. For these two reasons, then, $|v_g'|$ is as large as shown on Fig. 9-20 for the higher values of space charge.

The equations for the three cavities of Fig. 9-18 are now developed. The circuit equation for the left cavity is just Eq. 9-29a without the I_g term, with $v_{\text{tot}}^{(+2)}$ replaced by $v_g\epsilon^{\gamma_g} + v_b\epsilon^{\gamma_b}$, and with $v_{\text{tot}} = v_g + v_b$ and $\hat{J}_{\text{tot}} = \hat{J}_g + \hat{J}_b$. If we now add the coupling terms $(pM/2)(v_g\epsilon^{-\gamma_g} + v_b\epsilon^{-\gamma_b})$ to both sides of this equation we obtain, just as we obtained Eq. 9-29c,

$$v_g\epsilon^{-\gamma_g} + v_b\epsilon^{-\gamma_b} = 0, \quad v_g = 1 \quad \text{for reference} \quad (9-32)$$

With v_b now known the kinetic voltage $\hat{V}_k(1)$ and beam current density $\hat{J}(1)$ entering the detuned cavity are found by means of Eq. E-5:

$$\begin{aligned} \hat{V}_k(1) &= A_g\epsilon^{-\gamma_g} + A_bv_b\epsilon^{-\gamma_b} \\ Z_0\hat{J}(1) &= B_g\epsilon^{-\gamma_g} + B_bv_b\epsilon^{-\gamma_b} \end{aligned} \quad (9-33)$$

These excitations transform through the drift angle θ_a into

$$\begin{aligned} \hat{V}_k(2) &= \hat{V}_k(1) \cos \theta_a + jZ_0\hat{J}(1) \sin \theta_a \\ Z_0\hat{J}(2) &= j\hat{V}_k(1) \sin \theta_a + Z_0\hat{J}(1) \cos \theta_a \end{aligned} \quad \theta_a = \frac{\omega_q d}{u_0} \quad (9-34)$$

according to the transformation properties of Eq. E-2 with $e_z \equiv 0$. We have neglected a common phase shift of $\epsilon^{-j\omega d/u_0}$ in Eq. 9-34. Excitations $\hat{V}_k(2)$ and $\hat{J}(2)$ excite the cavity following the detuned one according to Eq. 9-32 with $-\gamma$ changed to $+\gamma$:

$$v_g'\epsilon^{\gamma_g} + v_a'\epsilon^{\gamma_a} + v_f'\epsilon^{\gamma_f} = 0 \quad (9-35a)$$

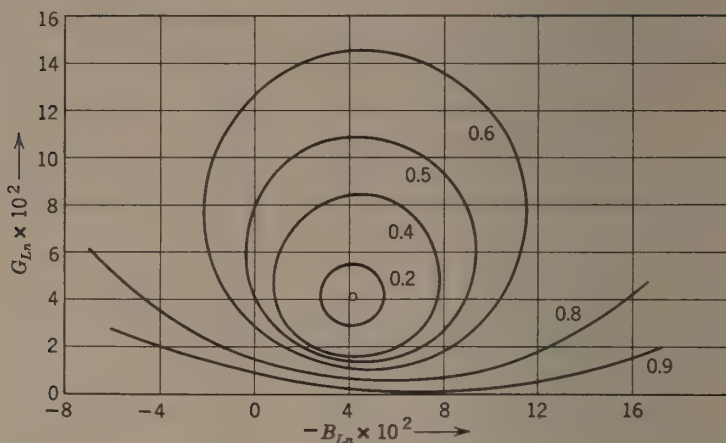


Fig. 9-22. Contours in the Y_{Ln} plane of constant amplitude of reflected backward wave. The incident growing wave has amplitude $v_o = 1$. Synchronism at $3\pi/4$. $-M/p = 0.10$, $C_B^2 = 0.001$, $4(QC)_B = 1$, and $b_B = 0$.

about maximum at the synchronous voltage. Assuming $v_o = 1$ in Fig. 9-21 we computed from the equations to follow the magnitude of the constant-amplitude backward wave $|v_b|$ generated by various values of the normalized load admittance defined in Eq. 9-40 as

$$Y_{Ln} = G_{Ln} + jB_{Ln} = \frac{\omega\mu Y_L}{\tau} \left(\int_{\text{loop}} \mathbf{h} \cdot d\mathbf{S}_\ell \right)^2$$

Here Y_L is the external load admittance Y_L' transformed to the plane S_o of Fig. 9-21. The loci of constant $|v_b|$ appear in Fig. 9-22. A “hot” match occurs for $Y_{Ln} \cong 0.04 - j0.04$, which is somewhat different from the “cold” circuit match. That match for these conditions is obtained from Eq. 9-41, with $\gamma = j\varphi^0$, as

$$\frac{M}{2p} e^{-j\varphi^0} - jY_{Ln}^0 = 0, \quad \varphi^0 = \frac{3\pi}{4}, \quad \frac{-M}{p} = 0.10$$

or $Y_{Ln}^0 = 0.035 - j0.035$ inductive. To estimate practical magnitudes of the unnormalized load Y_L , suppose $\omega \sim 3$ kMc, $\tau = 5 \times 10^{-5} m^3$, $\bar{h}_{\text{loop}} = 1$ (see the remark following Eq. 7-10), and the loop diameter = 1 cm. Then

$$\frac{\omega\mu}{\tau} \left(\int_{\text{loop}} \mathbf{h} \cdot d\mathbf{S}_\ell \right)^2 = 0.73$$

and the external matching load admittance on S_o is $0.055 - j0.055$. The load Y_L can be conveniently changed by altering the loop size.

The magnitude $|v_b|$ increases rather rapidly as the load departs from the

matched value in Fig. 9-22 and is likely to be 0.25 or more unless we match the "hot" circuit carefully. Similar loci of constant $|v_b|$ for the same $C_B^2 = 0.001$ but with $4(QC)_B = 10$ and a b_B for optimum gain are almost the same.

We then chose $C_B^2 = 0.010$, $4(QC)_B = 1$, and b_B for optimum gain. Now the loci appear as shown in Fig. 9-23 and the normalized matched load $Y_{Ln} = 0.042 - j0.058$. This differs considerably from the "cold" circuit match of $0.035 - j0.035$. Again the reflected $|v_b|$ is apt to be 0.25 or more unless we match carefully. The loci for the same C_B^2 but for $4(QC)_B = 9$ and a new value of b_B for optimum gain closely resemble those of Fig. 9-23.

Curiously, the matched load in the "hot" circuit which does not reflect the backward wave is *not* the load which maximizes the load power! This latter load is nearly $0.04 - j0.04$ on Fig. 9-23. The ready explanation is the fact that the presence of the backward wave can create more negative beam power leaving the last cavity than is attributed to the growing wave alone. Calculations for the data of Fig. 9-23 verify the fact that, with the nonmatched load of $0.04 - j0.04$, the amount of net power flow into the last cavity is nearly the same as if the growing wave alone were present, but the beam power leaving the last cavity is less and the load power consequently greater by about 10%.

This discrepancy between matched load and maximum power load is more apparent if we change the operating conditions so as to operate near the π synchronous point. Our first such example is that of $C_B^2 = 0.0025$,

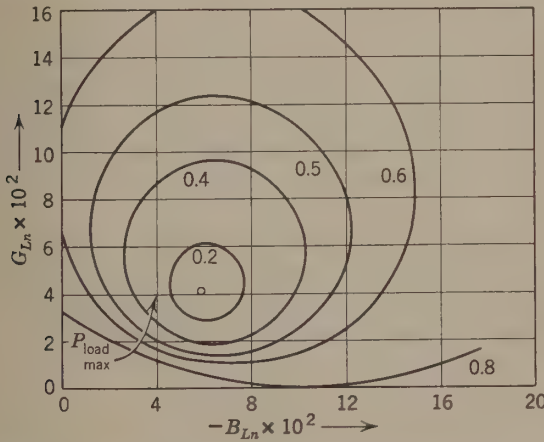


Fig. 9-23. Contours in the Y_{Ln} plane of constant amplitude of reflected backward wave. The incident growing wave has amplitude $v_g = 1$. Synchronism at $3\pi/4$. $-M/p = 0.10$, $C_B^2 = 0.010$, $4(QC)_B = 1$, and $b_B = 0.5$.

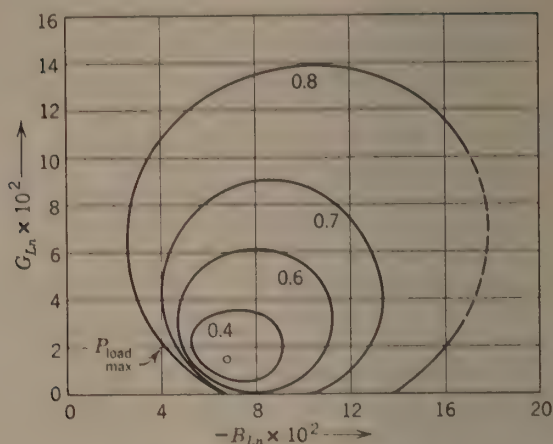


Fig. 9-24. Contours in the Y_{Ln} plane of constant amplitude of reflected backward wave. The incident growing wave has amplitude $v_o = 1$. Synchronism at π , $-M/p = 0.10$, $C_B^2 = 0.0025$, $4(QC)_B = 4$, and $b_B = 5$.

$4(QC)_B = 4$, and a value of $b_B = 5$ chosen on Fig. 9-10 for a growth rate of 0.33 neper per cavity as well as a healthy amount of power flow per unit amplitude v_o as specified by $\text{Im}(\epsilon^{v_o}) = -0.314$. The loci of constant $|v_b|$ appear in Fig. 9-24 and the matching load of $0.016 - j0.068$ is obviously more critical than for operation near the $3\pi/4$ point. The maximum power load is now nearly $0.02 - j0.04$. We have also examined the case of $4(QC)_B = 16$ for the same C_B^2 and a new optimum value of b_B but the loci very closely resemble those of Fig. 9-24.

Now let us maintain synchronism near the π cutoff but choose $C_B^2 = 0.010$, $4(QC)_B = 1$, and an optimum b_B such that X of the growing wave is 0.51 neper and $\text{Im}(\epsilon^{v_o}) = -0.435$. Figure 9-25 shows the loci of constant $|v_b|$. The matching Y_{Ln} is $0.022 - j0.081$, and this is apparently somewhat less critical than in the preceding case. Upon increasing $4(QC)_B$ to 4 and readjusting b_B for optimum gain the loci essentially did not change.

Figures 9-22 through 9-25 are descriptive of *loop* output coupling: the circuit equation for hole output coupling in which tangential electric field from the waveguide excites the last cavity on surface S_o of Fig. 9-21 would be somewhat different. We point out this difference in a moment.

It is almost if not just as important to match the input cavity of Fig. 9-15 to the reflected backward wave as it is to match the output cavity to the growing wave.¹⁰ By an input cavity match we do not mean a generator

¹⁰ Transversely magnetized ferrite posts may be inserted into the coupling apertures to attenuate the backward wave. The simple theory of their effect is discussed at the end of Section 11-3.

load which will absorb all the backward wave, but rather one which will prevent reexcitation of the growing wave. Usually the backward wave will carry appreciable beam excitation back to the first cavity so that growing, decaying, and perturbed fast waves will be reexcited in order to match the input conditions of no initial beam kinetic voltage and current. We now ask: Does an appropriate generator load exist such that only the decaying and perturbed fast beam waves will be reexcited? To answer this question we examined two representative cases: one for $-M/p = 0.10$, synchronism of beam and circuit at $3\pi/4$, $C_B^2 = 0.005$, $4(QC)_B = 4$, operating frequency at the synchronous frequency, and the other for operation at the π synchronous point such that $b_B = +6$, $C_B^2 = 0.0025$, and $4(QC)_B = 16$. In both cases a reasonable normalized load Y_{Ln} exists on surface S_i of Fig. 9-15 such that no growing wave is reexcited by a backward wave. It is $0.033 - j0.057$ for the first case and $0.011 - j0.069$ for the second case. Therefore we make the broad statement that

In general, a matched load exists for the first or input cavity of an amplifier tube such that a backward wave reflected into this cavity will regenerate no additional growing wave.

The circuit equation for the backward wave induced in the last cavity by the incident growing wave of amplitude $v_o = 1$ is developed as follows. With loop current I_L flowing out toward the load in Fig. 9-21 we have,

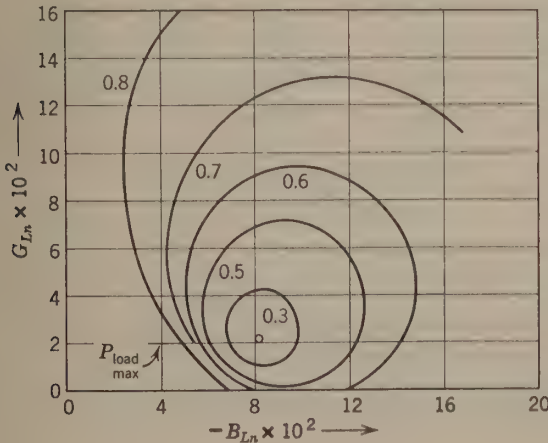


Fig. 9-25. Contours in the Y_{Ln} plane of constant amplitude of reflected backward wave. The incident growing wave has amplitude $v_o = 1$. Synchronism at π , $-M/p = 0.10$, $C_B^2 = 0.010$, $4(QC)_B = 1$, and $b_B = 3$.

from Eq. 9-29,

$$(p^2 - k^2 + pM)v_{\text{tot}}^{(N)} + \frac{pM}{2} v_{\text{tot}}^{(N-1)} + \frac{j\omega\mu}{\tau} \int_V \hat{\mathbf{j}}_{\text{tot}} \cdot \bar{\mathbf{e}}_z dV + \frac{j\omega\mu}{\tau} \oint_{\text{loop}} \hat{\mathbf{l}}_L \cdot \bar{\mathbf{e}} d\ell = 0 \quad (9-36)$$

with

$$v_{\text{tot}}^{(N)} = 1 + v_b, \quad v_{\text{tot}}^{(N-1)} = \epsilon^{\gamma_a} + v_b \epsilon^{\gamma_b}$$

If we add $(pM/2)(\epsilon^{-\gamma_a} + v_b \epsilon^{-\gamma_b})$ to both sides of this equation the left side, with the exception of the loop term, is identically zero for the reason following Eq. 9-29a. Therefore this equation is simply

$$-\frac{pM}{2} (\epsilon^{-\gamma_a} + v_b \epsilon^{-\gamma_b}) + \frac{j\omega\mu}{\tau} I_L \oint_{\text{loop}} \hat{\mathbf{l}}_L \cdot \bar{\mathbf{e}} d\ell = 0 \quad (9-37)$$

We can introduce the load admittance by writing Faraday's law for voltage V_L induced on surface S_o :¹¹

$$V_L = I_L Z_L = -j\omega\mu i \int_{\text{loop}} \bar{\mathbf{h}} \cdot d\bar{\mathbf{S}}_\ell = p v_{\text{tot}}^{(N)} \int_{\text{loop}} \bar{\mathbf{h}} \cdot d\bar{\mathbf{S}}_\ell \quad (9-38)$$

where $Z_L = Y_L^{-1}$ is the external load defined on S_o . Since

$$\int_{\text{loop}} \bar{\mathbf{e}} \cdot d\bar{\ell} \equiv \int_{\text{loop}} \nabla \times \bar{\mathbf{e}} \cdot d\bar{\mathbf{S}}_\ell = p \int_{\text{loop}} \bar{\mathbf{h}} \cdot d\bar{\mathbf{S}}_\ell$$

we can write the circuit Eq. 9-37 as

$$-\frac{pM}{2} (\epsilon^{-\gamma_a} + v_b \epsilon^{-\gamma_b}) + \frac{j\omega\mu}{\tau} p^2 \left(\int_{\text{loop}} \bar{\mathbf{h}} \cdot d\bar{\mathbf{S}}_\ell \right)^2 Y_L v_{\text{tot}}^{(N)} = 0 \quad (9-39)$$

If we define a normalized load Y_{Ln} on surface S_o as

$$Y_{Ln} \triangleq \frac{\omega\mu}{\tau} Y_L \left(\int_{\text{loop}} \bar{\mathbf{h}} \cdot d\bar{\mathbf{S}}_\ell \right)^2 \quad (9-40)$$

the expression for v_b from Eq. 9-39 is

$$v_b = \frac{(M/2p)\epsilon^{-\gamma_a} - jY_{Ln}}{-(M/2p)\epsilon^{-\gamma_b} + jY_{Ln}} \quad (9-41)$$

From this expression we computed the curves of Figs. 9-22 through 9-25.

The last term of Eq. 9-36 implies that the time-average power flow into the load is

$$\frac{1}{2} \text{Re} \oint_{\text{loop}} (v\bar{\mathbf{e}})^* \cdot I_L d\ell = \frac{1}{2} \text{Re} v^* I_L \int_{\text{loop}} \bar{\mathbf{e}} \cdot d\bar{\ell}$$

¹¹ There is no loop self-inductance term in Eq. 9-38 because $i\bar{\mathbf{h}}$ measures the *total* magnetic field threading the loop.

However, since

$$\int_{\text{loop}} \bar{\epsilon} \cdot d\ell = p \int_{\text{loop}} \bar{h} \cdot dS_\ell \quad \text{and} \quad pv^* = +j\omega\mu i^*$$

this power flow can also be written

$$\frac{1}{2} \text{Re} \left(j\omega\mu i^* \int_{\text{loop}} \bar{h} \cdot dS_\ell \right) I_L = \frac{1}{2} \text{Re} V_L^* I_L$$

which is the correct power as measured by the voltage and current in the coaxial tube just beyond surface S_o . Therefore our treatment implies continuity of power flow across this surface.

Suppose, now, the last cavity couples to the load by way of a rectangular or circular orifice instead of a loop through surface S_o of Fig. 9-21. If an electric field is maintained in the waveguide on S_o perpendicular to $\bar{h}$ in the cavity, the cavity will couple to the load. If we modify the derivation of Eqs. 4-34 and 4-35 for the cavity mode amplitudes so as to include the effect of this field as well as the beam, we obtain

$$pv_{\text{tot}}^{(N)} = -\frac{1}{\tau} \int_{S_o} \bar{n} \times \bar{E}_L \cdot \bar{h} dS - j\omega\mu i_{\text{tot}}^{(N)} \quad (9-42a)$$

$$pi_{\text{tot}}^{(N)} = -Mi_{\text{tot}}^{(N)} - \frac{M}{2} i_{\text{tot}}^{(N-1)} + \frac{1}{\tau} \int_V \bar{J}_{\text{tot}} \cdot \bar{\epsilon}_z dV + j\omega\epsilon v_{\text{tot}}^{(N)} \quad (9-42b)$$

If we denote the waveguide fields by $\bar{E}_L = V_L \bar{E}_0(S_o)$ and $\bar{H}_L = I_L \bar{H}_0(S_o)$ and naively ignore the cutoff waveguide modes excited by the tangential cavity magnetic field on S_o , we can say $V_L = I_L Z_L$ by definition of the load impedance of the propagating mode as measured on S_o , whereupon the equality of power flow leaving the cavity and entering the waveguide per unit of V_L means

$$i_{\text{tot}}^{(N)} \int_{S_o} \bar{E}_0 \times \bar{h} \cdot \bar{n} dS = I_L \int_{S_o} \bar{E}_0 \times \bar{H}_0 \cdot \bar{n} dS \quad (9-43)$$

Apparently, then, we need only express the V_L drive of Eq. 9-42a in terms of I_L and therefore i_{tot} in order to have a well-behaved pair of equations for the N th cavity mode amplitudes as influenced by load Z_L .

Unfortunately there is one inconsistency in this treatment which did not arise for loop coupling; namely, Eqs. 9-42 do not imply conservation of power flow across the coupling surface between cavities $(N-1)$ and (N) . The reason stems from the fact that $pv_{\text{tot}} = -j\omega\mu i_{\text{tot}}$ in cavity $(N-1)$ but not so in cavity (N) ; this means when we write the power flow out of cavity $(N-1)$ as proportional to $\text{Re } v^{(N-1)}(I^{(N-1)*} + I^{(N)*})$ (see Eq. 9-19) there is an extra component of power in the term $v^{(N-1)}I^{(N)*}$ due to the load term in Eq. 9-42a which is entirely different from the extra

components in the expression for power flowing into the N th cavity $\propto \text{Re } I^{(N)}(I^{(N-1)*} + I^{(N)*})$. The same statement is true if we measure power flow by the magnetic field on one side or the other and use the averaged electric field on the coupling surface.

The ambiguity in power flow is resolved if we cast the previous equations into equivalent circuit form, with the definitions in Section 4-7. The result is a circuit for the N th cavity of the form of either Fig. 4-4a or 4-4b depending on the symmetry of the mode patterns. The coupling to the $(+1)$ cavity should be removed and a current generator inserted in parallel with C_{nc} to represent the effect of the beam in Eq. 9-42b. The effect of the Z_L load is an appropriate impedance in parallel with L_{nc} . In this circuit there is obvious continuity of power flow across the coupling inductor from the $V_{nc}^{(-1)}$ terminal to the $V_{nc}^{(0)}$ terminal.

The interpretation of the equations with respect to power flow across the coupling surface between cavities is, however, somewhat unsatisfactory, and so we do not consider any further the cavity chain coupled to a load through a hole in the cavity wall.

The equations for the input cavity with loop coupling are developed as follows. Equations 9-36 through 9-40 are modified to apply to the first cavity by including the decaying and perturbed fast waves, changing γ to $-\gamma$, and by setting incident $v_b = 1$:

$$-\frac{pM}{2}(\epsilon^{-\gamma_b} + v_o\epsilon^{-\gamma_o} + v_d\epsilon^{\gamma_d} + v_f\epsilon^{-\gamma_f}) + jY_{L1}p^2(1 + v_o + v_d + v_f) = 0 \quad (9-44)$$

The beam equations are just Eqs. 9-30 and 9-31 but including the backward wave components as well:

$$-A_b = A_o v_o + A_d v_d + A_f v_f \quad (9-45)$$

$$-B_b = B_o v_o + B_d v_d + B_f v_f \quad (9-46)$$

By setting $v_o = 0$ in these last three equations we can readily find v_d and v_f in the latter two knowing coefficients A_b and B_b and then determine whether or not Eq. 9-44 determines a physically realizable Y_{L1} .

9-11. Effect on the Growing Wave of the Impedance Walls

So far in the analysis of the waves we have assumed the walls of the cavity to be perfectly conducting. We now treat the effect of the impedance walls on the phase shift per cavity, γ_o of the growing wave, by the following

means. The cavity field in the presence of perfectly conducting walls would establish a surface current $-\hat{n} \times i\mathbf{h}$ on them, $\hat{n}$ being the unit normal outward. If, now, the wall impedance Z_n is finite, a surface electric field $\hat{E}_n = -(\hat{n} \times i\mathbf{h})Z_n$ is established. This excites correction fields within the cavity which can also be represented by the one open- and short-circuit mode of this passband. Equation 9-42 give the expressions for v' and i' of the correction field¹² in terms, now, of $\hat{E}_n$ with the beam current term dropped. The phase shift γ_u for both the original and the primed fields in the growing wave is now perturbed from that for perfectly conducting walls. The beam kinetic voltage is driven by the total field $(v + v')e_z$ within the cavity. Since v' is related to v we can modify the determinantal equation for γ and find out how much the impedance walls change γ_u . In particular, we want to know if $\text{Re}(\Delta\gamma_u)$ is reduced or goes negative, thus implying augmented growth [$e^{-\gamma}$ = excitation in cavity $(n + 1)$ /excitation in cavity (n)].

Shown in Fig. 9-26 is the parameter $\text{Re}(Q_0\Delta\gamma_u)$ vs. the space charge parameter $4(QC)_H$ for various values of coupling coefficient C_H^2 ; $-M/p = 0.10$ in the structure of Fig. 9-1 operated near the $3\pi/4$ synchronism point. As $4(QC)_H$ increases, the beam velocity is readjusted slightly so as to keep the gain in the absence of the impedance in the walls optimum. The ordinary quality factor Q_0 of the unloaded "cold" cavity is

$$Q_0 = \frac{\omega W_s}{P_{\text{diss}}} = \omega \frac{1}{2} \frac{\epsilon |v|^2 \tau}{P_{\text{diss}}} \quad (9-47)$$

In the "cold" chain we can express the average power dissipated into the walls of the cavity of reference phase zero as

$$P_{\text{diss}} = P_{\text{in}} - P_{\text{out}} = -\frac{pM\tau |v|^2}{4\omega\mu} [\text{Im}(\epsilon^\gamma) + \text{Im}(\epsilon^{-\gamma})] \quad (9-48)$$

from Eq. 9-18. If the attenuation per cavity is small, and "cold" circuit phase shift φ^0 is not too near π , we can say that $\gamma = j\varphi^0 + \Delta\gamma \sim j\varphi^0 + \alpha$, α being the attenuation per cavity in nepers. Then Eq. 9-48 becomes

$$\begin{aligned} P_{\text{diss}} &= -\frac{pM\tau |v|^2}{4\omega\mu} [(1 + \alpha) \sin \varphi^0 - (1 - \alpha) \sin \varphi^0] \\ &= -\frac{pM\tau |v|^2}{2\omega\mu} (\alpha \sin \varphi^0) \end{aligned} \quad (9-49)$$

Since $\omega^2\mu\epsilon \cong p^2$ for this narrow passband, Eq. 9-47 tells us

$$Q_0\alpha = Q_0 \text{Re}(\Delta\gamma) = \frac{1}{(-M/p) \sin \varphi^0} \quad (9-50)$$

¹² This treatment is consistent with continuity of power flow on the coupling surfaces according to subsequent discussion.

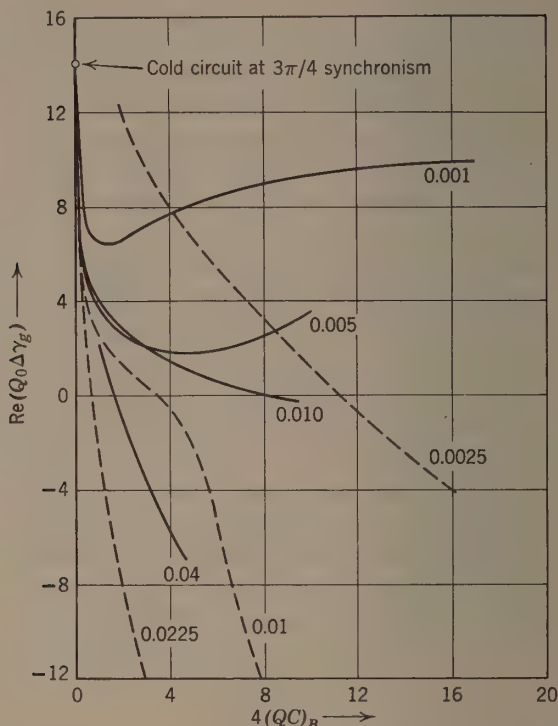


Fig. 9-26. Effect of the impedance walls on the growing wave. $-M/p = 0.10$. θ_s is chosen for optimum gain for each value of $4(QC)_B$, with C_B^2 as the parameter. Two conditions of synchronism are represented. Solid lines: Synchronism at $3\pi/4$. Dashed lines: Synchronism at π . Negative values of the ordinate imply enhancement of the rate of growth, $\text{Re } -\Delta\gamma_g$.

in the "cold" chain. For $-M/p = 0.10$ and $\varphi^0 = 3\pi/4$ we have the point 14.14 on Fig. 9-26.

Figure 9-26 shows us that, for weakly coupled beams, the "hot" circuit attenuation due to the impedance wall effect is diminished at the low space charge values but rises as the space charge increases. For sufficiently high values of C_B^2 , however, the attenuation will go negative (greater gain due to the impedance in the walls) for large values of space charge. This is the *resistive wall amplifier* effect, which is more pronounced for beam-circuit synchronism near the π cutoff point.

Figure 9-26 makes clear the fact that, for most beams of interest, the "cold" circuit attenuation per cavity is not descriptive of the effect of the impedance walls upon the growing wave. Indeed, the walls may actually enhance the gain.

We now outline the derivation of the expression for $Q_0\Delta\gamma$. From Eq. 9-42 the expressions for the open-circuit mode amplitudes v' and i' excited by $E_s = Z_s(-\bar{n} \times i\hbar)$ on the walls are

$$pv' = -\frac{1}{\tau} \int_{\text{walls}} (-Z_s \bar{n} \times i\hbar) \times \hbar \cdot \bar{n} dS - j\omega\mu i' \quad (9-51)$$

$$pi' = -MI'(1 + \cosh \gamma) + j\omega\epsilon v', \quad I' \cong i'$$

where γ is $\gamma_0 + \Delta\gamma$. Equation 9-51 gives us v' and i' both in terms of i . The beam equations remain the same except that $(v + v')$ is substituted for v , so we want to write the new circuit equation in terms of $v + v'$. Our original circuit equation 8-26 on page 258 was

$$[p^2 - k^2 + pM(1 + \cosh \gamma)]v = -\frac{j\omega\mu}{\tau} \int_V \mathcal{J} \cdot \bar{e}_z dV$$

From Eq. 9-51 we obtain v' in terms of v as

$$\frac{p^2 - k^2 + pM(1 + \cosh \gamma)}{p + M(1 + \cosh \gamma)} v' = \frac{pv}{j\omega\mu} Z_s \frac{1}{\tau} \int_{\text{walls}} |\bar{h}|^2 dS \quad (9-52)$$

where we used $pv = -j\omega\mu i$. Now observe that Q_0 of a single cavity is

$$Q_0 = \frac{\omega W_s}{P_{\text{diss}}} = \frac{\omega(\frac{1}{2})\mu |i|^2 \tau}{\frac{1}{2} \int_{\text{walls}} |i\hbar|^2 R_s dS} = \frac{\omega\mu\tau}{R_s \int_{\text{walls}} |\bar{h}|^2 dS} \quad (9-53)$$

Since $Z_s = R_s(1 + j)$ at microwave frequencies we can write Eq. 9-52 as

$$\frac{p^2 - k^2 + pM(1 + \cosh \gamma)}{p + M(1 + \cosh \gamma)} v' = \frac{pv(1 + j)}{jQ_0} \quad (9-54)$$

so our circuit equation can be manipulated into the form

$$\frac{p^2 - k^2 + pM(1 + \cosh \gamma)}{1 + \xi} (v + v') = -\frac{j\omega\mu}{\tau} \int_V \mathcal{J} \cdot \bar{e}_z dV \quad (9-55)$$

with

$$\xi = \frac{p[p + M(1 + \cosh \gamma)]}{p^2 - k^2 + pM(1 + \cosh \gamma)} \cdot \frac{1 - j}{Q_0}, \quad |\xi| \ll 1$$

Equation 9-55 together with the beam equations on page 276, with $v + v'$ replacing v , now furnish the same determinantal equation for γ as before except that $(1 + \xi)^{-1} \cong 1 - \xi$ multiplies the first bracket of Eq. E-9 in Appendix E. Then it is a simple matter to compute $\Delta\gamma_g$ caused by ξ . We made one more assumption to the effect that $|(M/p)(1 + \cosh \gamma_g)| \ll 1$ in the numerator of ξ ; with $-M/p = 0.10$ this is an excellent approximation for Fig. 9-26 because $|1 + \cosh \gamma_g| < 1$ for all those cases. This term should be ignored, however, according to the following consistency argument.

In order to have continuity of the first-order power flows proportional to $v'I$ and vI' on a coupling surface we need $i' \sim jv'$ (see Prob. 2). We can arrange this by expanding the primed fields with the short-circuit mode, in which case $PI' = j\omega\epsilon V'$ in Eq. 9-51b and the M term goes into Eq. 9-51a. But Eq. 9-55 again emerges, in terms of $v + V'$ provided we ignore the $M(1 + \cosh \gamma)$ term in ξ . It is also necessary to say the passband is narrow in the sense that $|M/p| \ll 1$. The conclusion is simply this: The single mode-pair correction to the fields due to wall loss is consistent and good enough to represent the quantitative effect of the wall impedance upon the growing wave.

Problems

1. Compare the gain per cavity of Fig. 9-7 with the rate of growth for low space charge in the helix traveling wave tube (Chapter 8 of Ref. [1]).
2. List the necessary and sufficient conditions for all the power flow expressions of Eq. 9-19 to be equivalent.
3. Verify the zero circuit power flow at the pips on Fig. 9-4a, for example.
4. Approximately what is the value of b_B for optimum gain and low space charge in Fig. 9-12?
5. Compare the growth rates with sinusoidal e_z field, Fig. 9-14, with the corresponding growth rates with uniform field, Figs. 9-3 through 9-6 and 9-10 through 9-12. Is there a definite advantage in using sinusoidal field with the same C_B value as for uniform field?
6. Show that the equivalent electric field remains finite as θ_e approaches π in Eq. 9-27.
7. Why does the growth rate go to zero at $b_B \cong 2.6$ for $C_B^2 = 0.005$, $4(QC)_B = 1$ in Fig. 9-16?
8. Compare the effect of a detuned cavity, Fig. 9-19, with the effect of a severed helix with drift space in the helix traveling wave tube (Chapter 9 of Ref. [1]).
9. Verify the matched load of Fig. 9-23 with respect to Eq. 9-41 and the data of Fig. 9-6.
10. From Eqs. 9-40 and 9-41 and related remarks show that the power flowing out of the last cavity into a matched load is $-\frac{1}{2}\omega W_s |M/p| \text{Im}(\epsilon^{-\gamma q})$, W_s being the time-average stored energy in the last cavity and M/p the fractional passband width.

References

1. J. R. Pierce, *Traveling Wave Tubes*, 1st ed., D. Van Nostrand and Co., Princeton, N.J., 1950.
2. R. M. Bevensee, "A Unified Theory of Electron Beam Interaction with Slow Wave Structures, with Application to Cutoff Conditions," *J. Electron. Control*, **9**, 401 (Dec. 1960).

Chapter X

Design Considerations for Cavity-Chain and Multicavity Klystron Beam Amplifiers

10-1. Synopsis¹

In the preceding chapter we studied many of the individual small-signal characteristics of the coupled-cavity beam amplifier without regard for the overall performance of the tube. In this chapter we study the small-signal power gain and input resistance of various amplifier models in a frequency range near the cavity resonant frequencies. We list the assumptions of the analysis in Section 10-2 and develop the equations for the input, intermediate, and output cavities so as to allow stagger-tuning and external loading of various kinds. The complete set of equations is presented as it was programmed for the computer, and expressions for the power flows and input impedance are developed.

The first portion of our program is restricted to a study of high-gain, low-bandwidth cavity-chain amplifiers in which the cavities are all stacked together with no drift spaces, field-free or otherwise, between them. The cavities are either decoupled or lightly coupled. The second phase of the program is devoted to the characteristics of various chains used as bunchers to feed r.f. beam excitation across a drift space to an optimized output cavity. These models have relatively low gains but large bandwidths. Lastly, we describe in Section 10-6 the properties of constituent cavities which may be used in multicavity klystrons. The computations

¹ The author gratefully acknowledges the help of Mr. Y. Satoda who launched the preliminary investigation in his M.S. thesis entitled "The Stagger-tuned Cavity-Chain Traveling Wave Tube Amplifier," Electrical Engineering Dept., University of California, Berkeley, 1961. Thanks go also to the Board of Regents of the University of California for the use of the IBM 704 and 7090 computers.

describe the gain and bandwidth characteristics of various physical tubes well enough to warrant the statement that the analysis is realistic. As a check on each computation there was a conservation-of-power sub-computation.

We first report, in Section 10-3, on the gain and bandwidth properties of various synchronously tuned amplifier models with weak beam-circuit coupling specified by C_B . The cavities are not externally loaded—a good assumption if their Q 's are several thousand or more—except for the last one, which is either matched to the growing wave at the $3\pi/4$ “cold” circuit phase shift frequency if the cavities are coupled or else loaded to a Q of 50 if they are decoupled. For representative values of θ_e and θ_o we vary the passband parameter m (which measures the electromagnetic cavity-cavity coupling) and the beam-circuit coupling coefficient C_B . These decoupled-cavity models are not very interesting because, despite their tremendous gains, they have virtually no bandwidths. What is worse, the coupled-cavity models exhibit widely fluctuating input resistance which usually goes negative in the frequency range of interest. External loading sufficient to keep the tubes stable at all frequencies will often allow gains of 30 db or so (ten-cavity length), but the bandwidths remain $\leq 1\%$. Nor does an increase of cavity-cavity coupling increase the bandwidths to any noticeable extent. Negative input resistance at frequencies near the π cutoff frequency explains the “cutoff” oscillations observed in many high-power tubes.

In Section 10-4 we answer the question of whether these high-gain power amplifiers, with high- Q intermediate cavities, can be broadbanded by stagger-tuning the cavities. Many stagger-tuning schemes were tried on a representative five- and ten-cavity tube model in which the cavities were either decoupled or else very lightly coupled, but they all had the effect of reducing the transducer gains tremendously without providing a significant increase in bandwidth. External loading of the intermediate cavities did not increase the bandwidths above 1% or so for reasonable stable gains of 30–40 db.

The reasons for the narrow bandwidths in these models are two: The last cavity, although of simple design, is not optimum with respect to cavity voltage generation per unit of entering r.f. beam current, and a drift space between the last cavity and its neighbor has not been inserted. Therefore the bandwidths of these cavity-chain models may be considerably increased by externally loading them and using them as *bunchers* to excite, across a drift space, an optimized output cavity. The loss of gain caused by the loading may be offset by an increase of d.c. beam admittance so as to increase C_B . In addition, the optimization of the output cavity allows an increase in gain on the order of $(\omega/\omega_q)^2$.

In Section 10-5 we present characteristics of various five-cavity buncher models designed with a low value of C_B to emphasize bandwidth rather than gain. For one typical choice of d.c. beam parameters, output cavity shunt resistance equal to the d.c. beam conductance, and Q of ten these various models have gains on the order of 20 db and bandwidths ranging from 1 to 10%. The bandwidth of each tube has been optimized with respect to gain by appropriate choice of generator resistance. For a model of given cavity-cavity coupling and θ_e , with $\theta_q/\theta_e = 0.1$, the bandwidth is strongly dependent on the stagger-tuning scheme employed. One of these models studied is a five-cavity "skirtron" tube, so-called because it realizes high gain on the skirts of the output power curve where the input power falls proportionately so as to keep the transducer gain high. This particular model has a bandwidth of more than 10% and a maximum transducer gain which can be 20 db or more. Computations indicate that neither stagger-tuning of the intermediate cavities nor coupling between the cavities has more than a negligible effect on its maximum gain and bandwidth.

In the last section, Section 10-6, we describe the effects of varying the parameters θ_e , θ_q , and C_B of an input, intermediate, and output cavity suitable for a multicavity klystron. Of interest is the kinetic voltage produced by the input cavity, the kinetic voltage produced by the intermediate cavity per unit of entering beam current, and the load power per unit of beam current entering the output cavity.

10-2. Outline of the Cavity-Chain Analysis²

A. Assumptions

We now outline the various steps in the analysis of the beam amplifier model shown schematically in Fig. 10-1. The beam enters the first cavity (1) initially unmodulated, and the generator feeds current I_g in through the first coupling loop. A large axial magnetic field constrains the beam to flow longitudinally through N cavities, each one with the same length L and loaded externally. No field-free drift spaces lie between successive cavities although the beam equations could easily be modified to include them. The r.f. magnetic field in each cavity induces a current to flow into the external load if one is coupled. The fields of each cavity are excited by the fields of its neighbors, the a.c. beam current in the cavity, and the

² Only the gist of this section is necessary for the ensuing discussion.

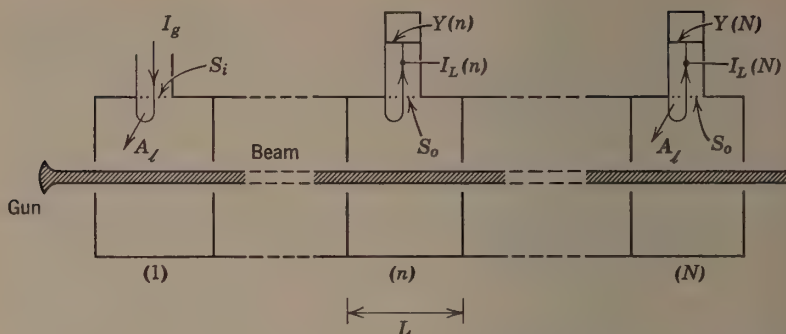


Fig. 10-1. Schematic diagram of the amplifier or buncher model. The quantity in parentheses denotes the cavity.

current in the loop. The beam is driven by the axial electric field in each cavity.

We have studied the small-signal a.c. power gain and input impedance as functions of frequency for various output loads and cavity parameters under the following assumptions.

1. There is only loop coupling from the cavities to coaxial lines, as shown in Fig. 10-1. The cavity dimensions are uniform unless otherwise stated.
2. We represent the solenoidal fields with the open- and short-circuit modes of the first passband alone, at frequencies presumed to be far from the resonances of all other cavity modes. We have only examined stagger-tuned chains in which the maximum open-circuit mode resonant frequency is within 15% of the minimum short-circuit mode resonant frequency.
3. The cavity length is presumed to be about three times the centerhole diameter so the axial electric field within the cavity is approximately uniform across the cavity. What is more important, the electric field pattern driving the beam is essentially the same whether the operating frequency is near the open- or the short-circuit mode resonant frequency. These two resonant frequencies would be the π cutoff and the zero cutoff frequencies, respectively, in an infinitely long and uniform chain. Figures 10-2 and 10-3 show the nearly uniform nature of the $\vec{e}_z$ pattern near the axis. The amplitude of the electric field is, of course, determined by the beam current and by the center-hole fields.
4. We neglect the fringing of the electric field near each centerhole and simply say the axial electric field of one cavity acts upon the beam right up to one centerhole, just beyond which the field of the next cavity "takes over" and drives the beam. This assumption was invoked in Chapter 9. These cavities are long compared to the centerhole diameter so this assumption is reasonable.
5. The preceding assumption does not mean the electromagnetic power flow is discontinuous on each hole. It is continuous according to the considerations

of Section 9-5 except now there is no explicit phase shift introduced from one cavity to the next.

6. To keep the analysis as realistic as possible we work with Maxwell's equations directly and do not introduce equivalent circuit elements except for the input and output admittances, both of which are well defined in the coaxial waveguides.
7. The longitudinal beam excitations are represented by the usual small-signal expressions for kinetic voltage and current (Eq. 8-17). We do not account for nonlinear effects even if the power gain is considerable.
8. We also assume a thin beam and neglect the radial variation of electric field pattern in any transverse cross section. Since the electric field varies about as $J_0(PR)$, with $PR = 2.40$ if R is the cavity radius, radial corrections are rather unimportant. We assume as before that the space charge reduction factor ω_q/ω_p is constant across each cavity and that the space charge electric field is proportional to the a.c. beam current by the factor $jZ_0\beta_q$, according to Eq. D-14 in Appendix D.

Granting that these assumptions are reasonable for a description of the amplifier we turn now to the formulation of the cavity equations.

B. Equations for the Fields and Beam Excitations in Each Cavity

We choose to expand the solenoidal portion of the fields in the n th cavity, $1 \leq n \leq N$, by the open-circuit mode. The primary reason for so doing is the simple form which the first amplitude equation takes, namely $p\dot{v} = -j\omega\mu i$. The secondary reason is the fact that the mode is excited by tangential magnetic field on the centerholes, whereas the beam and cavity fields excite only irrotational *electric* field given by the open-circuit modes. The discussion in Section 5-4 is relevant here. In order to derive the expressions for the open-circuit mode amplitudes $v(n)$ and $i(n)$, where the quantity in parentheses denotes the cavity under consideration, for a nonperiodic chain we proceed as follows. Equations 4-34 and 4-35 augmented for the presence of the beam give us

$$p(n)v(n) = -j\omega\mu i(n)$$

$$p(n)i(n) = \frac{1}{\tau(n)} \int_{\text{holes}} \vec{e}(n) \times \vec{H} \cdot \vec{n} dS + \frac{1}{\tau(n)} \int_V \vec{J} \cdot \vec{e}(n) dV + j\omega\epsilon v(n) \quad (10-1)$$

in which $\vec{J}$ can include a loop current as well as the beam current. Resonant frequency $p(n)$ may vary from cavity to cavity if the cavity radii are staggered, according to the footnote 3. We represent $\vec{H}_{\text{tan}}$ in the usual way as

$$\vec{H}_{\text{tan}}(M-) = \frac{1}{2}[I(n-1)\vec{H}(n-1) + I(n)\vec{H}(n)], \quad I(n) \cong i(n) \quad (10-2)$$

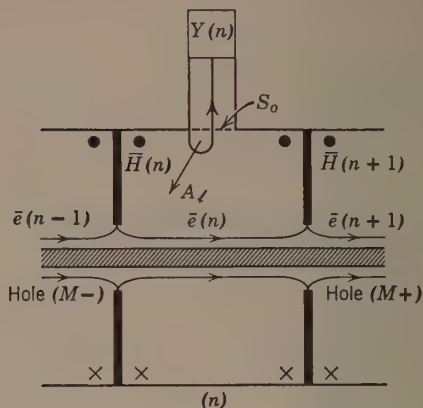


Fig. 10-2. Intermediate and output cavities. Magnetic field $I(n)\bar{H}(n)$ or $i(n)\bar{h}(n)$ threads the loop of area A_L and induces current to flow to the load $Y(n)$.

with a similar expression for $\hat{H}_{\tan}$ on the forward hole ($M+$). The directions of these patterns are shown in Fig. 10-2; $\bar{H} = \bar{H}_0$ always maintains the same direction, but we must account for the reversals of $\bar{e}$ across each centerhole.

We can proceed to eliminate $i(n)$ from Eqs. 10-1 and employ Eq. 10-2 for the hole excitation so as to write the general n th cavity equation for electric field amplitude $v(n)$ as

$$\begin{aligned} \left[p(n)^2 - k^2 + \frac{p(n)}{\tau(n)} \int_{M-} \bar{e}(n) \times \bar{H}(n) \cdot \bar{i}_z dS \right] v(n) \\ + \frac{p(n-1)v(n-1)}{2\tau(n)} \cdot \int_{M-} \bar{e}(n) \times \bar{H}(n-1) \cdot \bar{i}_z dS \\ - \frac{p(n+1)v(n+1)}{2\tau(n)} \int_{M+} \bar{e}(n) \times \bar{H}(n+1) \cdot \bar{i}_z dS \\ + \frac{j\omega\mu}{\tau(n)} \int_V \bar{J} \cdot \bar{e}(n) dV = 0 \end{aligned} \quad (10-3)$$

The integrals over the holes ($M\pm$) are left in this general form to allow for a possible change of dimensions from one cavity to the next, as discussed in footnote 3.

To account for the loop coupling to the external load in Fig. 10-2 we proceed as follows. Tangential electric field on coupling surface S_o does not excite the cavity mode because the former is asymmetric with respect to $\bar{h}(n) = \bar{h}_0(n)$. The loop excitation of the cavity is

$$\frac{j\omega\mu}{\tau(n)} I_L \oint_{\text{loop}} d\bar{\ell} \cdot \bar{e}(n) \quad (10-4)$$

since the loop current is essentially uniform along the loop. From the defining relation $\nabla \times \bar{e}(n) = p(n)\bar{h}(n)$ we see

$$\oint_{\text{loop}} \bar{e}(n) \cdot d\bar{\ell} = \int \nabla \times \bar{e}(n) \cdot d\bar{A}_\ell = p(n) \int \bar{h}(n) \cdot d\bar{A}_\ell \quad (10-5)$$

$$\cong p(n) |\bar{h}(n)|_\ell A_\ell$$

A_ℓ being the loop area inside surface S_o of Fig. 10-2. Subscript ℓ denotes a parameter measured at a loop. Therefore Eq. 10-4 can be written

$$\frac{j\omega\mu}{\tau(n)} I_L \oint d\bar{\ell} \cdot \bar{e}(n) = \frac{j\omega\mu}{\tau(n)} I_L p(n) |\bar{h}(n)|_\ell A_\ell \quad (10-6)$$

$I_L(n)$ can be related to the external load $Y_L(n)$ measured on surface S_o as $I_L(n) = V_L(n)Y_L(n)$. Lastly, V_L is given by Faraday's law for the loop:

$$V_L(n) = \oint \bar{e} \cdot d\bar{\ell} = v(n)p(n) |\bar{h}(n)|_\ell A_\ell \quad (10-7)$$

Equation 10-3 for cavity (n) will therefore add the following term to the left side:

$$+ \frac{j\omega\mu}{\tau(n)} Y_L(n) [p(n) |\bar{h}(n)|_\ell A_\ell]^2 v(n) \quad (10-8)$$

The last cavity would not have the $v(N+1)$ term.

Equation 10-3 for the very first cavity is modified further by dropping the $v(n-1)$ term (with $n=1$ now) and adding a generator excitation term to the left side, derived as follows. With reference to Fig. 10-3 the additional term now in Eq. 10-3 is the same term which accounts for the

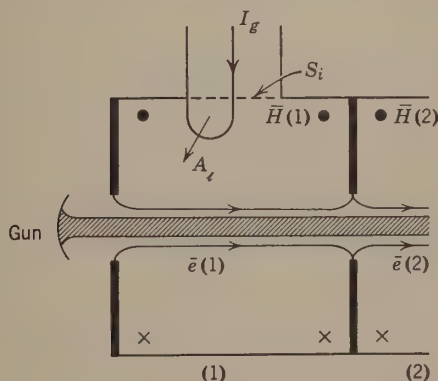


Fig. 10-3. Input cavity, with possible external loading not shown. Generator current I_g excites the first cavity by virtue of the magnetic field $I(1)\bar{H}(1)$ or $i(1)\bar{h}(1)$ threading the loop of area A_ℓ .

loop current in Eq. 9-29, that is,

$$+ \frac{j\omega\mu}{\tau(1)} I_g \oint_{\text{loop}} \bar{d}\ell \cdot \bar{e}(1) = + \frac{j\omega\mu}{\tau(1)} I_g p(1) |\bar{h}(1)|_\ell A_\ell \quad (10-9)$$

according to Eq. 10-5. This term is added to the left side of Eq. 10-3 to represent the generator excitation of cavity (1).

The solutions to the longitudinal beam equations were developed in Appendix E, Section E-1, for uniform axial $\bar{e}_z$ pattern in each cavity. In Eq. E-2 $\hat{V}_k(z=L)$ and $\hat{J}(L)$ are now the kinetic voltage and current density leaving a cavity in terms of electric field amplitude v within that cavity and $\hat{V}_k(0)$ and $\hat{J}(0)$ entering the cavity. These equations may be abbreviated into the form

$$\begin{aligned} \hat{V}_k(n+1) &= A_1 \hat{V}_k(n) + A_2 \hat{J}(n) + A_3 v(n) \\ \hat{J}(n+1) &= B_1 \hat{V}_k(n) + B_2 \hat{J}(n) + B_3 v(n) \end{aligned} \quad (10-10)$$

in which $\hat{V}_k(n)$ and $\hat{J}(n)$ enter the n th cavity, $\hat{V}_k(n+1)$ and $\hat{J}(n+1)$ enter the $(n+1)$ cavity, and the A_i and B_i coefficients are given by Eqs. E-2. The continuity of $\hat{V}_k$ and $\hat{J}$ across each coupling hole is explicit.

To evaluate the

$$\int \hat{J} \cdot \bar{e}_z dV$$

integral for each cavity equation we must integrate Eq. E-2b across each cavity. The integral for any n th cavity is obtained directly in terms of $\hat{V}_k(n)$ and $\hat{J}(n)$ entering that cavity and $v(n)$. For lack of space we do not quote the integrals.

C. The Normalized Set of Equations for the Stagger-Tuned Chain

We have now outlined the development of all the equations relevant to the analysis. These normalized parameters prove to be convenient:

$$\phi(n) \triangleq v(n)L$$

is the n th cavity voltage per unit of $\bar{e}_z$ field pattern

$$I_b(n) \triangleq Z_0 J(n)$$

is the beam current density in volts

$$\rho_n \triangleq \frac{p(n)}{P_r}$$

is the normalized open-circuit mode resonant frequency.

$$P_r \triangleq \frac{\omega_r}{c}$$

is any convenient reference frequency

$$\kappa \triangleq \frac{k}{P_r}$$

is the normalized operating frequency

$$\begin{aligned} m_{nn} &\triangleq \frac{1}{P_r \tau(n)} \int_{M-} \bar{e}(n) \times \bar{H}(n) \cdot \bar{i}_z dS \\ &= -\frac{1}{P_r \tau(n)} \int_{M+} \bar{e}(n) \times \bar{H}(n) \cdot \bar{i}_z dS < 0 \end{aligned} \quad (10-11)$$

is the normalized self-coupling coefficient for cavity (n),

$$m_{n,n+1} \triangleq +\frac{1}{P_r \tau(n)} \int_{M+} \bar{e}(n+1) \times \bar{H}(n) \cdot \bar{i}_z dS < 0$$

is a normalized cross-coupling coefficient, and

$$m_{n+1,n} \triangleq -\frac{1}{P_r \tau(n)} \int_{M+} \bar{e}(n) \times \bar{H}(n+1) \cdot \bar{i}_z dS < 0$$

is another cross-coupling coefficient.

If the dimensions are uniform from cavity to cavity, all the m_{ij} are equal.

$$D \triangleq \frac{p(1) |h(1)|_\ell A_\ell}{P_r^2 \tau(1)}$$

is the loop parameter for the generator coupling in the first cavity. A_ℓ is the loop area, and pattern $\bar{h}(1)_\ell$ threads the loop

$$I_G \triangleq \omega \mu I_g L$$

is the normalized input-loop current in volts

$$Y_{Ln} = \frac{\omega \mu}{\tau(n)} \left[\frac{p(n) |\bar{h}(n)|_\ell A_\ell}{P_r} \right]^2 Y_L(n)$$

is the normalized load admittance in cavity (n), just that of Eq. 9-40 for a nonstaggered chain.

In terms of these variables the complete set of equations for the beam tube is given in Table 10-1. There are $3N-2$ equations in terms of the variables $\phi(n)$, $V_k(n)$, and $I_b(n)$, excluding $V_k(1) = I_b(1) = 0$.

D. Tuning, Power Flow, and Input Impedance

In Table 10-1 only the parameter α_{nn} for the n th cavity contains the normalized resonant frequency ρ_n and the normalized load admittance Y_{Ln} of that cavity. The effective normalized resonant frequency is clearly

$$(\rho_n^2)_{\text{eff}} = \rho_n^2 - B_{Ln} \quad (10-12)$$

Table 10-1. Equations for the Stagger-Tuned Beam Amplifier

$$\begin{aligned} \alpha_{11}\phi(1) + \frac{1}{2}\rho_2 m_{21}\phi(2) - jDI_G &= 0 \\ \alpha_{nn}\phi(n) + \frac{1}{2}\rho_{n-1}m_{n-1,n}\phi(n-1) + \frac{1}{2}\rho_{n+1}m_{n+1,n}\phi(n+1) + f_n V_k(n) + g_n I_b(n) &= 0 \\ aV_k(n) + cI_b(n) + b\phi(n) - V_k(n+1) &= 0 \\ cV_k(n) + aI_b(n) + d\phi(n) - I_b(n+1) &= 0 \\ \alpha_{NN}\phi(N) + \frac{1}{2}\rho_{N-1}m_{N-1,N}\phi(N-1) + f_N V_k(N) + g_N I_b(N) &= 0 \end{aligned}$$

in which these abbreviations have been introduced:

$$\begin{aligned} \alpha_{nn} &= \rho_n^2 - \kappa^2 + \rho_n m_{nn} + q_n + jY_{Ln} \\ q_n &= -K_n |\bar{e}_z(n)| \left\{ jS_e \left[1 - \epsilon^{-j\theta_e} \left(\cos \theta_q + j \frac{\theta_e}{\theta_q} \sin \theta_q \right) \right] \right. \\ &\quad \left. + S_q \left[\epsilon^{-j\theta_e} \left(\sin \theta_q - j \frac{\theta_e}{\theta_q} \cos \theta_q \right) + j \frac{\theta_e}{\theta_q} \right] + 1 \right\} \\ f_n &= K_n \left[\left(\cos \theta_q + j \frac{\theta_e}{\theta_q} \sin \theta_q \right) \epsilon^{-j\theta_e} - 1 \right] \\ g_n &= jK_n \left[\left(\sin \theta_q - j \frac{\theta_e}{\theta_q} \cos \theta_q \right) \epsilon^{-j\theta_e} + j \frac{\theta_e}{\theta_q} \right] \\ a &= \cos \theta_q \epsilon^{-j\theta_e} \\ b &= |\bar{e}_z(n)| (-jS_e + jS_e \cos \theta_q \epsilon^{-j\theta_e} - S_q \sin \theta_q \epsilon^{-j\theta_e}) \\ c &= j \sin \theta_q \epsilon^{-j\theta_e} \\ d &= |\bar{e}_z(n)| (-S_e \sin \theta_q \epsilon^{-j\theta_e} + jS_q \cos \theta_q \epsilon^{-j\theta_e} - jS_q) \\ S_e &= \frac{\theta_e}{\theta_e^2 - \theta_q^2} \\ S_q &= \frac{\theta_q}{\theta_e^2 - \theta_q^2} \end{aligned}$$

(θ_q^2/θ_e^2 is not neglected with respect to 1)

$$K_n = - \frac{S_q \theta_e^2}{|\bar{e}_z(n)|} C_B^2(n)$$

C_B given by Eq. 9-3 relative to reference frequency P_r

Thus an external capacitive admittance will lower the effective resonant frequency.³

In order to check the computed excitations for each cavity-chain model we also computed the output beam power for a comparison with the generator input power minus all the output load power. With reference

³ We could also stagger-tune the cavities by inserting metal collars in order to stagger their radii. Since the short-circuit mode resonates in the n th cavity at frequency $P(n)$ such that $J_0[P(n)R_n] = 0$, R_n being the radius, we have $P(n+1)/P(n) = R_n/R_{n+1}$. According to Eq. 3-17 the short-circuit mode patterns $\bar{E}(n)$ and $\bar{H}(n)$ are such that $\bar{E}(n)$ is nearly uniform with radius near the axis while $\bar{H}(n) = \bar{H}_0(n)$ is nearly linear in r . Since the open-circuit mode pattern $\bar{e}(n)$ is a perturbation of $\bar{E}(n)$ near the holes we might assume $|\bar{e}(n)| = |\bar{e}(n+1)| = |\bar{e}(n+2)| = \dots$ at the same transverse point on the various holes. By the definition of the m_{ij} in Eq. 10-11 and the fact that volume $\tau(n)$ is proportional to R_n^3 we find that

$$m_{n+1,n+1} = \left(\frac{R_n}{R_{n+1}}\right)^3 m_{nn}, \quad m_{n,n+1} = \left(\frac{R_n}{R_{n+1}}\right)^2 m_{nn}$$

$$m_{n+1,n} = \frac{R_n}{R_{n+1}} m_{nn}$$

All the m_{ij} may be computed from any one of the m_{nn} , knowing the various R_n .

To relate the normalized open-circuit mode frequencies ρ_n to the R_n we can substitute $\bar{A} = \bar{e}(n)$ and $\bar{B} = \bar{E}(n)$ into Green's vector theorem, Eq. 3-43b, and use the definitions of the mode patterns in Eqs. 5-5 and 5-10 to obtain

$$\frac{P(n)}{P_r} - \rho_n = m_{nn}$$

valid for small centerholes, where $P(n)$ is the short-circuit mode resonant frequency for the n th cavity.

However, when we determine the $m_{n,n+1}$ and $m_{n+1,n}$ in this way we find they are not quite right to guarantee conservation of electromagnetic power flow from cavity (n) to ($n+1$). According to Eq. 9-19a the flow out of cavity (n) is

$$P_{av} = \frac{1}{2} \operatorname{Re} \int_{M+} v(n) \bar{e}(n) \times \frac{1}{2} [I(n)^* \bar{H}(n) + I(n+1)^* \bar{H}(n+1)] \cdot \bar{i}_z dS$$

The expression for the flow into cavity ($n+1$) is analogous. When we introduce the relation $p(n)v(n) = -j\omega\mu i(n)$, with $i(n) \cong I(n)$ and similar relations for cavity ($n+1$), we find the two power expressions are only equal provided

$$\frac{\rho_{n+1}}{\rho_n} \left(\frac{R_n}{R_{n+1}}\right)^2 m_{n+1,n} = m_{n,n+1}$$

This last relation is slightly incompatible with the previous expressions for the m_{ij} .

Therefore it would be necessary to readjust each $m_{n+1,n}$ and $m_{n,n+1}$ slightly in order to guarantee conservation of power flow in a chain of cavities with staggered radii. The single mode-pair description of the chain is not perfect for there is no unique way that these m_{ij} need be changed. The ambiguity is greater for greater disparity in resonant frequencies between adjacent cavities. However, computations indicate that if all the open-circuit mode resonant frequencies lie within a range of 10% the tube model behaves qualitatively just like a tube with uniform dimensions but external reactive loading to achieve the same resonant frequencies.

to Fig. 10-3 and Eq. 10-5 the input power is

$$P_{\text{in}} = \frac{1}{2} \text{Re } I_g^* V_g = \frac{1}{2} p(1) |\bar{h}(1)|_{\ell} A_{\ell} \text{Re } I_g^* v(1) \quad (10-13)$$

The load power in cavity (n) is

$$P_L(n) = \frac{1}{2} \text{Re } Y_L(n) |V_L(n)|^2 = \frac{1}{2} |v(n)|^2 (p |\bar{h}|_{\ell} A_{\ell})^2 G_L(n) \quad (10-14)$$

and the beam power leaving the last cavity (N) is

$$P_{\text{beam}} = \frac{1}{2} \text{Re } A_b \hat{V}_k(N+1) \hat{I}_b(N+1)^*, \quad A_b = \text{beam area} \quad (10-15)$$

Each of these powers can be written in terms of normalized parameters by multiplying them by the factor $\omega \mu L^2 / [P_r^2 \tau(1)]$. With the definitions of the various parameters in Eq. 10-11 the power conservation relation can be written very generally in terms of normalized voltage ϕ , loads Y_{Ln} , and the final beam excitations as

$$DI_G \text{Re } \phi(1) = \sum_n |\phi(n)|^2 \frac{\tau(n)}{\tau(1)} G_{Ln} - \frac{K_n}{e_z(n) S_q \tau(1)} \text{Re } [\hat{V}_k(N+1) \hat{I}_b(N+1)^*] \quad (10-16)$$

The error in this check expression was always negligible.

The input impedance is given by (see Eq. 10-7)

$$Z_{\text{in}} = \frac{V_g}{I_g} = \frac{\oint_{\text{loop}} \hat{e}(1) \cdot d\vec{\ell}}{I_g} = \phi(1) \frac{p(1) |\bar{h}(1)|_{\ell} A_{\ell} \omega \mu}{I_G} \quad (10-17)$$

To know the constant of proportionality between Z_{in} and $\phi(1)$ we must know both loop and cavity parameters. A particular tube of interest had the parameters $\omega = 2\pi(3 \text{ kMc})$, $|\bar{h}|_{\text{loop}} \cong 1$, loop diameter = 1 cm, cavity volume $\tau = 10^{-4} \text{ m}^3$, and cavity length $L = 1.8 \text{ cm}$. We choose I_G numerically equal to $p(1) |\bar{h}(1)|_{\ell} A_{\ell} \omega \mu$ so that $Z_{\text{in}} \equiv \phi(1)$, whereupon this tube had the loop excitation parameter $DI_G = 0.48$.

All the computed results in this chapter correspond to the excitation $DI_G = 0.48$.

10-3. Decoupled and Coupled Synchronously Tuned Tube Models with High Gain and Narrow Bandwidth

In this section we study the frequency response characteristics of various five- and ten-cavity tube models as a function of the passband parameter $m = M/p$ (which would be the fractional passband width in an

infinite chain of cavities) and beam-circuit coupling coefficient C_B . There are two basic reasons for the small bandwidths of these models. First, the value of C_B is not greater than 0.10, whereas in conventional high-gain broadband amplifiers it would be 0.30 or more. The higher C_B is, the more external loading we can apply for reasonable gain of 30 db or more and the broader the bandwidth of the beam kinetic voltage and current at the output cavity. Second, the output cavity, although easy to design and build, is not optimum with respect to load voltage generated per unit of entering beam current,⁴ which means the model cannot be externally loaded as heavily for a given gain with a consequent reduction of bandwidth. The absence of a drift space between the last cavity and its neighbor also causes a reduction in gain. The more interesting tube models which provide for greater gain and hence wider bandwidth for a given gain are discussed in Section 10-5.

We selected, for our study of these small-bandwidth tubes, the typical parameters $\theta_e = 2.24$ and $\theta_q = 0.235$ and initially regarded the intermediate cavities as lossless. The effects of loading the intermediates to various degrees are summarized in Section 10-3C. We chose these values of θ_e and θ_q so that the beam would be nearly synchronous with the "cold" circuit at the $3\pi/4$ phase shift frequency of a tube with coupled cavities, near which most tubes are operated for gain. For the representative value of $C_B = 0.10$ the $4(QC)_B$ parameter is 1 and the synchronism parameter b_B is 0.5 in Fig. 9-6, such that the growing wave is about optimum with respect to both the rate of growth and the power flow per unit amplitude. For smaller values of C_B and an $m = -0.10$ the value of b_B increases (by definition in Eq. 9-8) but the growing wave remains about optimum according to Figs. 9-3, 9-4a, and 9-5a.

We now proceed to discuss the following observations made from the computer analysis. The cavity chain with no coupling between cavities and the last one loaded to a Q of 50 exhibits positive input resistance varying from a few ohms to hundreds or even thousands of ohms throughout the passband and must be operated for high gain only in a very narrow frequency range. But the amplifier with finite cavity-cavity coupling and a matched load (matched always with the beam present) at the $3\pi/4$ phase shift frequency has vastly different characteristics. The coupled tube exhibits widely fluctuating input resistance which usually goes negative in

⁴ An optimized output cavity would have a Q of about 50 or smaller for large bandwidth and an effective shunt resistance such that an entering r.f. beam current of I_0 amperes would generate V_0 volts across the gap. This is because the r.f. beam current is comparable to the d.c. beam current in the large-signal regime (which we do not treat in this book) and the limit on r.f. voltage generated is the d.c. voltage. But in a conventional output cavity of low C_B , such as used in these models, the gap voltage is smaller by about a factor ω_0/ω .

one or more frequency ranges in the passband. It must be operated as an amplifier with either sufficiently large generator resistance or external loading in order to prevent the occurrence of oscillations at an unwanted frequency. The large negative resistances which develop at frequencies near π cutoff in all these coupled tubes explain the experimental "cutoff" oscillations.

Neither an increase of electromagnetic coupling nor external loading sufficient to keep the input resistance positive will raise the bandwidth up to 1% or so, along with a reasonable gain of 30 db. Analysis indicates that the gain-bandwidth product can be arbitrarily high, provided the bandwidth is vanishingly small. The effect of adding external loading to the intermediate cavities is a rapid decrease in gain accompanied by a slow increase in bandwidth, such that the gain-bandwidth product diminishes rapidly. As we mentioned several paragraphs back, the low bandwidth is attributed to the nonoptimization of the output section, so that 30 db cannot be obtained with external loading sufficient for bandwidths of several percent or more.

A. Gain and Input Resistance Characteristics

We first present the frequency response characteristics of two *decoupled* ten-cavity amplifier models which are constructed like the one shown in Fig. 10-1. Only the output cavity is loaded; the others are assumed to be lossless. This approximation is a good one for cavities with internal Q 's of several thousand or more. The decoupling is effected by cutting slots or holes between cavities on the outer portions of the partitions so as to cancel the centerhole coupling. Experimentally, the zero and π phase shift frequencies of a finite number of cavities properly terminated with shorting planes will then coincide. These two tube models have C_B values of 0.028 and 0.10 and a normalized output load of $Y_{LN} = 0.02 + j0$, for a loaded Q_L of 50. This value of Q_L gives about the optimum frequency response, which, however, is not very critical with the precise load. The fact that $Q_L = (k/P_r)^2/G_{LN} \cong 1/G_{LN}$ may be verified by substituting Eq. 10-14 for load power P_L into the expression for Q_L ,

$$Q_L = \frac{\omega W_s}{P_L} = \omega \frac{1}{2} \frac{\epsilon |v(N)|^2 \tau(N)}{P_L} \quad (\text{in the absence of the beam}) \quad (10-18)$$

and by invoking the definition of Y_{LN} from the last of Eq. 10-11. The term Q_L , as represented by external load G_{LN} , may be considered to be the total loaded Q in the absence of the beam provided G_{LN} includes the

equivalent loss conductance of the cavity walls. Thus Q_L obeys the usual relation $Q_L^{-1} = Q_0^{-1} + Q_{\text{ex}}^{-1}$, where Q_0 is internal and Q_{ex} is external, due to the load.⁵

The gain $\mathfrak{G}$ is defined in the usual way as

$$\mathfrak{G} = \frac{\text{load power}}{\text{power entering first cavity}}, \quad \begin{array}{l} \text{generator current} \\ \text{always constant} \end{array} \quad (10-19)$$

The more meaningful gain, however, is the transducer gain, which takes into account a possible mismatch between generator resistance and input resistance of the amplifier.

The transducer gain (of a two-port linear transducer) is defined by the Institute of Radio Engineers at a specified frequency as the ratio of the actual signal power transferred from the output port of the transducer to its load to the *available* signal power from the source driving the transducer [1]. Each of the gains $\mathfrak{G}$ has been computed on the basis of the same input current I_g . Suppose the generator is adjusted to pass I_g amperes into the input cavity so the load power is precisely that which we have computed. Then if we calculate the maximum available generator power for this value of generator voltage, we find the transducer gain $\mathfrak{G}_T$ is related to the computed $\mathfrak{G}$ as

$$\mathfrak{G}_T = |\mathfrak{G}| \frac{4R_g |R_{\text{in}}|}{(R_g + R_{\text{in}})^2}, \quad (10-20)$$

provided only that $R_g + R_{\text{in}} > 0$. Experimentally, the transducer gain is the measured output power divided by the reference power $V_g^2/8R_g$, V_g being the generator voltage. The gain $\mathfrak{G}_T$ is infinite for $R_g + R_{\text{in}} = 0$ because the generator supplies infinite current to the first cavity, whereupon the load power is infinite.

We see from Table 10-2, which summarizes the frequency responses of these tube models, that $\mathfrak{G}$ is very high at a frequency near the π cutoff frequency ($\kappa = 1$) where the input impedance Z_{in} is on the order of hundreds of ohms. The bandwidths, of course, are infinitesimal, nor will they improve very much when the various cavities are stagger-tuned (Section 10-4) so as to maintain a gain of about 30 db or more. It is perhaps surprising that the gains are so high with the cavities merely stacked together without optimized drift spaces between them.

Now consider how the responses of these two amplifiers change when

⁵ To include the effect of cavity wall loss we would include in Y_{LN} a term $(1+j)/Q_0$ according to Eqs. 9-55 and 9-36. Then the total normalized cavity conductance $G_{LN} = G_{LN\text{ex}} + 1/Q_0$, which implies the usual relation among the Q 's.

Table 10-2. Response of the Synchronously Tuned Uncoupled Ten-Cavity Amplifier Model. $m = 0$, $\theta_e = 2.24$ and $\theta_a = 0.235$, $Y_{Ln} = 0$ ($n = 1, \dots, 9$), $Y_{L,10} = 0.02 + j0$ ($Q_{10} = 50$). κ is the Normalized Operating Frequency.*

$C_B = 0.028$			$C_B = 0.10$		
κ	$\mathcal{G}$ (gain)	Z_{1n} (ohm)†	κ	$\mathcal{G}$ (gain)	Z_{1n} (ohm)
0.9985	2.5×10^2	$7 + j163$	0.9975	1.7×10^8	$36 + j100$
0.9990	1.1×10^3	$16 + j247$	0.9980	1.1×10^9	$57 + j121$
0.9993	5.7×10^3	$34 + j356$	0.9985	1.1×10^{10}	$99 + j147$
0.9995	3.6×10^4	$68 + j503$	0.9990	1.3×10^{11}	$188 + j155$
0.9997	1.0×10^6	$199 + j846$	0.9995	2.0×10^{11}	$308 + j51$
1.0000	2.1×10^7	$3050 - j1507$	1.0000	2.2×10^8	$254 - j126$

* $\kappa = 1$ corresponds to the resonant frequency of each cavity in this chain of zero passband width.

† $Z_{1n} \equiv \phi(1)$, the first cavity voltage generated, according to remarks following Eq. 10-17.

we introduce very slight electromagnetic coupling as specified by $m = -0.01$. Each model is matched to the growing wave at the $3\pi/4$ "cold" circuit phase shift frequency in order that the amplitude of the backward wave be zero by Eq. 9-41. Table 10-3 shows, for $C_B = 0.028$, large negative input resistance of about -300 ohms developing at $\kappa = 0.9991$ while, for $C_B = 0.10$ (no table) negative input resistance of about -20 ohms exists over the entire frequency range surveyed. These tubes would

Table 10-3. Response of the Synchronously Tuned Ten-Cavity Model. $m = -0.01$, $\theta_e = 2.24$, $\theta_a = 0.235$, $C_B = 0.028$, $Y_{Ln} = 0$ ($i = 1, \dots, 9$)*

κ	$\mathcal{G}^\dagger$	Z_{1n}
0.999 06	889	$540 + j187$
0.999 08	1611	$1204 + j58$
0.999 09	2801	$1661 - j954$
0.999 11	-5166	$-252 - j1039$
0.999 12	-2083	$-282 - j683$
0.999 16	-587	$-158 - j289$
0.999 20	-329	$-99 - j196$

* The output cavity is matched to the growing wave at that frequency for which the "cold" circuit phase shift would be $3\pi/4$ in the infinite chain.

† Negative gain means negative input power, $R_{in} < 0$.

be stable high-gain amplifiers only if the generator resistance were slightly greater than the magnitude of the most negative input resistance and would have very small bandwidths. Clearly the effect of this slight cavity-cavity coupling is to alter completely the input impedance characteristics to the extent that each coupled tube could be operated either as an amplifier or as an oscillator. Comparison of the minimum values of R_{in} for these two values of C_B indicates the effect of an increase in beam-circuit coupling is a *decrease* in the maximum magnitude of negative input resistance; however, an increase of coupling often increases the tendency to oscillate.

Table 10-4. Response of the Ten-Cavity Model. $m = -0.04$, $\theta_e = 2.24$ and $\theta_q = 0.235$, $Y_{ln} = 0$ ($n = 1, \dots, 9$)*

κ	$C_B = 0.0707$		$C_B = 0.10$	
	$\mathcal{G}$	Z_{in}	$\mathcal{G}$	Z_{in}
0.983	24	$3.6 - j9.3$	-152	$-2.7 - j3.3$
0.986	-114	$-12.5 + j30.3$	-58	$-21.6 - j15.1$
0.988	-726	$-3.7 - j55.6$	-240	$-5.4 - j20.6$
0.990	205	$3.0 - j19.6$	-1,869	$-0.8 - j12.7$
0.993	334	$15.3 + j0.3$	-5,014	$-10.0 + j16.7$
0.995	448	$9.0 - j6.4$	-14,300	$-0.5 - j7.7$
0.996	-1374	$-4.8 + j7.3$	-473	$-8.0 - j7.4$
0.998	-59	$-7.3 - j18.0$	-105	$-6.6 - j13.7$
1.000	-18	$-5.4 - j13.2$	-28	$-4.3 - j12.6$

* The output cavity is matched at the $3\pi/4$ phase shift frequency.

Now let us discuss several models with somewhat larger cavity-cavity coupling specified by $m = -0.04$. If $C_B = 0.028$, the tube is uninteresting: The gain is only 20 or so. For $C_B = 0.0707$ we have the curious characteristics listed in Table 10-4. Whereas the decoupled-cavity tube (response not tabulated) should be operated at a frequency very close to cutoff with a matched generator resistance of about 100 ohms, the coupled-cavity tube must be operated for large gain near $\kappa = 0.986$ with a generator resistance of about 15 ohms to prevent oscillation at all frequencies. If R_g were chosen equal to 15 ohms this model could have a bandwidth of about 1% in the range around $\kappa = 0.99$, but with gain peaks near the two extreme frequencies. If we now increase C_B to 0.10 the generator resistance must change to about 22 ohms in order that the tube be operated at the same frequency as before without oscillations at other frequencies.

If we increase the cavity-cavity coupling still further to $m = -0.07$, and rematch the model at the $3\pi/4$ frequency, we obtain the response of Table 10-5 with $C_B = 0.0707$. The increase of $|m|$ has changed the

characteristics greatly from those of Table 10-4. Now there is an abrupt transition from a positive input resistance to about -300 ohms at $\kappa = 0.9941$, which would necessitate operating the tube for high gain around this frequency with a somewhat greater generator resistance. Also in Table 10-5 we present the response of one of the models in the $m = -0.10$ class—the tube for which $C_B = 0.10$. Despite the larger beam-circuit coupling this tube has a minimum input resistance of only -28 ohms near the π cutoff frequency.

Tables 10-3 through 10-5 prove that the frequency response of these amplifier models can vary erratically as the m and C_B parameters are

Table 10-5. Response of the Ten-Cavity Amplifier. $\theta_e = 2.24$ and $\theta_q = 0.235$, $Y_L(n) = 0$ ($n = 1, \dots, 9$)*

$m = -0.07, C_B = 0.0707$			$m = -0.10, C_B = 0.10$		
κ	$\mathcal{G}$	Z_{in}	κ	$\mathcal{G}$	Z_{in}
0.9935	343	$19.4 + j14.3$	0.985	267	$14.3 - j8.1$
0.9939	1394	$48.0 + j80.5$	0.991	1,485	$6.2 + j12.8$
0.9940	8516	$44.3 + j231$	0.9912	4,509	$3.1 + j18.7$
0.9941	-1991	$-302 - j86$	0.9913	-65,480	$-0.3 + j22.1$
0.9942	-876	$-85 - j80$	0.9914	-3,848	$-5.5 + j25.1$
0.9945	-316	$-22.6 - j36$	0.9920	-531	$-28.4 - j4.2$

* The output cavity is matched at the $3\pi/4$ "cold" circuit phase shift frequency.

changed. Even though each amplifier is matched at the $3\pi/4$ phase shift frequency the input resistance can go sufficiently negative at other frequencies to allow troublesome cutoff oscillations which are observed in many tubes. The load required for match to the growing wave changes so rapidly with frequency about the $3\pi/4$ frequency that the backward wave engendered in the last cavity may cause the input resistance to go negative.

B. Cutoff Oscillations

Data taken for tubes that employ the centipede and cloverleaf slow wave structures discussed in Sections 7-6 and 7-7 show that the cutoff oscillations exist at frequencies near the π cutoff such that the beam is nearly synchronous with the "cold" circuit. As the beam voltage is lowered, the oscillations cease rather suddenly when the power carried by the growing

wave changes from negative to positive. This occurs when two conditions are both satisfied: $\theta_e \cong \pi$ at a frequency such that $\theta_e + \theta_q$, the slow beam mode phase shift per cavity, is about equal to the "cold" circuit phase shift in the backward wave region of the ω - q curve [2]. We now present evidence that the minimum value of input resistance can indeed go increasingly negative as the beam voltage increases above the critical voltage. Table 10-6 summarizes the behavior of the minimum value of R_{in} for a representative tube model with the parameters $m = -0.07$, $C_B = 0.10$ and with the output load matched at the $3\pi/4$ frequency when $\theta_e = 2.24$. Note that $\theta_q = 0.235$ throughout the range of variation of θ_e .

Table 10-6. Minimum Value of R_{in} vs. θ_e for a Ten-Cavity Tube. $m = -0.07$, $C_B = 0.10$, and $\theta_q = 0.235^*$

θ_e	$R_{in, min}$ (ohm)	κ at $R_{in, min}$
3.30	>0	<0.980
3.15	-0.25	0.980
2.73	-3.1	0.989
2.56	-4.1	0.991
2.24	-11.2	0.994
2.09	$\cong -20$	0.990

* The load on the last cavity is matched at the $3\pi/4$ phase shift frequency when $\theta_e = 2.24$.

The table not only indicates a tendency for oscillation to develop only when $\theta_e < \pi$ but also shows us that the frequency of oscillation might rise somewhat as θ_e diminishes; this effect is, in fact, observed experimentally.

We could maintain a positive R_{in} by decoupling the first cavity electromagnetically from its neighbor, but the remaining cavities would still tend to oscillate. The conventional cure for cutoff oscillations involves the insertion of ohmic loss along the tube or perhaps a severed cavity near the center. However, let us consider an unconventional procedure which involves a monotonic taper of the open-circuit mode resonant frequencies in the more heavily coupled tube of $C_B = 0.10$ in Table 10-5, as an example. This model tends to oscillate near the π cutoff frequency with a minimum input resistance of about -30 ohms. It turns out that the taper raises the minimum value of R_{in} somewhat (to -22 ohms) but moves it to a frequency nearer the center of the untapered passband. The response of the tapered model is given in Table 10-7 with the effective normalized open-circuit mode resonant frequencies tapered monotonically from $0.98\rho_s$

in the first cavity to $1.024\rho_5$ in the tenth one.⁶ The taper has successfully prevented cutoff oscillations if we are willing to operate such a tube as a high-gain, narrowband amplifier at $\kappa = 0.968$ with a generator resistance of about 22 ohms.

Stagger-tuning or tapering of an ordinary centerhole-coupled cavity chain, Fig. 10-1, has a pronounced effect throughout most of the passband. It is difficult to design such a structure to avoid the cutoff oscillations without affecting the useful gain at lower frequencies. However, the

Table 10-7. Response of the Ten-Cavity Tube Which Is a Tapered Version of the Tube of Table 10-5. ($m = -0.10$, $C_B = 0.10$)

κ	$\mathcal{G}$	Z_{in}
0.962	61.5	$0.3 - j1.2$
0.964	-67.4	$-0.8 + j2.3$
0.965	-39.0	$-2.5 + j5.1$
0.966	-29.0	$-7.5 + j8.7$
0.967	-24.2	$-19.8 + j6.7$
0.968	-21.2	$-21.2 - j10.2$

resonant-slot and resonant-loop coupled structures discussed in Chapter 7 have effective coupling coefficients for the first passband which are strongly frequency dependent and much larger in magnitude near the π cutoff frequency than at lower frequencies. The possibility therefore exists of staggering the open-circuit mode resonant frequencies in such chains so as to raise R_{in} and suppress the cutoff oscillations and yet not adversely affect the amplifier properties at lower frequencies. Preliminary analysis of the effect of taper upon the growing and constant-amplitude backward waves in the cloverleaf and centipede chains indicates that a monotonic taper of these resonant frequencies so as to increase them from the gun (input) end to the output end of the tube would indeed reduce the level of the cutoff oscillations or eliminate them entirely, except, perhaps, at isolated frequencies for a given beam voltage. Furthermore, the effect of a taper appears to be more favorable at higher beam voltages in the range where the tube is operated for gain. However, these conclusions have not as yet been substantiated by experiment.

⁶ The effective resonant frequency has been tapered by means of external susceptance B_{Ln} in the n th cavity, according to Eq. 10-12. If we taper the cavity radii instead, according to the preceding footnote, the short-circuit mode resonant frequencies would start from $0.885\rho_5$ in the first cavity to $0.918\rho_5$ in the tenth one. Upon readjusting the calculated $m_{n+1,n}$ and $m_{n,n+1}$ a few percent each so as to guarantee continuity of power flow through the cavities, the computed response qualitatively agreed with Table 10-7. But the minimum input resistance dropped to about -55 ohms at $\kappa = 0.967$. Thus the performance of the model is sensitive to the scheme of tapering.

C. Gain and Bandwidth Performance as Compared with Experiment. Effect of Intermediate Cavity Loading

Up to this point we have discussed the characteristics of power amplifier models (which could be operated as oscillators in many cases) with lossless intermediate cavities. The effect of gradually increasing the intermediate loading will be discussed at the end of this subsection. First we want to compare experimental characteristics of tubes which employ the ring-bar helix cavities with models of similar parameters but lossless intermediate cavities.

Preliminary to this comparison we wish to point out a rather interesting deduction made on the basis of the computations for amplifiers with very high- Q intermediate cavities. As a high-gain amplifier is lengthened its gain goes to infinity faster than its bandwidth goes to zero, so that the gain-bandwidth product is unlimited. For example, in a five-cavity tube model with no drift spaces and the parameters

$$\begin{aligned}\theta_e &= 2.36, & \theta_a &= 0.40, \\ C_B^2 &= 0.010, & Y_{L5} &= 0.02 + j0 \quad (Q_L = 50) \\ Y_{Li} &= 0 \quad \text{for } i = 1, \dots, 4; \\ m &= 0 \quad (\text{no electromagnetic coupling})\end{aligned}\tag{10-21}$$

the transducer gain was 44 db and the fractional bandwidth was 0.0015 in a range just below the π cutoff frequency. This gain is computed with the input cavity matched to the generator resistance at midband frequency. Thus the transducer gain-fractional bandwidth product (abbreviated $\mathcal{G}_T FB$) is 37. However, the ten-cavity model had a gain of 110 db and a fractional bandwidth of 0.001 for a $\mathcal{G}_T FB$ of 10^8 .

The tube models summarized in Tables 10-3 through 10-5 can be operated as high-gain amplifiers provided the generator resistance is slightly greater in magnitude than the most negative input resistance. There is no formula for the $\mathcal{G}_T FB$ of a tube with negative R_{in} , and, in fact, the product can be arbitrarily high if the generator resistance is sufficiently close to R_{in} . For the stable, decoupled-cavity tube (Table 10-2) an increase of C_B from 0.028 to 0.10 causes the $\mathcal{G}_T FB$ product to increase from about 10^2 to about 10^7 . But the $\mathcal{G}_T FB$ for practical tubes with some bandwidth lies typically in the range from 1 to 100. A product of 1 is the order of magnitude reported by Chodorow and Wessel-Berg [3] for an experimental three-cavity tube in which each cavity is a full electronic wavelength section of ring-bar helix, separated from the neighboring helices by short drift spaces of about 20 space charge degrees. The

parameters of their particular tube are

$m \cong 0$ (very slight cavity-cavity coupling)

$\theta_e = \pi$ for each half of a given helix cavity. Each cavity is a full electronic wavelength long

$$\frac{\theta_g}{\theta_e} = 0.14$$

$$\theta_e$$

$$k \sim 3.2 \text{ kMc}$$

$$\frac{R_{sh}}{Q} \text{ of a cavity} \cong 250 \quad (10-22)$$

Q_L , the loaded Q of the output cavity, = 60 (the best all-round experimental value)

$$V_0 \cong 13 \text{ kv}$$

$$I_0 \cong 2.3 \text{ amp}$$

The value of C_{Bs}^2 for this tube (Eq. 9-26) with a full sinusoid of $\bar{e}_z = e_m \sin(2\pi z/L)$ pattern in each helix cavity is calculated by first determining e_m^2/τ . By definition, the effective shunt resistance across the cavity gap is

$$R_{sh} = \frac{V^2}{2P_L}, \quad V^2 \triangleq L \int_0^L e_m^2 \sin^2 \frac{2\pi z}{L} dz = \frac{1}{2} e_m^2 L^2$$

so

$$\frac{R_{sh}}{Q} = \frac{L^2 e_m^2}{4\omega W_s} = \frac{L^2}{2\omega \epsilon} \frac{e_m^2}{\tau} \quad (10-23)$$

The parameters of this tube determine C_{Bs}^2 to be about 0.008 for each cavity, somewhat on the high side as ordinary cavity chains go. However, the wide-bandwidth high-power multicavity klystrons often have C_B^2 values ten times this or more.

Chodorow and Wessel-Berg have obtained a gain of 27 db and a bandwidth of 0.02 kMc with only three cavities. H. Golde [4] has done still better with five decoupled cavities of the same helix variety. Under the same operating conditions he has reported a synchronous small-signal gain of 53 db and a fractional bandwidth of 0.009 for a $\mathcal{G}_{TFB}$ of 1800. By stagger-tuning the cavity resonant frequencies he obtained 50-db small-signal gain over a fractional bandwidth of 0.021 for a $\mathcal{G}_{TFB}$ of 2000. Each cavity was loaded to a Q_L of 45 in these tubes.

These ring-bar helix cavities appear to be very good for use in high-power multicavity klystrons, judging by computer analyses of tube models using cavities of comparable parameters. We analyzed a model of the Fig. 10-1 variety with no drift spaces between cavities and initially no external loading of the intermediate ones. Each cavity of the model was half an electronic wavelength long ($\theta_e = \pi$) to correspond to each half of the ring-bar helix cavity. Since $C_{Bs}^2 \sim \tau^{-1}$ was 0.008 for the full ring-bar

cavity, it is 0.016 for the half-cavity. The model had the same θ_o/θ_e for $\theta_e = \pi$ but a C_{Bu}^2 for uniform e_z field of 0.0095. The $e_{z, \text{uniform}}$ in the model cavity is therefore $\sqrt{9.5/16} = 0.77$ of e_{max} in the half-ring-bar cavity. The normalized output load was chosen to be $0.02 + j0$ for a loaded Q_L of 50, and the intermediate cavities were initially taken to be lossless. We computed 33-db gain over a fractional bandwidth of 0.003 centered at $\kappa = 0.9985$ just below the π cutoff frequency. The input impedance varied from $50 + j46$ ohms near the low-frequency half-power point (relative to transducer gain), to $69 + j39$ ohms at midband and $91 + j6$ ohms at the upper half-power point. This tube does not, however, compare directly with Golde's tube, because ten of our cavities correspond to five of his full-wavelength cavities. Upon computing the frequency response for this ten-cavity model we were surprised to find that the gain had increased to 85 db with two-thirds of the former bandwidth, instead of the 2×33 db expected. The $\mathcal{G}_{TFB}$ product was about 7 in the five-cavity model and 7×10^5 in the ten-cavity model.

Since the Q of each intermediate cavity in Golde's tube was 45 we gradually increased the intermediate cavity loading in the tube model and studied the gain and bandwidth. As the intermediate Q_{Li} decreased from infinity the gain dropped rapidly at first, then less and less rapidly. For a Q_{Li} of 100, $i = 1, \dots, 9$, the model had a gain of 40 db and a fractional bandwidth of 0.007; a lower value of Q_{Li} implies a lower gain.

Judging from the measured synchronous gain of 53 db and fractional bandwidth of 0.009 as compared to the computed values of 40 db and 0.007, respectively, in the tube model we suspect that the full-wavelength ring-bar helix cavity may be nearly the ideal one to use in high-power multicavity klystrons. Apparently with a stronger value of C_B and increased external loading we could obtain a fractional bandwidth above 0.05 and gains of 30 db or more.

A computer study of the effect of increased intermediate cavity loading of the amplifier models discussed earlier in this section revealed the following facts. As the loading increases in the high-gain, narrow bandwidth amplifier of positive input resistance throughout the frequency range near π cutoff the gain drops considerably more rapidly than the bandwidth increases, if it increases at all. As loading increases in the coupled-cavity model which has negative input resistance at some frequencies it eventually becomes stable at all frequencies with a gain which can be appreciable. However, the bandwidth tends to remain about 1% or less with the external loading adjusted for a gain of 30 db, for reasons given in the initial paragraphs of this section.

In conclusion, the computations demonstrate conclusively that these *synchronously* tuned ten-cavity power amplifier models with relatively low

values of C_B and conventional, nonoptimized output cavities tend to have bandwidths of only 1 % or less with a reasonable gain of 30 db. Since the bandwidth of the experimental tube we have discussed more than doubled when the appropriate stagger-tuning scheme was invoked, let us study the effect of various stagger-tuning arrangements on these models.

10-4. Effect of Stagger-Tuning on the Performances of the High-Gain, Narrow-Bandwidth Tube Models

In the preceding section we studied the large gains and small bandwidths obtainable with decoupled-cavity models of the type shown in Fig. 10-1 in which the cavities were synchronously tuned. We now present evidence that various stagger-tuning schemes can be expected to increase the bandwidth by only a factor of 2 or so without undue decrease of gain. This means that the models which are very narrowband tend to remain so. We chose to stagger the resonant frequencies of the decoupled cavities in a five-cavity model with the following parameters:

$$\begin{aligned}\theta_e &= 2.36, & \theta_q &= 0.40, & C_B^2 &= 0.01, \\ G_{Li} &= 0 & (i &= 1, 2, 3, 4), & & (10-24) \\ Y_{L5} &= 0.02 + j0 & (Q_{L5} &= 50), & \text{no drift spaces}\end{aligned}$$

Only the output cavity is externally loaded. We selected this model because of its large transducer gain of 44 db, with an associated fractional bandwidth of 0.0015. The input impedance is $77 + j79$ at the lower half-power point, $132 + j51$ at the center frequency, and $157 - j4$ at the upper half-power frequency.

Many stagger-tuning schemes were tried, including monotonically increasing and decreasing tapers of the open-circuit resonant frequencies and some rather haphazard arrangements. But they all had the effect of lowering the gain by a factor of 10 or more without providing a smooth gain vs. frequency curve of commensurately larger bandwidth. Some of the schemes for staggering the resonant frequencies only slightly resulted in a loss of gain in the upper frequency range near the synchronous π cutoff. The remainder of the schemes involved staggering the frequencies more heavily, but they yielded response curves with unwanted "holes" which prevented an increase in bandwidth. The addition of external loading to the intermediate cavities had the expected effect: The gain dropped sharply and the bandwidth at best increased only slightly. Mere loading of the input cavity so as to increase its bandwidth is not sufficient, for the remaining cavities will still give a narrowband response.

Just to verify the fact that we missed no interesting effects we also

computed the gain frequency response of various staggered five- and ten-cavity amplifier models with the parameters of Eq. 10-24 but with the very light coupling specified by $m = -0.01$ and with the output cavity matched at the unstaggered $3\pi/4$ "cold" circuit phase shift frequency. Some of the five-cavity models developed minimum input resistance less than zero near the $3\pi/4$ phase shift frequency, and as the frequency increased toward the unstaggered cutoff frequency R_{in} increased until it was only slightly negative or else positive. Other models had negative input resistance throughout the frequency range near the unstaggered cutoff frequency.

And so we have found that these various stagger-tuned models, like the synchronously tuned ones of the preceding section, have reasonable gains in the range of 30 db or more but only for bandwidths of $\leq 1\%$. By contrast, many of the models discussed in the next section develop significantly larger bandwidths (at reduced gains) when the appropriate stagger-tuning scheme is adopted. The fundamental reason for the larger bandwidths is traced to the optimized output cavity separated by a drift space from its neighbor, the last buncher cavity. *The amplifier with optimized output section allows greater external loading for a given gain, say 30 db, and hence larger bandwidth.*

10-5. Various Tube Models with an Optimized Output Section

In this section we describe the small-signal gain and bandwidth characteristics of two varieties of models: those in which the tuned resonant frequencies of all the cavities are clustered in a narrow frequency range, and the "skirtron" tubes in which the first cavity is tuned low and the three intermediates are tuned high with respect to the output cavity tuned to the center frequency. All the cavities except the output cavity in each model are stacked together to form a buncher which feeds a strongly excited beam across a drift space to the optimized output cavity. We make no attempt to design the optimized output cavity but refer to the next section for a discussion of the responses of various output cavities. All the five buncher cavities in each tube are loaded to a Q of 50, excluding the beam, except for the "skirtron" model in which the four buncher cavities were loaded to a Q of 25 in accord with the experimental tubes. A comment will be made later about the qualitative effect of reducing the loading.

We have used the computer program based on the equations of Table 10-1 to search for models with bandwidths in the range of 5–10%. We

took a median value of $C_B = 0.10$ for all these models; consequently the maximum transducer gain will turn out to be about 20 db, even with an optimized output cavity. The gain may be raised and the bandwidth maintained by increasing the value of C_B and the external loading; some high-power tubes have $C_B = 0.30$ or more. We also fix the ratio θ_q/θ_e at 0.10; the smaller this ratio, the larger the gain tends to be, provided the d.c. beam current is maintained.

Various bunching sections, exclusive of the output cavity, were analyzed both for $\theta_e = \pi/2$ and $3\pi/4$. They are categorized as

- 1a. A five-cavity decoupled tube ($m = 0$), synchronously tuned
- 1b. The same decoupled tube with the cavities stagger-tuned in three different ways
- 2a. The same tube but coupled to $m = -0.04$, synchronously tuned
- 2b. The same tube of $m = -0.04$ but stagger-tuned in the same three different ways
- 3a. The same tube but coupled now to $m = -0.10$, synchronously tuned
- 3b. The same tube of $m = -0.10$ but stagger-tuned in the same three different ways

Lastly, we computed the characteristics of

4. The four-cavity skirtron buncher, with three intermediate cavities synchronously tuned and also stagger-tuned in three different ways. $\theta_e = \pi/2$ and $\theta_q/\theta_e = 0.11$

To describe the inherent transducer gain and bandwidth of these tubes it is convenient to refer to a "buncher power" defined as the magnitude of the kinetic voltage squared emerging from the last buncher cavity divided by the input resistance and also multiplied by the last factor in Eq. 10-20. The input cavity voltage is proportional to the input impedance by Eq. 10-17. Buncher power P_B is

$$P_B \triangleq \frac{|V_{k \text{ out}}|^2}{R_{in}} \cdot \frac{4R_g R_{in}}{(R_g + R_{in})^2}, \quad R_{in} > 0 \quad (10-25)$$

All buncher power responses vs. frequency have been computed for the excitation $DI_G = 0.48$ in the first cavity according to the equations of Table 10-1. Knowing the geometry of the coupling loop and other parameters we can determine the actual loop input current I_g and the input resistance R_{in} which is proportional to the computed normalized voltage $\phi(1)$ in the first cavity. Hence we can evaluate the total input power $P_{in} = \frac{1}{2} I_g^2 R_{in}$ at constant I_g . To estimate the actual transducer gain we assume that the output cavity is located 90 space charge degrees beyond the output cavity (less for large-signal operation), whereupon a beam

Table 10-8. Approximate Buncher Powers P_B and Bandwidths of the Various Tube Models. $\theta_q/\theta_e = 0.10$, $C_B^2 = 0.01$. All Cavities Loaded to a Q of 50

Scheme	$\theta_e = \pi/2$			$\theta_e = 3\pi/4$		
	$P_{B, \max}$	B.W. (%)	R_g^* (ohm)	$P_{B, \max}$	B.W. (%)	R_g
$m = 0$, sync.	165	3.2	6	300	3.5	4
1 [†]	140	5.3	7	135	5.2	2
2	150	3.7	12	205	4.5	5
3	110	7.2	0.8	200	3.5	0.7
$m = -0.04$, sync.	180	4.1	7	238	2.9	9
1	103	8.2	3	104	7.2	4
2	238	3.3	16.5	206	3.2	16
3	98	6.0	5.5	94	6.3	4
$m = -0.10$, sync.	246	5.3	6	196	2.1	5
4	215	4.5	8	118	1.5	3
5	237	6.0	8	290	2.1	2
6	185	5.5	8	160	1.5	3

* R_g is the generator resistance if $\text{Re } \phi(1)$ is numerically equal to R_{in} ; otherwise the true generator resistance is larger by a factor $R_{in}/\text{Re } \phi(1)$.

† The numbers denote stagger tuning schemes. Scheme 1 corresponds to normalized resonant frequencies $\rho_{1,2,3,4,5} = 0.98, 0.99, 1.00, 0.99, 0.98$. Scheme 2 $\sim 0.98, 0.99, 0.99, 1.0, 1.0$. Scheme 3 $\sim 1.0, 1.0, 0.99, 0.99, 0.98$. Scheme 4 $\sim 0.970, 0.985, 1.0, 0.985, 0.97$. Scheme 5 $\sim 0.970, 0.985, 0.985, 1.0, 1.0$. Scheme 6 $\sim 1.0, 1.0, 0.985, 0.985, 0.97$.

current of $Y_0 V_{k, \text{out}}$ will excite that cavity of self-admittance Y_c , load admittance Y_L , and gap-coupling coefficient S_c . The transducer gain $\mathcal{G}_T$ will then be

$$\mathcal{G}_T = \frac{1}{2} \left| \frac{Y_0 V_{k, \text{out}} S_c}{Y_L + Y_c} \right|^2 \frac{G_L}{\frac{1}{2} I_g^2 R_{in}} \cdot \frac{4 R_g R_{in}}{(R_g + R_{in})^2} \quad (10-26)$$

$$= P_B \left[\left| \frac{Y_0 S_c}{Y_L + Y_c} \right|^2 G_L \frac{1}{I_g^2} \right], \quad I_g \text{ a fixed reference}$$

It is sufficient to measure the relative gain as well as the inherent bandwidth by P_B at fixed driving current I_g . In a moment we estimate the magnitude of the bracketed factor in Eq. 10-26 and with it the transducer gain for a practical tube.

With respect to the buncher power P_B , then, the performance characteristics of the aforementioned tubes are summarized in Table 10-8.

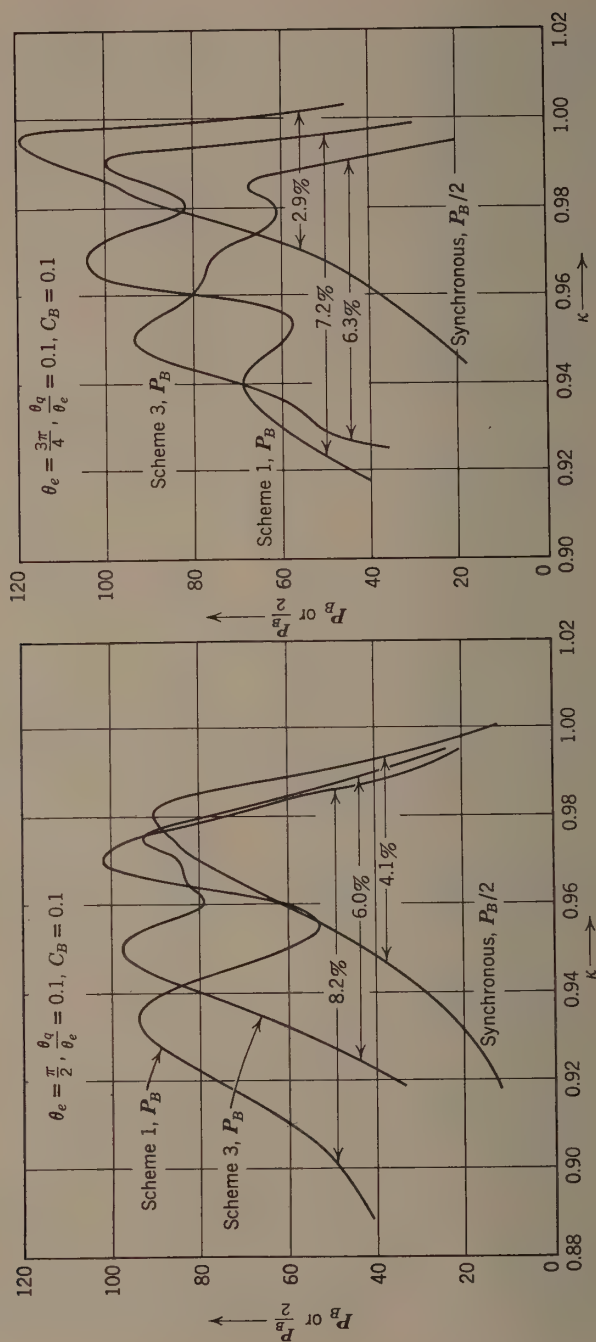


Fig. 10-4. Buncher power vs. normalized frequency for the tube models of $m = -0.04$ in Table 10-8. Stagger-tuning scheme 2 of narrow bandwidth is excluded for both $\theta_e = \pi/2$ and $3\pi/4$.

Maximum buncher power always occurs at a frequency below the highest resonant frequency.⁷ For most tubes the generator resistance R_g was adjusted so as to optimize the bandwidth without too great a reduction in the maximum buncher power. For some of the tubes—notably the $m = -0.10$, $\theta_e = 3\pi/4$ tubes—the bandwidth was determined primarily by the buncher power vs. frequency and only slightly by the frequency dependence of R_{in} , in which case we had no choice but to choose R_g so as to match R_{in} at the frequency of maximum buncher power. It is rather surprising that the bandwidths for all the decoupled and all the $m = -0.04$ models are *larger* than those of the more heavily coupled $m = -0.10$, $\theta_e = 3\pi/4$ tubes. We also observe that the bandwidth is a rather sensitive function of the manner of stagger-tuning the cavities; and a change in the coupling parameter m or drift angle θ_e may change not only the gain and bandwidth but also the tuning scheme necessary for a given bandwidth.

The buncher power vs. normalized frequency response curves appear on Fig. 10-4 for all varieties of the $m = -0.04$ models listed in the table except the narrow-bandwidth models stagger-tuned according to scheme 2. There $\kappa = 1$ may be regarded as a reference frequency, about which the cavities are tuned.

The computations reveal a fact which may not be generally known; namely, that the kinetic voltage after the initial buildup increases very slowly along these tubes, whereas the beam current builds up steadily until it is comparable to the kinetic voltage in magnitude at the output of the fifth and final buncher cavity. The kinetic voltage may increase by a factor of 2 or even 3 from the first to the fifth cavity output at frequencies near the upper half-power point in the $m = 0$, $\theta_e = 3\pi/4$, staggered and unstaggered models, but in the others it increases by less than a factor of 2. The skirtron performs somewhat differently because of the different tuning arrangement. These remarks mean that we can improve the transducer gain a few decibels by relocating the output cavity so as to maximize the total beam current as composed of both a fraction of the output buncher current and a portion of the kinetic voltage which has drifted into current.

The general effect of reducing the external conductance to 0.01 ($Q = 100$) in all the buncher cavities is as expected: an increase in the maximum buncher power and a slight reduction in bandwidth. The input resistance

⁷ We certainly expect this in a synchronously tuned tube because the impedance looking from the beam edge toward the cavity walls is inductive ($p_v = -j\omega\mu i$) and the electron power flow is in the direction of growth. It can be shown very generally that a beam propagating a slow wave which grows with distance must “see” an inductive impedance as presented by the surrounding medium. Therefore in a stagger-tuned tube it is rather plausible the frequency of maximum growth would always lie below the highest tuned frequency (open-circuit mode cutoff frequency) of any cavity.

remains positive in all of the synchronously tuned, coupled tubes but it does go negative in certain of the *coupled*, stagger-tuned models. So we must take care to maintain stability when reducing the external loading.

We now estimate the magnitude of transducer gain $\mathcal{G}_T$ from typical parameters in a high-power tube. At the end of Section 10-2 we estimated parameters for a 3-kMc tube such that the generator current I_g turns out to be $\frac{1}{8}$ amp for the reference excitation $DI_G = 0.48$. Beam current I_0 might correspond to a perveance of 4×10^{-6} amp/v $^{3/2}$ at 85 kv, say, so that $I_0 = 100$ amp and the beam admittance $Y_0 = 0.60(\omega/\omega_q)10^{-3}$. The output cavity conductance G_c will probably be on the order of 10^{-5} mhos, much smaller than the load conductance G_L which, for large-signal operation (not treated in this book) should be about $I_0/V_0 = 1.2 \times 10^{-3}$. For a loaded Q_L of 10 in the output cavity, R_{sh}/Q_L is therefore 85. With zero output cavity susceptance at midband and $P_B = 100$, say, we find the transducer gain is about

$$\mathcal{G}_{T,\max} \cong 1.0 \left(\frac{\omega}{\omega_q} \right)^2 \quad (10-27)$$

The $(\omega/\omega_q)^2$ dependence arises from the fact the r.f. power in the last cavity is proportional to $|Y_0 V_{k,\text{out}}|^2$ from the buncher, and $Y_0 \sim \omega/\omega_q$. We should not forget that a precise estimate of transducer gain would account for the implicit dependence of the buncher performance—and hence R_{in} and P_B —upon ω/ω_q .

Equation 10-27 predicts a maximum transducer gain of only 100 or 20 db for these tube models, whereas 30 db is more typical of practical tubes. The gain is low, as we mentioned earlier, because of the median value of $C_B = 0.10$ selected for the models, as compared to values of 0.30–0.40 in really high-power tubes. An increase of C_B in conjunction with an increase in external loading of the buncher cavities would tend to maintain the bandwidth but increase the gain. A larger value of C_B also tends to optimize the output cavity with respect to gap voltage generated per unit of entering beam current, as explained in the next section.

We should remark at this point that a tube which is built with drift tubes connecting the various cavities may perform quite differently from a model of the tube with field-free drift spaces between cavities and the same value of θ_q/θ_e . Calculations made on the broadband megawatt hollow-beam multicavity klystron designed by A. Bers and L. B. Anderson [5] indicate that a model with the same parameters but with field-free drift spaces between cavities had a (power out/power in)-bandwidth of about 5.5%, as compared to the experimental small-signal bandwidth of 8.7%. This discrepancy suggests the presence of fields in the drift tubes

between the cavities that couple them electromagnetically to an extent which cannot be ignored in an accurate computation of the performance.

Each of the aforementioned tube models is characterized by a group of tuned cavity resonant frequencies clustered in a narrow frequency range. The skirtron tubes, on the other hand, are designed with the input cavity tuned to the approximate lower half-power frequency, whereas the three or more intermediate cavities are tuned to the approximate upper half-power point. The beam which emerges from the last buncher cavity drifts to the output cavity tuned to the midband frequency and with at least the tube bandwidth—on the order of 10% or more. Experimental tubes have been built with a reduced plasma transit angle of 20° or so between successive cavities, excluding the separation between last buncher and output cavity—but we have analyzed tube models in which the bunching cavities are stacked together. These models have performance characteristics similar to the experimental ones.

A representative four-cavity skirtron buncher will be described with the parameters

$$m = 0, \quad \theta_e = \frac{\pi}{2}, \quad \frac{\theta_a}{\theta_e} = 0.11, \quad C_B = 0.20$$

$$G_{Ln} = 0.04 \quad (\text{buncher cavities loaded to a } Q \text{ of } 25)$$

$$\rho_{1,2,3,4} = 0.96, 1.04, 1.04, 1.04$$

(normalized resonant frequencies of the input and three intermediate bunching cavities). In this skirtron tube model the kinetic voltage leaving the fourth cavity builds up with increasing frequency relative to the voltage leaving the first cavity, with a maximum ratio of about 2 obtained at maximum buncher power P_B on Fig. 10-5. The current leaving the fourth cavity remains always less than half of the kinetic voltage leaving that cavity. This model has the remarkable property that $|V_{k,\text{out}}|^2$ varies with frequency qualitatively just like $\text{Re } \phi(1) \propto R_{\text{in}}$ shown on Fig. 10-5 so the buncher power proportional to $|V_{k,\text{out}}|^2 / \text{Re } \phi(1)$ tends to stay constant over a wide frequency range. By choosing a median value of generator resistance we lose some gain in the low-frequency range but maintain good gain in the high-frequency range. The resultant bandwidth is 11.4% and the maximum buncher power is 90.

We have also examined the effect of stagger-tuning the intermediate-cavity resonant frequencies in various ways—as 1.03, 1.04, 1.05; 1.03, 1.05, 1.04; and 1.05, 1.04, 1.03—both for zero electromagnetic coupling and a coupling of $m = -0.02$. However, the effect of each stagger-tuning scheme on both the kinetic voltage and the current output is slight and apparently not beneficial over the frequency range of concern. The result of changing m to -0.02 is virtually no change in the maximum buncher power and bandwidth provided we change R_0 a few ohms. We

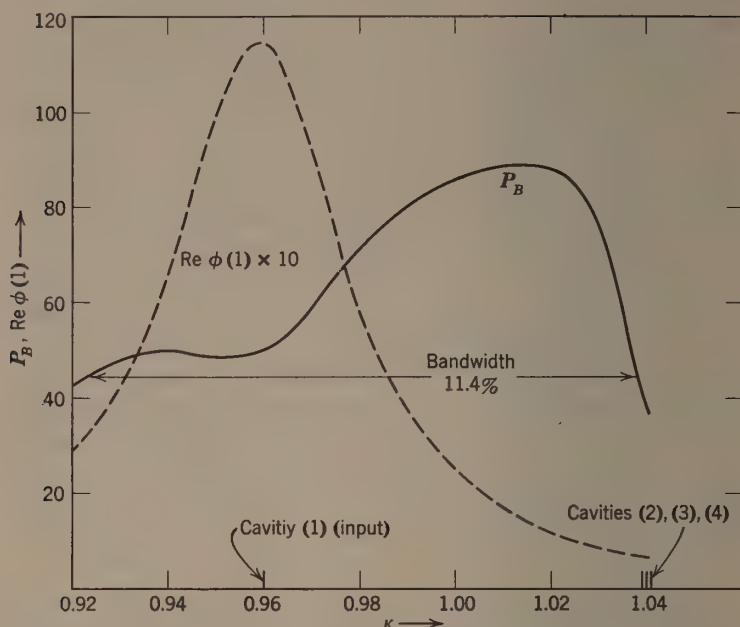


Fig. 10-5. Buncher power and input resistance of a skirtron buncher model consisting of four decoupled cavities with no drift spaces between them. The relative frequencies of the cavities are shown along the abscissa. $\theta_e = \pi/2$, $\theta_q/\theta_e = 0.11$, $C_B^2 = 0.04$, the loaded Q of each cavity is 25, and generator resistance R_g has been adjusted to 5 ohms. It is presumed that $\text{Re } \phi(1)$ is the input resistance according to remarks following Eq. 10-17.

suspect that the response curves will also be insensitive to changes in θ_q/θ_e and C_B as well, provided the generator resistance is varied so as to maintain both a reasonable bandwidth and reasonable gain.

10-6. Response Characteristics of Constituent Cavities Appropriate for Multicavity Klystrons

In this section we describe qualitatively the response characteristics of input and intermediate cavities in a range of small C_B^2 and the numerical characteristics of these cavities and the output cavity in a range up to $C_B^2 = 0.10$. The parameters which are varied are θ_e , θ_q and C_B . The $\bar{e}_z$ pattern within each cavity is uniform. Each cavity is assumed to be decoupled electromagnetically from its neighbors—an assumption that may be untenable if they are connected by drift tubes through which a strong beam passes. The information is intended to serve as an aid in the selection of cavities for a power amplifier or buncher.

We first discuss the qualitative effects of varying θ_e , θ_q , and C_B in the range

$$0 < \theta_e \leq \frac{3\pi}{4}, \quad 0 < \frac{\theta_q}{\theta_e} \leq 0.3, \quad \text{and} \quad 0 < C_B^2 \leq 0.02$$

With respect to the *input* cavity, computations indicate that the voltage generated is very insensitive to these parameters at a given frequency in the resonance region, for a fixed external loading. This means the input resistance, proportional to $\text{Re } \phi(1)$ by Eq. 10-17, will be insensitive. The amount of kinetic voltage generated is insensitive to changes of θ_q at a given θ_e and C_B in a frequency range throughout the resonance region but is more sensitive to changes of θ_e : As θ_e approaches π at fixed θ_q the kinetic voltage generated near resonance goes down slowly at first, then more rapidly until, at $\theta_e = \pi$, the V_k generated is on the average about two-thirds of the V_k for $\theta_e = \pi/2$. Thus we are not unduly constrained to use a narrow-gap input cavity. The effect of varying only C_B in the range indicated is—curiously enough—very little change in the cavity voltage and the kinetic voltage and current leaving the cavity. Of course, a larger C_B will usually imply a stronger d.c. beam current and hence a larger inherent a.c. power output proportional to $Y_0 V_k \sin \theta_q$ amperes of beam current, which would be created by kinetic voltage V_k drifting through reduced plasma transit angle θ_q .

Computations for typical *intermediate* cavities which are loaded externally and operated in the aforementioned ranges of θ_e , θ_q , and C_B and which are uncoupled to their neighbors reveal the following points. An increase of θ_q causes the kinetic voltage generated per unit of entering beam current to decrease at frequencies below resonance (which would be the midband frequency in an infinite chain of such cavities) and to increase at frequencies above resonance so that the behavior is not clear cut. However, an increase of θ_e alone with all other parameters fixed raises the kinetic voltage generated throughout the resonance range and the range θ_q . This is a valid reason for not designing narrow-gap intermediate cavities for a given θ_q . As a rule, both the entering kinetic voltage and current excite the cavity and we cannot neglect the contribution by V_k to the departing beam excitations.

The numerical response characteristics for the various cavities at their resonant frequencies are summarized in Tables 10-9 through 10-11 for C_B^2 in the range

$$0.025 \leq C_B^2 \leq 0.10$$

The first two tables are for the input and intermediate cavities, respectively, for various θ_e and θ_q and the external loading of $G_L = 0.02$ ($Q_L = 50$). These tables are compatible with the statements made about these cavities

Table 10-9. Input Cavity $|V_k|$ Response for Various Parameters at the Resonant Frequency. $G_{Ln} = 0.02$

θ_e	θ_a	C_B^2		
		0.025	0.06	0.10
1.57	0.175	39 (22.6)*	36 (20.7)	32.9 (18.7)
	0.525	33.5 (19.9)	26.9 (15.5)	21.6 (12.0)
2.35	0.175	31.1 (20.6)	25.9 (17.1)	21.7 (14.2)
	0.525	23.4 (15.9)	15.8 (10.6)	11.5 (7.7)
	1.05	15.7 (12.0)	9.2 (6.9)	6.2 (4.6)
π	0.175	23.0 (18.8)	17.6 (14.4)	13.9 (11.4)
	0.525	15.5 (13.1)	9.5 (8.0)	6.6 (5.6)
	1.05	9.8 (9.3)	5.2 (5.0)	3.4 (3.3)

* $\text{Re } \phi(1)$ appears in the parentheses; it equals R_{in} numerically for the parameters following Eq. 10-17.

in the lower range of C_B . Notice that the kinetic voltage generated by the input cavity for the reference excitation $DI_G = 0.48$ (defined in Eq. 10-11 and appearing in the first equation of Table 10-1) *decreases* with increasing C_B , whereas the kinetic voltage generated per unit of entering beam current (measured as $Z_0 I = 1$ v) by the intermediate cavity *increases* with increasing C_B . The beam current generated by the input cavity (not tabulated) increases with increasing θ_a/θ_e ratio and can easily be comparable to the kinetic voltage.

The responses of various output cavities to an entering beam current

Table 10-10. Intermediate Cavity V_k Response to an Entering Beam Current of $Z_0 I = 1$ at the Resonant Frequency. $G_{Ln} = 0.02$ *

θ_e	θ_a	C_B^2		
		0.025	0.06	0.10
1.57	0.175	2.3	5.2	7.9
	0.525	2.0	3.8	5.1
2.35	0.175	3.6	7.3	10.1
	0.525	2.7	4.4	5.3
	1.05	1.7	2.4	2.6
π	0.175	3.9	7.1	9.4
	0.525	2.5	3.7	4.3
	1.05	1.4	1.7	1.9

* The numbers are rounded off to the nearest tenth.

of $Z_0 I = 1$ v appear in Table 10-11 for the aforementioned range of C_B^2 . These cavities all have a loaded Q_L of 10. In footnote 4 we explained that, in the large-signal regime, the cavity gap voltage generated by 1 amp of r.f. current should be about V_0/I_0 volts. Therefore a beam current of

Table 10-11. Response of the Output Cavity Voltage ϕ to an Entering Beam Current of $Z_0 I = 1$ at the Resonant Frequency. $|\phi|$ Is Tabulated for Various Parameters and $G_{Ln} = 0.10$ ($Q_L = 10$)

θ_e	θ_q	C_B^2		
		0.025	0.06	0.10
1.57	0.175	0.28	0.67	1.09
	0.525	0.26	0.61	0.96
2.35	0.175	0.29	1.24	1.98
	0.525	0.24	1.04	1.54
	1.05	0.16	0.78	1.07
π	0.175	0.77	1.71	2.65
	0.525	0.67	1.34	1.88
	1.05	0.53	0.95	1.23

$Z_0 I = 1$ v should generate a voltage of about $Y_0 V_0 / I_0 = \theta_e / 2\theta_q$ volts. The computed gap voltage is $ve_z L = \phi e_z \cong 2\phi$ if we estimate e_z by the patterns of the "straight-through" short-circuit mode (Eq. 3-17) in these long cavities. We see that the cavities with $C_B^2 = 0.06$ and 0.10 and $\theta_q = 0.525$ and 1.05 do indeed generate about the optimum gap voltage or more. The long cavities of $\theta_e = \pi$ are not inferior to the shorter ones.

Problems

1. According to the Bethe theory of small-hole coupling, Chapter 3, show that there is very little radiation of power out the left hole of the first cavity and the last hole of the output cavity in a cavity-chain amplifier.
2. How would we use the computer program to study the noise figure of an amplifier?
3. How could the computer program be used to analyze the transient response of an amplifier? What effect would a negative input resistance have on the output response?
4. What does the transducer gain formula, Eq. 10-20, imply about the reactances of the generator and input impedances?
5. Show there is usually no finite gain-bandwidth product for an amplifier with negative input resistance; the value depends on the generator resistance.
6. What practical considerations prevent us from indefinitely increasing the

transducer gain by lowering the input resistance to a very small value and raising the ω/ω_q ratio to a large value?

7. Show that an electron beam will support a growing wave only if the impedance looking out from its boundaries is inductive.
8. Study the transducer gain, Eq. 10-26, for the amplifiers of Fig. 10-4 with respect to frequency, with $Y_L = G_L(1 + j2Q_L\delta)$ and $\delta = (\omega - \omega_0)/\omega_0$ the fractional change of frequency from resonance. $G_c \ll G_L$. How must the output cavity bandwidth $1/Q_L$ compare with the buncher power bandwidth in order that the overall bandwidth be nearly that of the latter.
9. Verify the fact that the buncher power response of the skirtron model of Fig. 10-5 is insensitive to the value of generator resistance R_g . Let $\text{Re } \phi(1) \equiv R_{in}$.

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Chapter XI

Parametric and Maser Slow Wave Systems

11-1. Synopsis

This chapter opens with a review of parametric power conversion on two transmission lines coupled through a nonlinear "pumped" inductance, as described by Tien. There are four basic states of operation of this three-frequency device, depending upon whether the pump frequency ω_p is higher or lower than the applied signal frequency ω_1 and whether the group velocities at frequencies ω_1 and $\omega_p - \omega_1 = \omega_2$ are in the same or opposite directions at corresponding points on the ω - q curves. In two of these states the fields at frequencies ω_1 and ω_2 are sinusoidal along the line; in the other two they are exponential. The sinusoidal or exponential nature of the waves may be inferred from knowledge of the directions of the group velocities and the Manley-Rowe power relations for lossless nonlinear media. The elementary expression for noise figure indicates the conditions for low-noise amplification.

We then analyze in some detail the three-frequency cavity-chain parametric amplifier, with ω_1 in the m th passband and ω_2 in the n th passband. The fields in each of these (narrow) passbands may still be described by the open- and short-circuit mode-pairs, defined with respect to the Hermitian (lossless) portion of the permittivity dyadic. Then Maxwell's equations are solved for the fields at each frequency coupled to the fields at the other frequency through the nonlinear pumped ferroelectric. A determinantal equation is obtained for the complex phase shift γ_1 of the ω_1 fields in terms of the nonparametric phase shifts on the ω - q curves and a parametric coupling coefficient W . The properties of the four independent waves of the amplifier are listed. If $\omega_p = \omega_1 + \omega_2$, there are *two* states

of exponential gain operation, depending on whether the two dispersion curves in which ω_1 and ω_2 lie have the same or opposite shapes. If they have the same shape, this state is the analog of gain operation on the transmission line; in the other state the phase shifts are such that the powers at frequencies ω_1 and ω_2 also flow in the same direction. If $\omega_p = \omega_1 - \omega_2$, two states of exponential gain also exist, one of which corresponds to the state of opposing group velocities on the transmission line. In the other state the phase shifts adjust themselves so that the powers at frequencies ω_1 and ω_2 also flow in opposite directions. Our field expressions, which yield a determinantal equation for γ_1 , are consistent with the Manley-Rowe relations if the ferroelectric is lossless.

The expression for the maximum growth factor per cavity, valid for low rates of growth, reveals a gain-passband width conservation relation which depends not only upon the parametric coupling coefficient W but also upon the nonparametric phase shifts at the frequencies ω_1 and ω_2 .

Currie and Gould have calculated the gain vs. frequency response for various six-cavity amplifiers and different operating conditions, such that the pump phase shift per cavity φ_p is about 2.9 radians. Both ω_1 and ω_2 lie in the first passband and $\omega_1 = \omega_2 = \omega_p/2$ at the nonparametric phase shift of $\varphi_1^0 = \varphi_2^0 \cong 1.2$ radians. We show, by means of curves of the growth factor vs. W and a pump phase shift of 3.0 radians, the ranges of φ_1^0 and φ_2^0 and W within which only one growing-decaying wave pair exists in the system.

Grabowski and Weglein show that the best condition for unilateral gain occurs for $\varphi_p = 5\pi/4$, provided $\omega_p = \omega_1 + \omega_2$ and both ω - φ curves have the same shapes. Unilateral performance is improved if the ω - φ curves are distorted from their single mode-pair shapes so as to increase the asynchronism between φ_p^- in the backward direction and $\varphi_1^0 + \varphi_2^0$ in that direction. On the other hand, if $\omega_p = \omega_1 + \omega_2$ and the ω - φ curves have opposing group velocities at corresponding points, the best pump phase shift is perhaps $\pi/4$, so that $\pi/4 \cong \varphi_1^0 + \varphi_2^0$ in the forward direction and $-\pi/4 \approx \varphi_1^0 + \varphi_2^0$ in the backward direction.

We present a table listing the phase shifts of the four independent waves at frequency ω_1 for $\varphi_p = 4.0$ and various values of φ_1^0 , φ_2^0 and W . With these waves and formulas for their relative amplitudes, as determined by the terminal loads and excitations, we can compute the parametric gain vs. frequency for an amplifier operating with ω_1 in any narrow passband and ω_2 in another (or the same) passband of the same shape.

We summarize the experimental behavior of the Grabowski-Weglein iris-loaded waveguide (cavity-chain) parametric amplifier. The unilateral performance is improved by (a) changing some of the pump phases,

particularly in the last cavity, (b) coupling the fields at a sum frequency in the backward wave to loss but not the fields of the forward wave, or (c) inserting a magnetized ferrite post into each coupling aperture between cavities. The simple theory of elliptically polarized r.f. magnetic field around each post is developed.

In the next section we review Suhl's analysis of the three-frequency ferromagnetic parametric amplifier, and cast his equations for electromagnetic operation between two resonant cavity modes at frequencies ω_1 and ω_2 and for semistatic operation between one resonant cavity mode and the sample magnetostatic modes into the form appropriate for the traveling wave amplifier. We include the effects of cavity wall losses upon the waves. The determinantal equation for phase shift γ_1 of the fields at frequency ω_1 is similar to the equation derived for the ferroelectric amplifier in the case of electromagnetic operation but is more complicated for semistatic operation because of the inclusion of more than one excited magnetostatic mode.

The reader is referred to W. H. Louisell [4] for a discussion of parametric beam amplifiers which employ the fast space charge modes and the fast cyclotron modes.

Section 11-5 develops the quantum electrodynamic equations for the cavity-chain maser amplifier with a continuously pumped active material of three energy levels in each cavity. The equations are formally similar to those of the ferromagnetic parametric amplifier. The pumping field maintains a surplus of molecules in the higher-energy states from which they radiate energy into the electromagnetic field at the signal frequency. The Heisenberg equations of motion are developed for the quantized electric and magnetic field operators of the resonant cavity mode belonging to the narrow passband within which the signal frequency lies. These operators in a given cavity are coupled to the fields in the adjacent cavities, magnetic dipole moments of the molecules, loss mechanism in the walls, and to the generator field in the first cavity and the dissipative external load in the last cavity. Then the equations of motion for the dipole moments are developed, including the T_2 or spin-spin coupling between molecules but excluding the T_1 or spin-lattice dissipation. From these operator equations we may obtain, in principle, all the signal frequency components of the matrix elements of the electric and magnetic field operators in each cavity, in terms of the matrix elements established by the d.c. and "pump" magnetic fields. Given a specified (constant) density matrix for the system, as determined by the temperature and pumping field, we can then compute the expectation values of the field energy and load power in the last cavity.

11-2. Parametric Power Conversion on Two Coupled Transmission Lines

The principles of power conversion from one frequency to another can be illustrated with reference to Tien's fundamental article [1]. Consider the pair of distributed transmission lines coupled through the nonlinear inductances which are varied with time along the lines in Fig. 11-1. Line 1 with the per unit length parameters X_1 and B_1 supports voltage V_1 and current I_1 at frequency ω_1 , and they propagate as $e^{-j\beta_1 z}$ in the absence of the pump wave. Line 2 supports a similar mode¹ at frequency ω_2 ; this line usually represents propagation on the same physical structure. Now let a high-level pumping wave propagate to the right along the lines and modulate the nonlinear inductance as

$$L_p(z, t) = L_{p0}(\epsilon^{j(\omega_p t - \beta_p z)} + \text{c.c.}), \quad (11-1a)$$

where c.c. means the complex conjugate of the preceding term so that $L_p(z, t)$ is a real quantity. We assume the pumping level is so high that L_p is unaffected by the power converted to other frequencies along the interaction length of interest. In the presence of the pump wave, V_1 and I_1 at frequency ω_1 will not exactly vary as $e^{-j\beta_1 z}$, and so we write

$$\begin{aligned} V_1(z, t) &= V_1(z) \epsilon^{j(\omega_1 t - \beta_1 z)} + \text{c.c.} \\ I_1(z, t) &= I_1(z) \epsilon^{j(\omega_1 t - \beta_1 z)} + \text{c.c.} \end{aligned} \quad (11-1b)$$

with analogous expressions for V_2 and I_2 at frequency ω_2 .

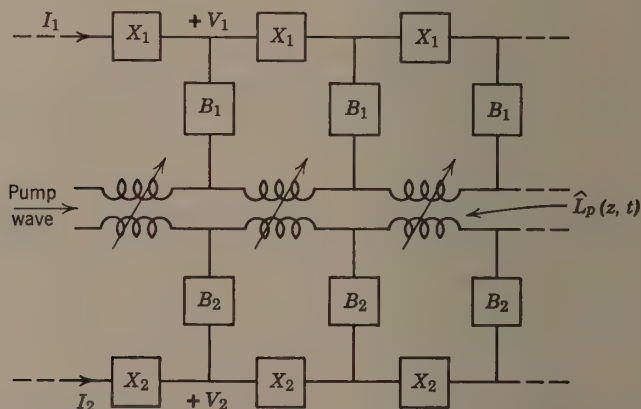


Fig. 11-1. Two coupled TEM transmission lines, in which the pumping wave modulates the common inductance between the lines.

¹ We distinguish the separate *modes* in the unpumped line which compose each wave of the pumped line. These terms are defined in the Introduction.

The transmission line equations for V_1 and I_1 are clearly

$$\frac{\partial}{\partial z} [V_1(z)\epsilon^{-j\beta_1 z}] = -jX_1 I_1(z)\epsilon^{-j\beta_1 z} - \frac{\partial}{\partial t} [L_p(z, t)I_2(z, t)]_{+\omega_1 \text{ comp}} \quad (11-2a)$$

$$\frac{\partial}{\partial z} [I_1(z)\epsilon^{-j\beta_1 z}] = jB_1 V_1(z)\epsilon^{-j\beta_1 z} \quad (11-2b)$$

At this point we must distinguish two states of operation. In the first one an applied frequency ω_1 higher than ω_p generates a difference frequency ω_2 and we neglect the sum frequencies $\omega_1 + \omega_p$ and $\omega_1 + \omega_2$. Interaction among the frequencies is most effective if the propagation constants of their uncoupled fields are related in the same way as the frequencies, whereupon this state is described by the following relations:

$$\text{I. } \omega_1 - \omega_p = \omega_2 > 0, \quad \beta_1 - \beta_p \cong \beta_2$$

The other state of operation is described by the relations

$$\text{II. } \omega_p - \omega_1 = \omega_2 > 0, \quad \beta_p - \beta_1 \cong \beta_2,$$

and we neglect the sum frequencies $\omega_p + \omega_1$ and $\omega_p + \omega_2$. State I is associated with inverter or mixer operation, in which power may be converted from the pump frequency to a higher frequency ω_1 in either a sinusoidal or an exponential manner depending on the relative group velocities at the two frequencies ω_1 and ω_2 . State II is associated with amplifier operation because pump power may be converted into the fields at the lower frequency ω_1 (or ω_2) either in a sinusoidal or an exponential manner without supplying additional power at the other frequency ω_2 (or ω_1). The ensuing discussion will clarify these points.

First, consider state I. The $+\omega_1$ component of $[L_p I_2]$ in Eq. 11-2a is just $L_{p0} I_2(z) \epsilon^{j(\omega_p + \omega_2)t - j(\beta_p + \beta_2)z}$. Now $\beta_p + \beta_2$ is not necessarily equal to β_1 because these are the propagation constants of the individual uncoupled frequencies, but we shall limit our discussion in this section to the case $\beta_p + \beta_2 = \beta_1$ because it is the most efficient from the point of view of exponential gain per unit length. If we differentiate Eq. 11-2b with respect to z , use Eq. 11-2a, and also assume

$$\frac{\partial^2 I_1(z)}{\partial z^2} \ll \frac{\beta_1 \partial I_1}{\partial z}$$

we obtain

$$\frac{\partial I_1(z)}{\partial z} = -j \frac{B_1 L_{p0} \omega_1}{2\beta_1} I_2(z) \quad (11-3)$$

By symmetrical reasoning, the equation for $\partial I_2(z)/\partial z$ is the same with interchange of subscripts 1 and 2. We now differentiate Eq. 11-3 once

more and use the equation for $\partial I_2/\partial z$ to arrive at

$$\frac{\partial^2 I_1(z)}{\partial z^2} = -\frac{\omega_1 \omega_2 B_1 B_2}{4\beta_1 \beta_2} L_{p0}^2 I_1(z) \quad (11-4)$$

We now must distinguish two substates of state I, depending on the signs of B_1 and B_2 . If B_1 and X_1 (which have the same signs in any case) have the same signs as B_2 and X_2 , the solution to Eq. 11-4 is

$$I_1(z) = a_1 e^{j\alpha z} + b_1 e^{-j\alpha z}$$

with

$$\alpha = \frac{1}{2} \left(\frac{\omega_1 \omega_2 B_1 B_2 L_{p0}^2}{\beta_1 \beta_2} \right)^{1/2} = \frac{1}{2} \left(\frac{\omega_1 \omega_2}{Z_{01} Z_{02}} \right)^{1/2} L_{p0}, \quad \text{real} \quad (11-5)$$

Therefore the complete solution for this substate such that $I_2(0)$, as obtained from Eq. 11-2 for $I_2(z)$ in terms of $I_1(z)$, is zero is

$$\begin{aligned} I_1(z, t) &= a \cos \alpha z \cos (\omega_1 t - \beta_1 z + \theta) \\ I_2(z, t) &= \left(\frac{\omega_2 B_2 \beta_1}{\omega_1 B_1 \beta_2} \right)^{1/2} a \sin \alpha z \sin (\omega_2 t - \beta_2 z + \theta) \end{aligned} \quad (11-6)$$

a and θ being the initial amplitude and phase of the injected $I_1(0, t)$.

These expressions reveal the fact that the power initially injected at frequency ω_1 converts to frequency ω_2 at distances $\alpha z = \pi/2, 3\pi/2, \dots$, and reverts to frequency ω_1 at distances $\alpha z = \pi, 2\pi, \dots$. The maximum power flows at these two frequencies are

$$\begin{aligned} P_{1, \max} &= \frac{1}{2} I_{1, \max}^2 Z_{01} = \frac{1}{2} a^2 Z_{01} \\ P_{2, \max} &= \frac{1}{2} I_{2, \max}^2 Z_{02} = \frac{1}{2} a^2 \frac{\omega_2 B_2 \beta_1}{\omega_1 B_1 \beta_2} Z_{02} \end{aligned} \quad (11-7)$$

Since $Z_0 = \sqrt{X/B} = \beta/B$ we have

$$\frac{P_{1, \max}}{\omega_1} = \frac{P_{2, \max}}{\omega_2} \quad (11-8)$$

This power relation is a special case of the Manley-Rowe relations [2, 3] which have become so famous we shall digress a moment to summarize them.

Suppose we have a lossless nonlinear medium characterized by macroscopic vector parameters $\hat{E}$ and $\hat{H}$ as well as polarizations $\hat{P}$ and $\hat{M}$ related to them as $\hat{P} = \hat{D} - \epsilon_0 \hat{E}$ and $\hat{M} = \hat{B}/\mu_0 - \hat{H}$, where $\hat{D}$ and $\hat{B}$ are the continuous electric and magnetic flux densities. Suppose further that $\hat{E}$ is a single-valued function of $\hat{P}$ (not necessarily in the same direction) and $\hat{H}$ is a single-valued function of $\hat{M}$ at every point. This often implies that the voltage $\oint \hat{E} \cdot d\vec{\ell}$ between any two points is a single-valued function of $\oint \hat{D} \cdot d\vec{\ell}$ along the same contour, and $\oint \hat{H} \cdot d\vec{\ell}$ around any contour is a

single-valued function of the flux $\int \hat{B} \cdot d\vec{S}$ through the contour. Under these conditions, there may be a number of fields at different frequencies specified by $m\omega_1 + n\omega_2$ for different positive and negative values of m and n . If P_{mn} denotes the net real power flow *out* of the medium at frequency $m\omega_1 + n\omega_2$, all the powers are constrained in two ways;

$$\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{mP_{mn}}{m\omega_1 + n\omega_2} = 0 = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{nP_{mn}}{m\omega_1 + n\omega_2} \quad (11-9)$$

Each excluded $-|m|$ term in the first expression, for example, is equal to a corresponding $+m$ term in the summation. Equations 11-9 are the Manley-Rowe relations. To obtain Eq. 11-8 for the three frequencies $\omega_p = 1\omega_1 - 1\omega_2$, ω_1 , and ω_2 we apply the relations to the region between two planes at $\alpha z = 0$ and $\pi/2$, whereupon we know $P_1(m=1, n=0) = -P_{1,\max}$ and $P_2(m=0, n=1) = P_{2,\max}$. The Manley-Rowe relations separately imply

$$\frac{P_1}{\omega_1} + \frac{P_p}{\omega_p} = 0 = \frac{P_2}{\omega_2} - \frac{P_p}{\omega_p}, \quad (11-10)$$

that is, $P_1/\omega_1 = -P_2/\omega_2$, which is Eq. 11-8.

Equation 11-8 tells us immediately that we must draw less power out at frequency ω_2 than we injected at frequency ω_1 since ω_2 is less than ω_1 ; the remainder of the power flows into the pumping fields. However, if we had injected power at frequency ω_2 and not at ω_1 , the sine and cosine dependences would just interchange in Eq. 11-6 and we could draw more power out at frequency ω_1 than we injected at frequency ω_2 . Power would then flow into the system by way of the pumped inductance.

Now we consider the other substate of state I operation, for which B_1 and B_2 in the transmission lines have opposite signs and the group velocities at frequencies ω_1 and ω_2 are opposed. The sign on the right side of Eq. 11-4 changes to plus and exponential waves result. If power is injected at frequency ω_1 and $I_2(0, t) = 0$, the currents for all z and t are now

$$\begin{aligned} I_1(z, t) &= a \cosh \alpha z \cos(\omega_1 t - \beta_1 z + \theta) & B_1 > 0 \\ I_2(z, t) &= -a \left(\frac{\omega_2}{\omega_1} \left| \frac{B_2}{B_1} \right| \frac{\beta_1}{\beta_2} \right)^{1/2} \sinh \alpha z \sin(\omega_2 t - \beta_2 z + \theta), & B_2 < 0. \end{aligned} \quad (11-11)$$

The Manley-Rowe relation $P_1/\omega_1 = P_2/\omega_2$ is true for any section of the line from $z = 0$ to $z = z$. Now the power flows grow exponentially at both frequencies, fed by the pumped inductance. We have assumed the best case of $\beta_p - \beta_1 = \beta_2$; if the synchronism is off by $\Delta\beta$ Tien shows

that the growth constant will change to

$$\alpha = \left[\alpha_0^2 - \frac{(\Delta\beta)^2}{4} \right]^{1/2} \quad (11-12)$$

and the *perturbed* propagation constants β_1' and β_2' will satisfy the relation $\beta_p = \beta_1' + \beta_2'$ exactly.

This completes our review of state I operation. State II operation in which the group velocities at the two frequencies ω_1 and ω_2 oppose each other is identical to state I operation in which the group velocities are in the same direction (Eq. 11-6 is now written with $|B_2/B_1|$). State II operation in which the group velocities both lie in the same direction is identical to state I operation in which they are opposed, Eq. 11-11. Here is a summary of the various states of operation:

Sinusoidal waves:

- I. $\omega_p = \omega_1 - \omega_2$, group velocities in the same direction
- II. $\omega_p = \omega_1 + \omega_2$, group velocities in opposite directions

(11-13)

Exponential waves:

- I. $\omega_p = \omega_1 - \omega_2$, group velocities in opposite directions
- II. $\omega_p = \omega_1 + \omega_2$, group velocities in the same direction

Each wave includes both ω_1 and ω_2 components of the field which propagate together according to the relation $\beta_p = \beta_1' \pm \beta_2'$ (β_1' and β_2' being, in general, the perturbed propagation constants), depending on the relative frequencies in the expression $\omega_p = \omega_1 \pm \omega_2$.

We should observe that Tien's solution neglects the two backward circuit *modes* because of the approximation preceding Eq. 11-3; each of these modes has ω_1 or ω_2 components propagating as $e^{j(\omega_1 t + \beta_1 z)}$ or $e^{j(\omega_2 t + \beta_2 z)}$, respectively. Strictly speaking, each of the two independent waves found by Tien, either of constant amplitude or of a growing-decaying pair, contains weak components of the uncoupled backward mode at both frequencies ω_1 and ω_2 . In addition, there are two more independent waves, each of which has a strong component of backward mode at frequency ω_1 or ω_2 and weak components of the other backward mode and of the forward modes at both those frequencies. We can solve for all four propagation constants of these four independent waves by substituting $I_1(z, t) = I_{1+} e^{j(\omega_1 t - \beta_1' z)} + \text{c.c.}$, and so on, into the transmission line Eq. 11-2 for the excitations at ω_1 and ω_2 whereupon a quartic equation for the perturbed β_1' develops with the aid of the relation $\beta_p = \beta_1' \pm \beta_2'$. Each of the allowed values of β_1' determines the associated β_2' and the relative amplitudes $I_{1+}, \dots, I_{2-}$ of the four components in a given wave. With weak parametric coupling and a transmission line output

matched at frequencies ω_1 and ω_2 these extra components in the forward waves are not important and the backward waves are not excited, but in the parametrically operated cavity chain all four independent waves are required, in general, to match the boundary conditions at frequencies ω_1 and ω_2 in the first and last cavities.

We close this section by demonstrating the simple theory of the low noise figures obtainable in the parametric line.² Suppose the line is operated for exponential gain, with the matched input at frequency ω_1 of temperature T_1 , the matched input at frequency ω_2 at T_2 , and both matched outputs at temperature T_L . We utilize the single-sideband gain from frequency ω_2 at the input to ω_1 at the output so the noise figure F is

$$F = \frac{(S/N)_{\text{in},2}}{(S/N)_{\text{out},1}} = \frac{N_{\text{out},1}}{N_{\text{in},2}} \frac{1}{G_{21}} \quad (11-14)$$

where N , S , and G_{21} are the noise power, signal power, and power gain from the input at frequency ω_2 to the output at frequency ω_1 , respectively. To obtain G_{21} we observe that the gain G of the amplifier is the same for the fields at frequency ω_1 as for the ω_2 fields; the ratio of signal power out at frequency ω_2 to that entering at ω_2 is

$$G \triangleq \frac{S_{\text{out},2}}{S_{\text{in},2}} \quad (11-15)$$

Thus, the net output power at frequency ω_2 is $S_{\text{in},2}(G - 1)$. From the Manley-Rowe Eq. 11-8 it follows that the net signal power out at ω_1 is

$$S_{\text{out},1} = \frac{\omega_1}{\omega_2} S_{\text{in},2}(G - 1)$$

Since no signal power enters the line at frequency ω_1 ,

$$G_{21} = \frac{\omega_1}{\omega_2} (G - 1) \quad (11-16)$$

The noise power into the matched load at ω_2 is $N_{\text{in},2} = kT_2B$, B being the receiver noise bandwidth. The total noise out of the line at frequency ω_2 is $kT_2B(G - 1)$, and by the same reasoning which gave us Eq. 11-16 the associated noise out at frequency ω_1 , excluding the entering kT_1B noise, is $(\omega_1/\omega_2)kT_2B(G - 1)$. Thus, the total noise out at frequency ω_1 is

$$N_{\text{out},1} = \frac{\omega_1}{\omega_2} kT_2B(G - 1) + kT_1BG \quad (11-17)$$

² W. H. Louisell [4, p. 142]. He discusses the single-sideband gain from frequency ω_1 at the input to frequency ω_1 at the output.

Therefore

$$F = 1 + \frac{\omega_2 T_1}{\omega_1 T_2} \frac{G}{G - 1} \quad (11-18)$$

In high-gain amplifiers $F \cong 1$ provided $T_1 \ll T_2$ and/or $\omega_2 \ll \omega_1$.

With this introduction to parametrically operated slow wave structures we turn now to the ferroelectric parametric amplifier.

11-3. The Three-Frequency Ferroelectric Parametric Amplifier

Consider the symmetric cavity chain shown schematically in Fig. 11-2 with lossless ferroelectric pills distributed throughout each cavity and with mirror symmetry about a transverse bisecting plane through the cavity. Such an arrangement guarantees symmetric open- and short-circuit resonant cavity mode patterns on the coupling surfaces, so that each of these modes will continue to represent a physical field at one of the cutoff frequencies in the absence of the pump excitation. The coupling surfaces are depicted as edge apertures only for definiteness. For the sake of generality, let frequency ω_1 lie in the m th passband and ω_2 lie in the n th passband (n may equal m) of the unpumped structure, and $\omega_p = \omega_1 + \omega_2$. The electric fields at these two frequencies are represented, in the single-passband approximation, by the m th and n th mode-pairs, respectively, and these electric fields are usually aligned within the ferroelectrics for best interaction. If the pump frequency ω_p lies in a stopband, external pumping is necessary to establish the requisite real phase shift φ_p in the pump fields between consecutive cavities. However, external pumping is advantageous for it allows us to vary the relative pump phases and enhance the performance somewhat, as we mention later. Frequencies $\omega_p + \omega_1$ and $\omega_p + \omega_2$ are presumed to lie deep within stopbands, so this device is essentially a three-frequency one.

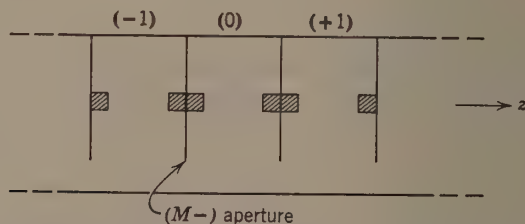


Fig. 11-2. Schematic diagram of a symmetric cavity chain, loaded periodically and symmetrically by ferroelectric pills.

We can define the solenoidal short- and open-circuit modes of Eqs. 5-5 and 5-10 with respect to Hermitian (lossless) dyadic permittivity ϵ_r and permeability μ_r merely by writing $\epsilon_r(\bar{r}) \cdot \bar{E}_m(\bar{r})$ in place of $\epsilon_r \bar{E}_m$, for example, whereupon the orthogonality statement, Eq. 5-9, involves

$$\int_V \bar{E}_m^\dagger \cdot \epsilon_r \cdot \bar{E}_n dV \quad \text{and} \quad \int_V \bar{H}_m^\dagger \cdot \mu_r \cdot \bar{H}_n dV$$

($\dagger$ denotes a transpose conjugate, in this case of column vectors $\bar{E}_m = \text{column}(\bar{E}_{mx}, \bar{E}_{my}, \bar{E}_{mz})$ and $\bar{H}_m$. For lossless media $\epsilon_r^\dagger = \epsilon_r$ and $\mu_r^\dagger = \mu_r$.) The normalized resonant frequencies P_m and p_m remain real but the patterns $\bar{E}_m$, and so on, may be complex. The open- and short-circuit solenoidal mode-pair so defined for the m th (or n th) passband furnishes a good description of the fields even if the ferroelectric is lossy, provided the resonant frequencies of the neighboring modes are far from the operating frequency as compared to the resonant frequencies of the m th (or n th) mode-pair.

Let us split the dyadic permittivity into an unpumped and a pumped portion,

$$\hat{\epsilon} = \epsilon_0 \epsilon_r + \hat{\epsilon}_p \quad (11-19a)$$

in which that portion of the pumped ϵ_p seen by the fields at frequency $(m\omega_p + s\omega_1)$ may contain more than one harmonic of the pumped field in general,

$$\hat{\epsilon}_p(m\omega_p + s\omega_1) = \sum_h \epsilon_{p,h}(m\omega_p + s\omega_1; h\omega_p) \epsilon^{jh\omega_p t} \quad (11-19b)$$

$\epsilon_{p,h}$ is the dyadic "seen" by the fields at this frequency $m\omega_p + s\omega_1$, varying at the frequency $h\omega_p$. We assume the pumping field is large compared to the ω_1 and ω_2 fields, so that each pump harmonic $\epsilon_{p,h}(\pm\omega_1; h\omega_p)$ "seen" by the $\pm\omega_1$ fields and $\epsilon_{p,h}(\pm\omega_2; h\omega_p)$ "seen" by the $\pm\omega_2$ fields are implicitly known and are not affected by the amplitudes of the ω_1 and ω_2 fields. The general frequency in the system is $m\omega_p \pm \omega_1$ ($s = \pm 1$) because the predominant harmonics include those resulting from the mixing of the low-level fields at frequency ω_1 with the high-level pump fields at frequencies $h\omega_p$, namely $h\omega_p \pm \omega_1$. These mix with the pump harmonics and produce comparable fields at the frequencies $h'\omega_p \pm \omega_1$ and so on.

The fields at frequency ω_1 are represented by the modes of the m th passband in which ω_1 lies as

$$\hat{e}(\omega_1) = v_m(1) \bar{e}_m \epsilon^{j\omega_1 t} + \text{c.c.} \quad (11-20a)$$

$$\hat{h}(\omega_1) = i_m(1) \bar{h}_m \epsilon^{j\omega_1 t} + \text{c.c.} \quad (11-20b)$$

$$\hat{H}(\omega_1) = I_m(1) \bar{H}_m \epsilon^{j\omega_1 t} + \text{c.c.} \quad (11-20c)$$

The fields at frequency ω_2 are similarly represented by the modes of the n th passband within which ω_2 lies. Hereafter we write $v_m(1)$ at frequency ω_1 and $v_m(-1) = v_m(1)^*$ at $-\omega_1$ as just v_m and v_m^* , respectively, for brevity. We expand the fields of Maxwell's equations with these open-circuit modes as driven by the (solenoidal) magnetic field on the coupling surfaces. This latter field is given by Eq. 11-20c, and we make the usual narrow-band single mode-pair approximation that $I_m \cong i_m$ and $I_n \cong i_n$. The equations at frequency $+\omega_1$ are

$$\begin{aligned}\nabla \times \hat{e}(\omega_1) &= -j\omega_1\mu\hat{h}(\omega_1) \\ \nabla \times \hat{h}(\omega_1) &= \frac{\partial}{\partial t}(\hat{\epsilon} \cdot \hat{e})_{+\omega_1}\end{aligned}\quad (11-21)$$

The procedure in Section 4-6 of evaluating amplitudes v_m and i_m yields straightforwardly, for $\omega_p = \omega_1 + \omega_2$,

$$p_mv_m = -j\omega_1\mu i_m, \quad \mu \text{ is a scalar here} \quad (11-22a)$$

$$\begin{aligned}p_mi_m &= -M_{mm}^*(1 \pm \cosh \gamma_1)I_m + j\omega_1\epsilon_0 v_m \\ &\quad + j\omega_1 \frac{1}{\tau} \int_{\text{ferro}} \bar{e}_m^{\dagger} \cdot \epsilon_{p,+1}(-1, 1; +\omega_p) \cdot v_n^* \bar{e}_n^* dV, \quad (11-22b)\end{aligned}$$

with

$$M_{mm} = \frac{1}{\tau} \int_{M-} \bar{e}_m \times \bar{H}_m^* \cdot \bar{i}_z dS \quad \begin{array}{l} \text{on the } (M-) \text{ coupling} \\ \text{surface of Fig. 11-2} \end{array} \quad (11-22c)$$

$$I_m \cong i_m \quad (11-22d)$$

Here the v_m and i_m amplitudes at frequency ω_1 phase shift from one cavity to the next as $\epsilon^{-\gamma_1}$ in the $+z$ direction. The electric displacement flux created by the $-\omega_2$ fields at frequency $+\omega_1$ is $\epsilon_{p,+1}(-1, 1; +\omega_p) \cdot \hat{e}(-2) = \epsilon_{p,+1}(-\omega_2; +\omega_p) \cdot \hat{e}(2)^*$.

Analogously we can develop the expressions for amplitudes v_n^* and i_n^* at frequency $-\omega_2$, as coupled to v_m :

$$p_nv_n^* = +j\omega_2\mu i_n^* \quad (11-23a)$$

$$\begin{aligned}p_ni_n^* &= -M_{nn}(1 \pm \cosh \gamma_2^*)I_n^* - j\omega_2\epsilon_0 v_n^* \\ &\quad - j\omega_2 \frac{1}{\tau} \int_{\text{ferro}} \bar{e}_n^t \cdot \epsilon_{p,-1}(0, +1; -\omega_p) \cdot v_m \bar{e}_m dV \quad (11-23b)\end{aligned}$$

$$I_n \cong i_n \quad (11-23c)$$

where

$$\gamma_{-2} \text{ (the phase shift of the } -\omega_2 \text{ fields)} = \gamma_2^* \quad (11-23d)$$

The superscript t denotes a transpose of column vector $\bar{e}_n$. The fields at

frequency $0\omega_p + \omega_1 = \omega_1$ create, through $\epsilon_{p,-1}(0, +1; -\omega_p)$, electric displacement flux at frequency $-\omega_2$. In these equations we have made the narrow-passband approximation of

$$T_{mm} = \frac{1}{\tau} \int_V \bar{E}_m^\dagger \cdot \epsilon_r \cdot \bar{e}_m dV \cong T_{nn} \cong 1 \quad (11-24)$$

with small percentage error.

Equations 11-22 combine to give us one relation for i_m as related to i_n^* , and Equations 11-23 combine to yield another relation. Therefore we can obtain one equation for i_m , say, in terms of itself, whereupon i_m cancels and leaves this determinantal equation for the allowed values of γ_1 :

$$[p_m^2 - k_1^2 + p_m M_{mm}^* (1 \pm \cosh \gamma_1)] \\ \times [p_n^2 - k_2^2 + p_n M_{nn} (1 \pm \cosh \gamma_2^*)] = |w|^2$$

with

$$w \triangleq \omega_1 \omega_2 \mu_0 \epsilon_0 \frac{1}{\tau} \int_{\text{ferro}} \bar{e}_m^\dagger \cdot \frac{\epsilon_p}{\epsilon_0} (-\omega_2; +\omega_p) \cdot \bar{e}_n^* dV \quad (11-25)$$

In deriving this equation we have employed the equality $\epsilon_{p,+1}(-1, 1; +\omega_p) \equiv \epsilon_{p,+1}(-\omega_2; \omega_p) = \epsilon_{p,-1}^\dagger(0, 1; -\omega_p) \equiv \epsilon_{p,-1}^\dagger(\omega_1; -\omega_p)$, which may be readily verified for lossless media with the aid of the Manley-Rowe relations. Therefore the two $\int_{\text{ferro}}$ integrals in Eqs. 11-22b and 11-23b are Hermitian conjugates of each other. The upper sign is taken in either of the brackets of Eq. 11-25 if the open-circuit mode of that passband resonates at the nonparametric $\varphi = \pi$ phase shift frequency; otherwise, the lower sign is taken.

The phase shift γ_2^* is related to γ_1 by the fact that the coupled-amplitude equations must be invariant with respect to the cavity for which they are written. If we write Eq. 11-22b, for example, in the next cavity, v_m and i_m each gain a factor of $\epsilon^{-\gamma_1}$, $\epsilon_p(-\omega_2; +\omega_p)$ gains a factor of $\epsilon^{-j\varphi_p}$ (φ_p real), and v_n^* becomes $v_n^* \epsilon^{-\gamma_2^*}$. It follows that

$$\gamma_1 = j\varphi_p + \gamma_2^* \quad (11-26)$$

Equation 11-25 is a quartic equation in $\epsilon^{-\gamma_1}$, the solutions to which yield the four values of γ_1 associated with each independent wave of the infinitely long structure. Once we know the four values of γ_1 we know, from either Eq. 11-22 or Eq. 11-23, the relative amplitudes v_m , i_m , v_n^* , and i_n^* comprising each of the waves. Then we can select the overall amplitudes of these various waves so as to satisfy the terminal boundary conditions. Before proceeding with a description of the waves we pause to note that the equations do imply the Manley-Rowe relations among the net powers flowing into a cavity, regardless of the number of waves present.

To appreciate this, let subscript α denote the α th wave of amplitudes $v_{m\alpha}$ and $i_{m\alpha}$ at frequency ω_1 and amplitudes $v_{n\alpha}$ and $i_{n\alpha}$ at ω_2 in Eq. 11-22. Multiply Eq. 11-22a by $i_{m\beta}^*$ of the β wave and Eq. 11-22b by $v_{m\beta}^*$, subtract the two, sum over α and β from 1 to 4, and take the real part. The terms remaining are ($I_{m\alpha} \cong i_{m\alpha}$)

$$\begin{aligned} & \operatorname{Re} \sum_{\alpha\beta} v_{m\beta}^* \tau M_{mm}^* (1 \pm \cosh \gamma_{1\alpha}) i_{m\alpha} \\ &= \operatorname{Re} \int_{\text{ferro}} \left(\sum_{\beta} v_{m\beta}^* \bar{e}_m^\dagger \right) \cdot \frac{\partial}{\partial t} \left(\hat{\epsilon}_p \cdot \sum_{\alpha} v_{n\alpha}^* \bar{e}_n^* \right) \Big|_{+\omega_1} dV \quad (11-27) \end{aligned}$$

The right side is the power flowing into the ferroelectric at frequency $+\omega_1$. In the left side we replace $i_{m\alpha}$ by $p_m v_{m\alpha} / -j\omega_1 \mu$ and observe that only the $\cosh \gamma_{1\alpha}$ terms contribute to the real sum, M_{mm} being nearly pure real. The left side now reads

$$\mp \operatorname{Re} \frac{p_m M_{mm}^* \tau}{j2\omega_1 \mu} \sum_{\alpha\beta} v_{m\alpha} v_{m\beta}^* (\epsilon^{\gamma_{1\alpha}} + \epsilon^{-\gamma_{1\alpha}})$$

Comparison with Eq. 9-22 for the beam tube indicates that this is indeed the total power flow carried into the cavity by all the wave components at frequency $+\omega_1$. The discrepancy of a factor of 2 arises because of our Eq. 11-20 expressions for the fields.

In order to solve Eq. 11-25 in terms of the phase shifts φ_1^0 on the m th ω - φ curve at frequency k_1 , and φ_2^0 on the n th ω - φ curve at frequency k_2 , we write the defining equations for these curves:³

$$\begin{aligned} p_m^2 - k_1^2 + p_m M_{mm}^* (1 \pm \cos \varphi_1^0) &= 0 \\ p_n^2 - k_2^2 + p_n M_{nn} (1 \pm \cos \varphi_2^0) &= 0 \end{aligned} \quad (11-28)$$

with which Eq. 11-25 can be written in the form

$$(\cosh \gamma_1 - \cos \varphi_1^0)(\cosh \gamma_2^* - \cos \varphi_2^0) = \frac{|w|^2}{(\pm p_m M_{mm}^*)(\pm p_n M_{nn})} \triangleq W \quad (11-29)$$

$$\gamma_2^* = \gamma_1 - j\varphi_p \quad (\text{Eq. 11-26})$$

For practical purposes M_{mm}^* is a real number, and so the dimensionless parametric coupling coefficient W is real. The coefficient W is positive if both dispersion curves have the same slopes over the range $0 \leq \varphi \leq \pi$ and is negative if they have opposite group velocities. The significance of W is apparent if we substitute the good approximation $k_1 \cong \omega_1 \sqrt{\mu \epsilon_0} \cong p_m$

³ More precisely, the real coefficients M_{mm}^*/U_{mm}^* and M_{nn}/U_{nn} occur here. But the U factors do not enter into dispersion Eq. 11-29.

and $k_2 \cong p_n$ into Eq. 11-25 for w . Then

$$W \cong \frac{1}{(\pm M_{mm}^*/p_m)(\pm M_{nn}/p_n)} \left| \frac{1}{\tau} \int_{\text{ferro}} \bar{e}_m^\dagger \cdot \frac{1}{\epsilon_0} \epsilon_p(-\omega_2; \omega_p) \cdot \bar{e}_n^* dV \right|^2 \quad (11-30)$$

in which M_{mm}^*/p_m is essentially the fractional passband width of the m th passband in which k_1 lies and M_{nn}/p_n is the n th fractional passband width for k_2 .

The four values of γ_1 and associated γ_2 which obey Eq. 11-29 for particular values of φ_1^0 , φ_2^0 , and W (some of which are listed in Table 11-1) have these properties:

1. $\text{Im } \gamma_1 + \text{Im } \gamma_2 \equiv \varphi_p$ in each wave *always*, even though $\varphi_1^0 + \varphi_2^0$ may not equal φ_p .
2. Interchange of φ_1^0 and φ_2^0 merely interchanges γ_1 and γ_2 in each wave. γ_1 and γ_2 remain unchanged if only the sign of φ_1^0 or φ_2^0 , or both, is changed, and the sign of W must not be changed.
3. If $y_1 = \epsilon^{-\gamma_1}$ and $y_2 = \epsilon^{-\gamma_2}$ are the phase factors for a particular wave, another wave exists with the phase factors $y_1' = 1/y_1^*$ and $y_2' = 1/y_2^*$. Therefore a growing wave always implies the existence of another independent decaying wave.
4. If $(y_1)_a, \dots, (y_1)_d$ denote the phase factors for the ω_1 fields in the four waves, their product $(y_1)_a \dots (y_1)_d = \epsilon^{-j2\varphi_p}$.
5. Examination of the real and imaginary portions of Eq. 11-29 reveals the fact that, for $W > 0$ and maximum growth in the forward direction we want $\text{Im } \gamma_{1,2} \equiv \varphi_{1,2}^0$, attained if $\varphi_1^0 + \varphi_2^0 \equiv \varphi_p$ and $\sin \varphi_1^0 \sin \varphi_2^0 > 0$. Such a situation corresponds to Tien's fourth state of operation in Eq. 11-13. If, instead, $W < 0$ so that the group velocities are opposed at corresponding points on the dispersion curves we can still have exponential growth for $\omega_p = \omega_1 + \omega_2$ which will be a maximum provided $\varphi_1^0 + \varphi_2^0 \equiv \varphi_p$ but now $\sin \varphi_1^0 \sin \varphi_2^0 < 0$.

This last state of operation does not correspond to one of Tien's states. Let us illustrate the case of $W < 0$ by taking $M_{mm} < 0$ and $M_{nn} > 0$; then if *either* $\sin \varphi_1^0 > 0$ and $\sin \varphi_2^0 < 0$ or $\sin \varphi_1^0 < 0$ and $\sin \varphi_2^0 > 0$, the power flows at the two frequencies will be in the *same* direction, because power flow in the $+z$ direction is proportional to $-M \sin \varphi$. Gain can then occur. To interpret a growing or decaying wave physically we must always examine the directions of the power flows at each frequency; the power flow in the $+z$ direction is always proportional to $-M \sin (\text{Im } \gamma)$, γ being the phase lag from cavity (n) to cavity ($n + 1$).

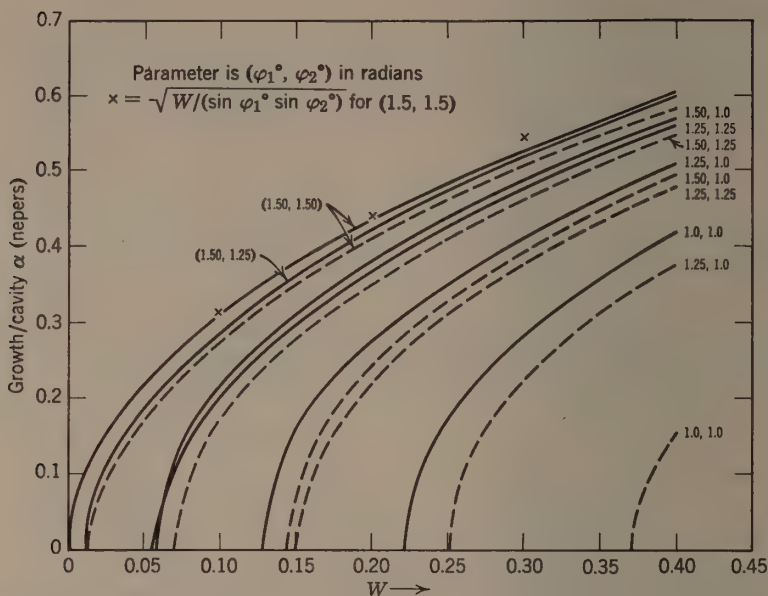


Fig. 11-3. Growth constant α of the one or two waves in the ferroelectric parametric amplifier, as a function of parametric coupling parameter W . The pumping wave traveling in the $+z$ direction phase shifts by $\varphi_p = 3.0$ radians per cavity. For $\omega_p = \omega_2 + \omega_1$ a solid line denotes power flow at both frequencies ω_1 and ω_2 in the $+z$ direction if both M_{mm} and M_{nn} are <0 ; in the $-z$ direction if both are >0 . A dashed line denotes power flow at both frequencies in the $-z$ direction if both M_{mm} and M_{nn} are <0 ; in the $+z$ direction if both are >0 .

For either situation $W \geq 0$ discussed in the preceding paragraph the approximate growth factor per cavity is

$$\alpha_{\max} \text{ (neper)} \cong \left| \frac{W}{\sin \varphi_1^0 \sin \varphi_2^0} \right|^{1/2} \varphi_1^0 \text{ and } \varphi_2^0 \neq n\pi \quad (11-31)$$

This expression develops error as $\alpha_{\max}$ increases but it is often surprisingly accurate up to 0.4 or so, as Fig. 11-3 shows. (The entry $\varphi_p = 4.0$, $\varphi_1^0 = 3.0$, $\varphi_2^0 = 1.0$, $W = 0.05$ in Table 11-1 is one exception.)

We can put $\alpha_{\max}$ into a form which involves the cavity impedances as they are naturally defined at frequencies ω_1 and ω_2 . By analogy with the cavity chain beam tube (Chapter 8) we define the interaction impedance at ω_1 in the m th passband as

$$Z_{01} = \frac{|\bar{e}_m|_{\text{ferro}}^2}{2(\beta_1^0)^2 |P_{av}|_1} = \frac{2|\bar{e}_m|_{\text{ferro}}^2}{(\beta_1^0)^2 \omega_1 \epsilon_0 |M_{mm}/p_m| \tau |\sin \varphi_1^0|}, \quad \varphi_1^0 = \beta_1^0 L, \quad (11-32)$$

in which we have used Eq. 8-32 for $(P_{av})_1$ and the approximation $P_m/\omega\mu \cong \omega\epsilon_0/p_m$. Impedance Z_{02} is defined the same way with respect to the parameters of the n th passband. Therefore $\alpha_{\max}^2$ in terms of W of Eq. 11-30 is

$$\alpha_{\max}^2 = \frac{\omega_1(\beta_1^0)^2 Z_{01} \omega_2(\beta_2^0)^2 Z_{02}}{4} \frac{\left| \int_{\text{ferro}} \bar{e}_m^\dagger \cdot \epsilon_p(-\omega_2; \omega_p) \cdot \bar{e}_n^* dV \right|^2}{|\bar{e}_m^2 \bar{e}_n^2|_{\text{ferro}}} \quad (11-33)$$

Equation 11-31 indicates an interesting gain-passband width conservation relation when $\alpha_{\max}$ is small, because W of Eq. 11-30 is inversely proportional to the product of the two fractional passband widths. The statement of this principle is

Parametric coupling of a frequency ω_1 in one narrow passband to one other frequency ω_2 in another narrow passband via the pump frequency ω_p , with $\omega_p = \omega_1 + \omega_2$ implies a maximum gain-fractional passband width (FPW) conservation of the form

$$\alpha_{\max} \left(\frac{\text{growth}}{\text{cavity}} \text{ in nepers} \right) \sqrt{(\text{FPW})_1 (\text{FPW})_2} = \frac{\left| \frac{1}{\epsilon_0} \int_{\text{ferro}} \bar{e}_m^\dagger \cdot \epsilon_p(-\omega_2; \omega_p) \cdot \bar{e}_n^* dV \right|^2}{\tau \sqrt{|\sin \varphi_1^0 \sin \varphi_2^0|}},$$

$$\alpha_{\max} \ll \sqrt{2} \quad (11-34)$$

where $\alpha_{\max}$ occurs for the condition $\varphi_p \equiv \varphi_1^0 + \varphi_2^0$. Equation 11-34 is valid whether the dispersion curves of the two passbands have group velocities in the same or in opposing directions. If, instead, $\omega_p = \omega_1 - \omega_2$, the optimum phase condition should read $\varphi_p = \varphi_1^0 - \varphi_2^0$.

Observe that this conservation relation is stated in terms of nonparametric passband widths and not the parametric *bandwidth* which depends upon the external loads and the lesser of the two passband widths. Equation 11-34 tends to be optimistic at higher values of $\alpha_{\max}$; the entries for $\varphi_1^0 = 3.0$ and $\varphi_2^0 = 1.0$ in Table 11-1 show that, as W increases, $\alpha_{\max}$ does not increase as fast as Eq. 11-31 specifies, and therefore the true $\alpha_{\max} \sqrt{(\text{FPW})_1 (\text{FPW})_2}$ is lower than specified by Eq. 11-34.

Let us now consider a state of operation for which we might at first expect to find $\alpha_{\max}$ quite large. We choose k_1 such that $\varphi_1^0 \equiv \pi$ at the edge of the m th passband, k_2 such that $\varphi_2^0 \neq n\pi$, and pump phase shift $\varphi_p = \varphi_1^0 + \varphi_2^0$. For either case of W the growth factor $\alpha_{\max}$ for that

growing-decaying wave pair with values of γ_1 which are perturbations of $j\varphi_1^0 = j\pi$ is, for small $\alpha_{\max}$,

$$\alpha_{\max} \cong \frac{\sqrt{3}}{2} \left| \frac{2W}{\sin \varphi_p} \right|^{1/2}, \quad \alpha_{\max} \ll \frac{1}{2} \quad \text{assumed} \quad (11-35)$$

Equation 11-35 is often a good approximation out to $\alpha_{\max} \approx 0.30$.

These remarks should not be interpreted to mean that the detailed nature of the four waves for the state $W > 0$ and $\sin \varphi_1^0 \sin \varphi_2^0 > 0$ is similar to the nature of the waves for the state $W < 0$ and either φ_1^0 or φ_2^0 negated. For instance, the state of operation for which $\varphi_p = \varphi_1^0 + \varphi_2^0$ with $\varphi_1^0 = \pi$ and $\varphi_2^0 = \pi/6$, $W > 0$, has waves quite different from the same state with $W < 0$ and φ_2^0 regarded as $-\pi/6$. The former state of operation has two growing-decaying wave pairs because the synchronism condition, in addition to being satisfied in the forward direction is sufficiently well satisfied in the backward direction in the sense that $\varphi_p^- = 5\pi/6 \approx \pi + \pi/6$. The latter state of operation unexpectedly has only *one* growing-decaying wave pair corresponding to the synchronism condition $\varphi_p^- = 5\pi/6 = \pi - \pi/6$ but does not develop another pair associated with synchronism in the forward direction. In each of these growing or decaying waves $\text{Im } \gamma_1$ and $\text{Im } \gamma_2$ sum to φ_p and also make the powers at both frequencies flow in the same direction as required by the Manley-Rowe relations.

The aforementioned properties of all the waves apply if $\omega_p = \omega_1 + \omega_2$. The effect of changing to the condition $\omega_p = \omega_1 - \omega_2$ is to couple the fields at frequency ω_1 to the fields at $+\omega_2$; Eqs. 11-22 and 11-23 are written in terms of $v_n(2)$, $i_n(2)$, and γ_2 and the relation among the γ 's is $\gamma_1 - \gamma_2 = j\varphi_p$. Examination of Eq. 11-29 shows us that for maximum (small) growth factor we want the synchronism condition $\varphi_p = \varphi_1^0 - \varphi_2^0$. It follows that, for maximum growth factor and $W > 0$, we now want $\sin \varphi_1^0 \sin \varphi_2^0 < 0$; physically this means the dispersion curves have the same shapes but the power flows are opposed. If $W < 0$ we want the sine product to be greater than 0, so that the power flows remain opposed. Then for low $|W|$, $\varphi_1^0 \cong \text{Im } \gamma_1$, $\varphi_2^0 \cong \text{Im } \gamma_2$, and the power flows in the two frequency components of the growing (or decaying) wave oppose each other in either case. The first-mentioned state of operation is not the analog of any of Tien's states; the $W < 0$ state is the analog of his third state in Eq. 11-13. The Manley-Rowe relations tell us that the power flows at frequencies ω_1 and $\omega_2 < \omega_1$ must be opposed in this state because we cannot extract power at frequency $\omega_1 > \omega_p$ without supplying some power at frequency ω_2 .

Now let us take the condition $\omega_p = \omega_1 + \omega_2$ again, for definiteness, and

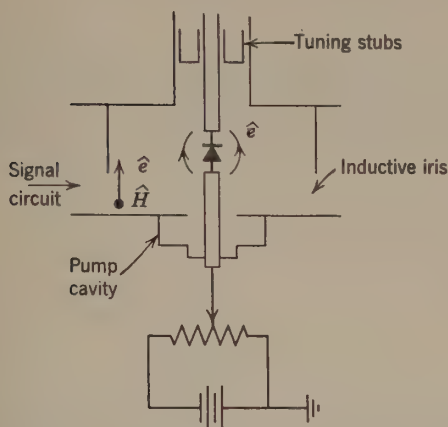


Fig. 11-4. Schematic diagram of a resonant cavity containing a diode in the coupled-cavity traveling wave parametric amplifier. At zero phase shift per cavity in the first passband an open-circuit cavity mode resonates with the apertures magnetically shorted and strong $\hat{e}$ field across the diode. At π phase shift per cavity in the first passband the short-circuit resonant cavity mode also has strong $\hat{E}$ field across the diode.

discuss the design principles we should adopt in order to obtain unilateral gain from an amplifier. Currie and Gould [5] have discussed this point and calculated the gain vs. frequency responses of various six-cavity amplifiers for different pump frequencies, pump phase shifts, pump amplitudes, and output loads. They consider states in which both ω_1 and ω_2 are in the first passband of a slow wave structure resembling Fig. 11-4 but loaded with variable-capacitance diodes at positions of maximum electric field within the cavities. Here $\omega_1 = \omega_2 = \omega_p/2$ occurs at a "cold" circuit phase shift of about $\varphi_1^0 = \varphi_2^0 \cong 1.2$ radians. The pump phase shift is 2.9 radians/cavity. When ω_1 and therefore ω_2 vary over about two-thirds of the passband width of 10% the transducer gain varies between 10 and 15 db when the output load is matched to the one growing wave. Gould's coupling coefficient ε_G equal to our $|WM_{mm}M_{nn}/(p_{mm}p_{nn})|^{1/2}$ is not large enough to allow two growing waves in the system.

In Fig. 11-3 we present a graphical description of the growth factors as computed by Eq. 11-29 for $\varphi_p = 3.0$ and for various values of φ_1^0 , φ_2^0 , and positive W . The solid curves show the behavior of growth factor α for these waves which occur because the synchronism condition $\varphi_p = 3.0 \approx \varphi_1^0 + \varphi_2^0$ is approximately satisfied in the forward direction; the dashed curves show the lesser growth factors for the waves which exist because the condition $\varphi_p^- = 3.28 \approx \varphi_1^0 + \varphi_2^0$ is almost as well satisfied in the reverse direction. A typical value of W is 0.10, corresponding to Gould's $\varepsilon_G = 0.032$ for fractional passband widths $-M/p$ of 0.10. We see the risks of

Table 11-1. The 4 Values of $\gamma_1 = \epsilon^{-\gamma_1}$ at Frequency ω_1 for the 4 Waves in the Ferroelectric Parametric Amplifier. $W > 0$ and $\varphi_p = 4.0^*$

φ_1^0	φ_p^0	W	$(y_1)_a$		α	$(y_1)_b$		α	$(y_1)_c$		α	$(y_1)_d$	
3.00	.25	.05	-1.456	-j.228	.387	-670	-j.105		-746	+j.199	.345	-374	+j.601
		.15	-1.708	-j.467	.571	-545	-j.149		-745	+j.561	.548	-249	+j.522
		.25	-1.855	-j.638	.674	-482	-j.166		-717	+j.1789	.656	-193	+j.482
	.50	.05	-1.461	-j.273	.396	-661	-j.124		-452	+j.892	0	-553	+j.833
		.15	-1.699	-j.517	.574	-539	-j.164		-629	+j.1422	.441	-260	+j.588
		.25	-1.840	-j.690	.675	-477	-j.179		-616	+j.1670	.577	-195	+j.527
	.75	.05	-1.453	-j.349	.402	-650	-j.156		-043	+j.999	0	-789	+j.614
		.15	-1.673	-j.597	.575	-530	-j.189		-431	+j.1113	.177	-303	+j.782
		.25	-1.805	-j.771	.674	-468	-j.200		-437	+j.1455	.423	-199	+j.624
1.00	1.00	.05	-1.414	-j.449	.394	-642	-j.204		-866	+j.500	0	236	+j.972
		.15	-1.617	-j.699	.566	-521	-j.225		-664	+j.748	0	116	+j.993
		.25	-1.742	-j.873	.667	-459	-j.230		-111	+j.994	0	-375	+j.927
	1.25	.05	-1.324	-j.553	.361	-643	-j.269		-906	+j.423	0	481	+j.877
		.15	-1.522	-j.805	.543	-513	-j.272		-767	+j.642	0	410	+j.912
		.25	-1.641	-j.982	.648	-449	-j.268		-624	+j.781	0	322	+j.947
	1.50	.05	-1.169	-j.632	.284	-662	-j.358		-929	+j.371	0	687	+j.726
		.15	-1.381	-j.898	.499	-509	-j.331		-823	+j.568	0	640	+j.768
		.25	-1.501	-j.1081	.615	-439	-j.316		-720	+j.694	0	587	+j.810
2.75	1.75	.05	-883	-j.608	.069	-769	-j.529		-942	+j.335	0	847	+j.532
		.15	-1.195	-j.956	.425	-510	-j.408		-857	+j.515	0	816	+j.579
		.25	-1.323	-j.1152	.562	-430	-j.374		-775	+j.632	0	780	+j.625
	.50	.05	-1.294	-j.281	.281	-738	-j.160		-540	+j.991	.121	-424	+j.778
		.15	-1.582	-j.525	.511	-570	-j.189		-602	+j.1455	.454	-243	+j.587
		.25	-1.735	-j.696	.626	-496	-j.199		-583	+j.1696	.584	-181	+j.527
	.75	.05	-1.319	-j.372	.315	-702	-j.198		-051	+j.999	0	-702	-j.198
		.15	-1.568	-j.613	.521	-553	-j.216		-420	+j.1190	.233	-264	+j.747
		.25	-1.709	-j.784	.631	-483	-j.222		-434	+j.1497	.444	-179	+j.616
1.00	1.00	.05	-1.310	-j.491	.336	-669	-j.251		-809	+j.588	0	233	+j.972
		.15	-1.526	-j.726	.525	-534	-j.254		-596	+j.803	0	101	+j.995
		.25	-1.655	-j.895	.632	-468	-j.253		-240	+j.1093	.112	-192	+j.873
	1.25	.05	-1.250	-j.616	.332	-644	-j.317		-846	+j.533	0	479	+j.878
		.15	-1.445	-j.846	.515	-516	-j.302		-704	+j.710	0	404	+j.915

1.50	.05	-1.130 - j.718	.292	-.631 - j.401	-.867 + j.498	0	.686 + j.727
	.15	-1.319 - j.953	.487	-.498 - j.360	-.761 + j.649	0	.637 + j.771
	.25	-1.434 - j.1.124	.600	-.432 - j.338	-.655 + j.755	0	.580 + j.814
1.75	.05	-.936 - j.756	.185	-.646 - j.522	-.879 + j.476	0	.846 + j.533
	.15	-1.151 - j.1.026	.433	-.484 - j.432	-.795 + j.607	0	.814 + j.581
	.25	-1.267 - j.1.208	.560	-.413 - j.394	-.711 + j.703	0	.777 + j.630
2.00	.05	-.537 - j.843	0	-.832 - j.555	-.887 + j.461	0	.952 + j.307
	.15	-.943 - j.1.042	.341	-.477 - j.527	-.816 + j.578	0	.932 + j.362
	.25	-1.072 - j.1.250	.499	-.395 - j.461	-.747 + j.665	0	.909 + j.416
2.50	.05	-.915 - j.403	0	-.961 - j.276	-.688 + j.998	0	-.614 + j.789
0.75	.15	-1.369 - j.626	.409	-.604 - j.276	-.379 + j.1.301	.304	-.206 + j.708
	.25	-1.530 - j.796	.545	-.514 - j.268	-.371 + j.1.566	.476	-.143 + j.604
1.00	.05	-1.107 - j.527	.204	-.736 - j.351	-.692 + j.722	0	.227 + j.974
	.15	-1.358 - j.758	.442	-.561 - j.313	-.453 + j.892	0	.063 + j.998
	.25	-1.496 - j.922	.564	-.484 - j.299	-.205 + j.1.274	.255	-.123 + j.765
1.25	.05	-1.111 - j.686	.267	-.651 - j.402	-.727 + j.686	0	.476 + j.879
	.15	-1.305 - j.899	.460	-.519 - j.358	-.581 + j.814	0	.391 + j.920
	.25	-1.425 - j.1.059	.574	-.452 - j.336	-.413 + j.911	0	.275 + j.961
1.50	.05	-1.039 - j.819	.280	-.594 - j.468	-.747 + j.665	0	.684 + j.729
	.15	-1.205 - j.1.028	.460	-.480 - j.410	-.640 + j.769	0	.630 + j.776
	.25	-1.313 - j.1.187	.571	-.419 - j.379	-.529 + j.849	0	.566 + j.824
1.75	.05	-.902 - j.902	.243	-.555 - j.555	-.758 + j.652	0	.845 + j.535
	.15	-1.062 - j.1.125	.436	-.443 - j.470	-.674 + j.739	0	.810 + j.587
	.25	-1.165 - j.1.290	.553	-.385 - j.427	-.588 + j.809	0	.769 + j.639
2.00	.05	-.692 - j.880	.113	-.552 - j.702	-.765 + j.643	0	.951 + j.309
	.15	-.883 - j.1.172	.384	-.410 - j.544	-.695 + j.719	0	.930 + j.368
	.25	-.987 - j.1.354	.516	-.352 - j.482	-.624 + j.781	0	.905 + j.425
2.25	.05	-.770 + j.638	0	.998 + j.068	-.296 - j.955	0	-.712 - j.702
	.15	-.681 - j.1.148	.288	-.382 - j.645	-.709 + j.705	0	.991 + j.136
	.25	-.795 - j.1.368	.459	-.318 - j.546	-.648 + j.762	0	.979 + j.201
2.50	.05	-.950 - j.312	0	-.710 - j.704	-.516 + j.857	0	.214 + j.977
1.00	.15	-1.115 - j.756	.298	-.614 - j.417	-.124 + j.1.057	.062	-.109 + j.933
	.25	-1.275 - j.929	.456	-.512 - j.373	-.118 + j.1.422	.356	-.578 + j.698

(Continued)

Table 11-1 (Concluded)

φ_1^0	φ_2^0	W	$(y_1)_a$	α	$(y_1)_b$	$(y_1)_c$	α	$(y_1)_d$
2.00	1.25	.05	-.872 - j .682	.102	-.712 - j .556	-.554 + j .832	0	.470 + j .883
		.15	-.111 - j .931	.371	-.529 - j .443	-.393 + j .920	0	.365 + j .931
		.25	-1.235 - j 1.090	.499	-.455 - j .402	-.164 + j .986	0	.185 + j .983
		.05	-.888 - j .881	.224	-.567 - j .563	-.574 + j .819	0	.681 + j .732
		.15	-1.046 - j 1.088	.412	-.459 - j .478	-.462 + j .887	0	.618 + j .786
1.75	1.50	.25	-1.148 - j 1.242	.526	-.401 - j .434	-.338 + j .941	0	.539 + j .842
		.05	-.798 - j 1.008	.252	-.483 - j .610	-.586 + j .811	0	.843 + j .538
		.15	-.929 - j 1.213	.424	-.398 - j .519	-.499 + j .867	0	.803 + j .596
		.25	-1.020 - j 1.369	.535	-.350 - j .470	-.408 + j .913	0	.755 + j .656
		.05	-.651 - j 1.064	.221	-.419 - j .684	-.593 + j .805	0	.950 + j .312
2.00	2.00	.15	-.775 - j 1.292	.410	-.342 - j .569	-.521 + j .853	0	.926 + j .377
		.25	-.861 - j 1.457	.526	-.301 - j .509	-.448 + j .894	0	.897 + j .442
		.05	-.452 - j .987	.082	-.383 - j .837	-.597 + j .802	0	.997 + j .072
		.15	-.600 - j 1.312	.367	-.288 - j .630	-.535 + j .845	0	.989 + j .147
		.25	-.686 - j 1.499	.500	-.252 - j .551	-.472 + j .881	0	.976 + j .219
1.75	1.25	.05	-.493 - j .870	0	-.875 - j .483	-.335 + j .942	0	.459 + j .888
		.15	-.853 - j .882	.205	-.566 - j .586	-.127 + j .992	0	.302 + j .953
		.25	-1.002 - j 1.075	.385	-.464 - j .498	.138 + j 1.274	.248	.840 + j .776
		.05	-.576 - j .817	0	-.666 - j .746	-.359 + j .933	0	.675 + j .737
		.15	-.847 - j 1.098	.327	-.440 - j .571	-.233 + j .973	0	.595 + j .804
2.00	1.50	.25	-.949 - j 1.261	.456	-.381 - j .506	-.075 + j .997	0	.481 + j .877
		.05	-.642 - j 1.046	.205	-.426 - j .695	-.372 + j .928	0	.840 + j .543
		.15	-.761 - j 1.261	.387	-.351 - j .581	-.278 + j .960	0	.791 + j .612
		.25	-.844 - j 1.417	.500	-.310 - j .521	-.175 + j .985	0	.730 + j .683
		.05	-.531 - j 1.160	.243	-.326 - j .713	-.379 + j .925	0	.948 + j .318
1.75	2.00	.15	-.628 - j 1.373	.412	-.276 - j .602	-.304 + j .953	0	.920 + j .393
		.25	-.701 - j 1.533	.522	-.247 - j .539	-.224 + j .975	0	.884 + j .468
		.05	-.257 - j .966	0	-.745 - j .667	.665 + j .746	0	-.112 + j .994
		.15	-.600 - j .991	.147	-.447 - j .738	.545 + j .838	0	.052 + j .999
		.25	-.725 - j 1.215	.347	-.362 - j .607	.376 + j 1.133	.177	.263 + j .795
1.75	1.50	.05	-.338 - j .941	0	-.491 - j .871	.835 + j .551	0	-.129 + j .992
		.15	-.569 - j 1.242	.312	-.304 - j .665	.770 + j .638	0	-.020 + j 1.000
		.25	-.47 - j 1.414	.442	-.268 - j .585	.114 + j .993	0	.677 + j .736

operating with both φ_1^0 and φ_2^0 in the range 1.25 to 1.50 for reasonable values of W , where two growing waves exist with comparable growth factors. Clearly the bilateral nature of $\varphi_p \approx \pi$ restricts the frequency range and the growth factor we can attain and still maintain unilateral gain.

For unilateral gain it is clear that we want (1) $\varphi_p \cong \varphi_1^0 + \varphi_2^0$ at each ω_1 and related ω_2 in the frequency range of operation, (2) $\varphi_p^- = 2\pi - \varphi_p \not\approx \varphi_1^0 + \varphi_2^0$ in the reverse direction, and (3) a parametric coupling coefficient W not so large that a growing-decaying wave pair will exist to amplify power reflected from a mismatched load. Grabowski and Weglein [6] show that if ω_1 and ω_2 both lie in the first passband the best pump phase shift for unilateral gain is $5\pi/4$ (see Fig. 11-5). Such a phase shift, however, requires a distortion of the ω - φ curve so that the phase shift of the fields at midfrequency, $\omega_1 = \omega_2$, is $5\pi/8$, whereupon the difference between $\varphi_p^- = 3\pi/4$ in the backward direction and $\varphi_1^0 + \varphi_2^0$ in that direction will be about $\pi/2$ throughout the frequency range of operation. This effect should be apparent by examination of the curve of $\varphi_1^0 + \varphi_2^0$ in the backward direction as compared to $\varphi_{p1}^- = 3\pi/4$, in the first passband region of Fig. 11-5. On the other hand, if the second passband is utilized as shown, the best pump phase shift is $\varphi_{p2} = \pi/4$. Then φ_1^0 (negative) $+ \varphi_2^0 \cong \pi/4$ in the indicated range of operation of ω_1 in the second passband and ω_2 in the first passband, for good forward gain. Operation is improved if the first dispersion curve is distorted as shown and also the second dispersion curve is depressed downward as shown so as to keep $\varphi_2^0 - |\varphi_1^0| \cong \varphi_p$ well above zero in the frequency range of operation. Under these conditions $\varphi_1^0 + \varphi_2^0$ in the backward direction is a lag of about $\pi/4$ radians whereas the pump phase shift φ_p^- in this direction is a *lead* of $\pi/4$, a discrepancy of $\pi/2$ which will prevent bilateral gain unless W is unduly high. Later, in Section 12-6, we point out the fact that resonant slots may be used to "shape" the curves in the desired manner.

Before discussing the experimental work of Grabowski and Weglein it seems appropriate to tabulate the values of γ_1 for the four waves which occur when the fields at frequency ω_1 and nonparametric phase shift φ_1^0 in any narrow m th passband interact parametrically with the fields at frequency ω_2 and phase shift φ_2^0 on the n th dispersion curve. A table of these roots will enable us to estimate the gain performance of many amplifiers with respect to various terminal loads and excitations. Table 11-1 lists the roots for a pump phase shift of 4.0 and various combinations of φ_1^0 , φ_2^0 , and the positive W values of 0.05, 0.15, and 0.25. Some of the roots not listed may be inferred from others; for example, the γ_1 roots for the case $\varphi_1^0 = 1.5$, $\varphi_2^0 = 3.0$, are just the γ_2 roots for the tabulated case $\varphi_1^0 = 3.0$, $\varphi_2^0 = 1.5$. Since $W > 0$ the roots are of interest when both

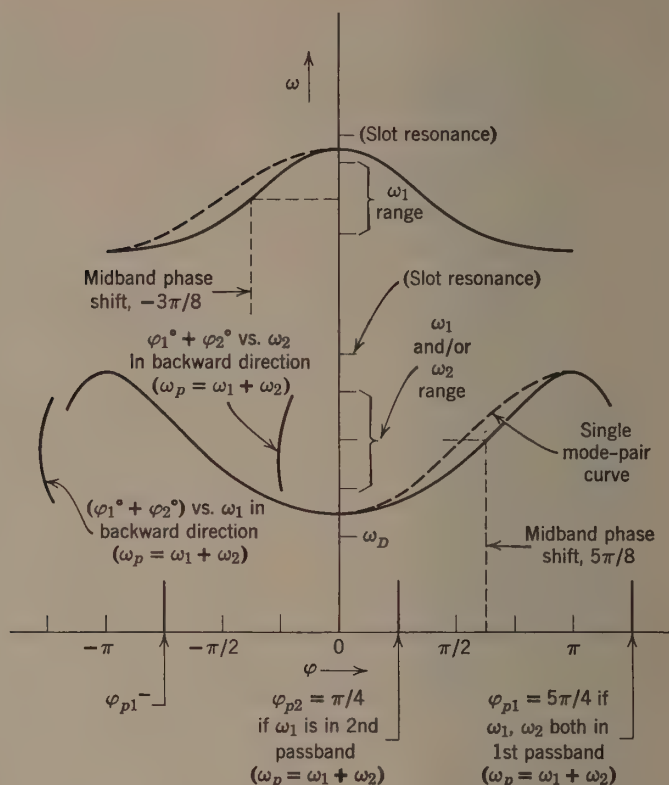


Fig. 11-5. The first two dispersion curves in the parametric amplifier of Fig. 11-4. The single mode-pair curves are shown dashed. In the first case $\varphi_{p1} = 5\pi/4$ if $\omega_p = \omega_1 + \omega_2$ so that $(\varphi_1^0 + \varphi_2^0)$ vs. ω_1 remains nearly equal to $5\pi/4$ in the forward direction but far out of synchronism with the pump phase shift of $3\pi/4$ in the backward direction. In the second case $\varphi_{p2} = \pi/4$ if $\omega_p = \omega_1 + \omega_2$ so that $\varphi_1^0 + \varphi_2^0$ remains nearly equal to $\pi/4$ in the forward direction but far out of synchronism with the pump phase shift of $-\pi/4$ in the backward direction. Resonant slots inserted into the partitions between cavities will help to perturb the dispersion curves as shown if their resonant frequencies lie along the ω axis as shown. This effect is discussed in Section 12-6.

ω - φ curves have the same shapes; a separate table would have to be computed for the $W < 0$ cases. Observe that many of these states of operation have two pairs of growing and decaying waves.

It should be clear now that the parametric amplifier behaves in an entirely different fashion than the traveling wave beam tube, with respect to the characters of the various independent waves.

To calculate the performance of a three-frequency amplifier, $\omega_p = \omega_1 + \omega_2$, with a table like Table 11-1 we would (1) determine from the

ω - φ curves the phase shifts φ_1^0 and φ_2^0 for a given ω_1 and $\omega_2 = \omega_p - \omega_1$ of interest, (2) read the four values of γ_1 from the table and calculate the associated $\gamma_2 = \gamma_1^* + j\varphi_p$, and (3) evaluate the relative amplitudes of the four waves by satisfying 4 boundary conditions in a manner to be outlined next.

The equation which must be satisfied in the first cavity, labeled with superscript (0), fed by an input loop current, let us say, of amplitude $I_{\varphi 1}$ at frequency ω_1 and $I_{\varphi 2}$ at frequency ω_2 is

$$\frac{1}{2} p_m M_{mm} v_{\text{tot},1}^{(-1)} = \frac{j\omega_1 \mu}{\tau} I_{\varphi 1} \int_{\text{loop}} \bar{e}_m \cdot d\ell \quad (11-36)$$

(see Eq. 9-29c, with ferroelectric in place of the beam) with

$$v_{\text{tot},1}^{(-1)} = v_{1a}\epsilon^{\gamma_{1a}} + v_{1b}\epsilon^{\gamma_{1b}} + v_{1c}\epsilon^{\gamma_{1c}} + v_{1d}\epsilon^{\gamma_{1d}}$$

Here v_{1a} , and so on, are the ω_1 amplitudes of the four independent waves and γ_{1a} , and so on, are the phase shifts of the ω_1 field components. Equation 11-36 and a similar equation for the amplitudes $v_{2a}, \dots, v_{2d}$ at frequency ω_2 in the n th passband furnish two relations for the four independent amplitudes (recall that each v_{2i} amplitude is related to its v_{1i} amplitude through the coupled-amplitude Eq. 11-22 or 11-23 in which γ_{1i} and γ_{2i} are both known.)

The remaining two relations for the amplitudes are furnished by the output cavity (N) load conditions; from Eqs. 9-39 and 9-40 and the ferroelectric in place of the electron beam we have

$$\frac{1}{2} p_m M_{mm} v_{\text{tot},1}^{(N+1)} = j p_m^2 Y_{LN,1} v_{\text{tot},1}^{(N)} \quad (11-37a)$$

where

$$\begin{aligned} v_{\text{tot},1}^{(N)} &= v_{1a}\epsilon^{-N\gamma_{1a}} + \dots + v_{1d}\epsilon^{-N\gamma_{1d}} \\ v_{\text{tot},1}^{(N+1)} &= v_{1a}\epsilon^{-(N+1)\gamma_{1a}} + \dots + v_{1d}\epsilon^{-(N+1)\gamma_{1d}} \end{aligned} \quad (11-37b)$$

and $Y_{LN,1}$ is the normalized output load at frequency ω_1 . There is a similar equation involving the total fields at frequency ω_2 in terms of $Y_{LN,2}$.

Suppose only the growing wave, signified by subscript a , impinges upon the output cavity (N); to avoid generating a backward wave (which may be either a constant-amplitude wave or a growing wave carrying power backward) we want to absorb both incident growing wave components at both frequencies. If we neglect all waves but the growing one in Eq. 11-37 we want the loads to be

$$Y_{LN,1} = -j \frac{M_{mm}}{2p_m} \epsilon^{-\gamma_{1a}} \quad (11-38a)$$

and

$$Y_{LN,2} = -j \frac{M_{nn}}{2p_n} \epsilon^{-\gamma_{2a}}, \quad (11-38b)$$

as verified by the numerator of Eq. 9-41 for the beam tube.

The input power to the first cavity may be calculated at each frequency by Eq. 10-13, and the output power by Eq. 10-14.

We conclude this section with a resumé of the experimental work of Grabowski and Weglein, together with some comments on the various methods of enhancing the amplifier performance. Their iris coupled-cavity amplifier is shown in Fig. 11-4; it is operated with both frequencies in the first passband in the so-called degenerate mode of operation because the interaction impedances are low at frequencies in the second passband. An open-circuit mode resonates at the $\varphi = 0$ cutoff frequency in the first passband with respect to magnetic shorts on the inductive apertures, and a short-circuit mode resonates at the $\varphi = \pi$ cutoff frequency. Both these modes have strong electric field across the diodes. In this structure the modes of the second passband have weak electric field across the diodes but in a modified version of the structure called the coupled coaxial-cavity amplifier (with a reported insertion gain of 13 db and 15% bandwidth for degenerate operation and somewhat poorer performance with ω_1 in the first passband and ω_2 in the second one) the diode supporting rods make the resonant cavity modes of both the first and the second passbands perturbations of a coaxial mode with strong voltage across the diodes. In the structure of Fig. 11-4 voltage gain-bandwidth products of 2000 Mc with a 350-Mc bandwidth at S-band have been realized with commercial diodes. The trend is toward higher products which are realized at higher gains, as we would expect from the performances of the high-gain amplifiers discussed in Chapter 10. The pump phase shifts may be externally adjusted for gain enhancement and a better match to the load. The first passband of this structure, shown in Fig. 11-5, is somewhat depressed in the direction favoring unilateral gain with both ω_1 and ω_2 in this passband and $\varphi_p = 5\pi/4$. This depression is caused by the fact that the diode series resonant frequency ω_D falls at about 2.8 kMc (2-v bias) just below the $\varphi = 0$ cutoff frequency, so the diodes are inductive at frequencies in the first passband. The tuning stubs enable us to tune out the diode lead inductance so that more r.f. voltage falls across the capacitive portion.

This passband, as well as the second one, could also be "shaped" more favorably by inserting long narrow slot(s) in each partition between neighboring cavities. If the slots are half a free-space wavelength long at the frequencies designated on Fig. 11-5, the interaction of their fields with the modes of the nearest passband will shift the lower dispersion curve

downward and the upper dispersion curve downward as shown, according to the discussion in Sections 7-5 and 12-6.

The experimental results proved, indeed, that the insertion gain was about the same in both directions for $\varphi_p = \pi$ as expected. As φ_p approached $5\pi/4$ the forward gain increased to 10–15 db in a four-cavity amplifier over most of the 8% passband width, while the reverse gain decreased to less than 0 db over almost all of the band. An increase of the pump power increased the forward gain most noticeably near the band edges, where the reverse gain also tended to rise. The increased gain demanded rematching; when this was done the forward gain was made quite flat in the 10–15 db range over a frequency range of nearly 10%. The reverse gain was then less than 0 db. The effect of adding two more cavities increased the forward gain to the range 20–25 db but reduced the frequency range for this gain to about 5% and also made the gain more sensitive to terminal conditions and irregularities in the cavities. However, the amplifier remained stable with a gain-bandwidth product of 2000 Mc.

There are several ways to reduce the regenerative effects due to a mismatched output cavity or the presence of a growing wave which carries power back to the input. One scheme involves changing the pump phase shift in the last cavity, and perhaps in others also, so as to flatten the gain curve near the edges without reducing its midband value. This also tends to lower the reverse gain without reducing the bandwidth. It is a method of stabilizing high-gain responses while maintaining low reverse gain. Another method of combating regeneration is to allow one of the sum frequencies, say $\omega_u = \omega_p + \omega_1$, to fall in a passband which has been shaped so that $\varphi_u \approx \varphi_p + \varphi_1^0$ in the forward direction but $\varphi_u^- \cong \varphi_p^- + \varphi_1^0$ in the backward direction. Then the backward growing wave with ω_1 and ω_2 phase shifts which satisfy the relation $\varphi_p^- = \text{Im}(-\gamma_1) + \text{Im}(-\gamma_2)$ will also support a component of frequency ω_u . If, then, loss is introduced at the frequency ω_u through resonant slots, for example, this backward wave will attenuate. At the same time, the forward growing wave supports very weak field components at frequency ω_u because they tend to propagate with the wrong phase shift per cavity.

The most interesting and perhaps the most promising means of obtaining nonreciprocal loss in the parametric amplifier, and in beam tubes as well, is with ferrites. Carlile and Sensiper [7] have demonstrated experimentally the nonreciprocal loss introduced by ferrite posts or wafers in the coupling apertures between neighboring cavities in a chain, and we now develop the simple theory of the elliptical polarization upon which the nonreciprocal nature of the loss depends.

Consider the rectangular aperture containing a ferrite post as shown in Fig. 11-6; the cavities may have general shapes but the aperture must lie

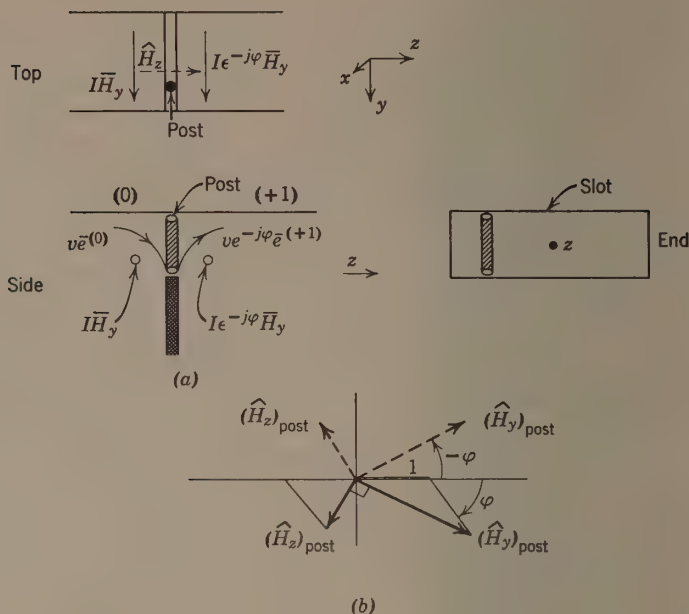


Fig. 11-6. Ferrite post in a coupling aperture. (a) The tangential electric and magnetic field components on each side of the post, (b) phase relations of magnetic field components showing elliptical polarization.

next to the outer wall as shown for the maximum effect. Now the cavity mode-pair for this passband will have one of two characters: Either the $\bar{H}$ pattern reverses direction across the aperture and $\bar{e}$ does not, or $\bar{e}$ reverses and $\bar{H}$ does not. Let us examine the latter situation first, as shown in the figure. The phase relations shown imply an elliptical polarization of a.c. magnetic field in the yz plane. With electric field amplitude v in cavity (0) as reference we deduce from the relation $p\bar{v} = -j\omega\mu i$, or $PI = j\omega\varepsilon V$, that amplitude I has the phase of jv . However, $\hat{H}_y$ on the aperture where the post lies is given by half the sum of the mode expressions on both sides; hence $(\hat{H}_y)_{\text{post}} \sim jv(1 + \epsilon^{-j\varphi})$. The normal component of magnetic field $\hat{H}_z$ is given by the curl of $\bar{e}$ on the aperture. Since $|\bar{e}|$ increases toward the center of the aperture and $\bar{e}$ on it is half the sum of the mode expressions on both sides, we deduce the fact that $(\hat{H}_z)_{\text{post}} \sim jv(\epsilon^{-j\varphi} - 1)$. The vector relation between these two components is shown in Fig. 11-6b (solid arrows); as seen from the top of the post $\hat{H}$ rotates counterclockwise in the yz plane with elliptical polarization. If the ferrite magnetization vector rotates clockwise with a period equal to the r.f. period it will not "see" this magnetic field. Now, if the phase shift per cavity φ is between $-\pi$ and 0, the vector relations are shown as dashed arrows, $\hat{H}_z$ at the post

leads $\hat{H}_y$ in time, and the polarization as seen from the top of the post is clockwise. The magnetization vector will therefore precess in step with this field and absorb energy from it.

If, instead, the $\vec{H}$ pattern reverses direction across the coupling aperture and $\vec{e}$ does not, the sense of the polarization is reversed.

For these reasons a properly magnetized ferrite post placed in a coupling aperture near the edge will establish a high ratio of reverse to forward loss over most of the passband width, excluding the edges, at the expense of additional pump power necessary to compensate for the power extracted from the forward wave. Ferrite elements also tend to alleviate the matching problem in the last cavity.

11-4. The Three-Frequency Ferromagnetic Cavity-Chain Amplifier

The ferromagnetic parametric amplifier operates on the same principles as the ferroelectric amplifier described in the preceding section but the interchanges of power through the nonlinear ferrite take place through the magnetization vector rather than through electric dipoles as in the ferroelectric. The relative permittivity of a ferrite may be treated as a scalar, usually about 10.

As Suhl [8] explains, the ferromagnetic amplifier converts power from one frequency to another by virtue of a relatively large uniform precession of the magnetization vector induced by a local oscillator or pump magnetic field in the plane perpendicular to the applied d.c. magnetic field. To conserve pump power the ferrite sample should have a precessional frequency as determined by the applied d.c. magnetic field equal or nearly equal to the pump frequency. If, then, suitably oriented magnetic fields are present at frequencies ω_1 and ω_2 such that $\omega_p = \omega_1 + \omega_2$ (the case $\omega_p = \omega_1 - \omega_2$ makes a trivial change in the coupled-amplitude equations), the precession of the magnetization vector will establish secular or time-average coupling between them. Generally both r.f. magnetic fields $\hat{H}_1$ and $\hat{H}_2$ at frequencies ω_1 and ω_2 have components parallel and perpendicular to the d.c. magnetic field $\vec{H}_{z0}$, but only one of these fields must have a component parallel to $\vec{H}_{z0}$. In electromagnetic operation the other field must then have a component perpendicular to $\vec{H}_{z0}$.

We shall discuss here the two most important states of operation, electromagnetic and semistatic. In the first, frequency ω_1 lies near both resonant frequencies of a solenoidal cavity mode-pair and ω_2 lies near the resonant frequencies of the same or a different mode-pair. This means that the fields are predominantly solenoidal and uniform through the

sample of dimensions considerably greater than the wavelengths of the spin modes and yet considerably smaller than the wavelengths at ω_1 and ω_2 . In addition, ω_1 and ω_2 lie below that frequency range between $|\gamma|(H_{z0} - N_z M_{z0})$ to $|\gamma|(H_{z0} + 2\pi M_{z0})$ within which there are a large number of magnetostatic mode (modified irrotational mode with a magnetic field which is the gradient of a scalar potential) resonances, each of which represents strong sample excitations [9]. Here γ is the gyromagnetic ratio, H_{z0} is the applied magnetic field in the z direction as contrasted with the effective d.c. field $H_{z0} - N_z M_{z0} = H_{\text{d.c.}}$, M_{z0} is the d.c. magnetization, and N_z is the demagnetizing factor which lies between 0 and 4π for ellipsoidal samples. Under these conditions these modes are not excited and the resonant cavity modes adequately represent the ω_1 and ω_2 fields outside and inside the sample, which couple together through the uniform-precessional component of the magnetization vector.

In semistatic operation one of the frequencies, say ω_2 , lies outside the magnetostatic (m.s.) mode spectrum near the resonances of a cavity mode-pair as before, but ω_1 lies within the m.s. range. Then a fair number of these modes will be excited at frequency ω_2 and contribute to the magnetic field and magnetization vector.

In the ensuing treatment the ω_1 and ω_2 fields are presumed to be at a sufficiently low level so as not to affect the pumped precession of the magnetization vector, which may take place at a pump frequency within the range of the m.s. mode resonances. The pumping level is presumed to lie below the threshold at which instabilities occur. The total vector $\hat{M}_{\text{tot}}$ is expanded as

$$\hat{M}_{\text{tot}} = \bar{M}_{z0} + \hat{M}_1 \epsilon^{j\omega_1 t} + \text{c.c.} + \hat{M}_2 \epsilon^{j\omega_2 t} + \text{c.c.} + \hat{M}_p \quad (11-39a)$$

where c.c. denotes a complex conjugate of the preceding term. The

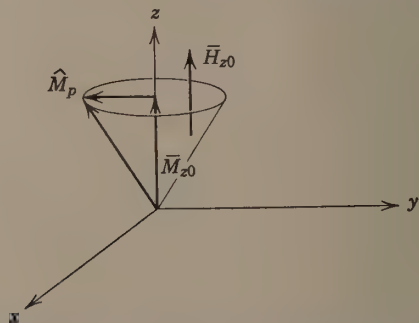


Fig. 11-7. Vector relations in the ferrite sample. Subscript 0 denotes a d.c. field quantity; p , a pumped quantity.

pumped component in the transverse plane is given by

$$\hat{M}_p = \hat{i}_x M_{p2} \frac{1}{2} (\epsilon^{j\omega_p t} + \text{c.c.}) + \hat{i}_y M_p \frac{1}{2j} (\epsilon^{j\omega_p t} - \epsilon^{-j\omega_p t}) \quad (11-39b)$$

(See Fig. 11-7). Here $\hat{M}_p$ is the circularly polarized component of the magnetization vector rotating from the x axis to the y axis. Similarly,

$$\hat{H}_{\text{tot}} = \bar{H}_{\text{d.c.}} + \hat{H}_1 \epsilon^{j\omega_1 t} + \text{c.c.} + \hat{H}_2 \epsilon^{j\omega_2 t} + \text{c.c.} + \hat{H}_p \quad (11-39c)$$

A. Electromagnetic Operation

We now follow Suhl to obtain the elements in the susceptibility dyadic χ defined by $\hat{M}_{1,2} = (\chi \cdot \hat{H})_{1,2}$ at frequency ω_1 or ω_2 . We do not expand $\hat{M}$ with the m.s. modes because $\hat{H}_1$ and $\hat{H}_2$ are essentially uniform through the sample, so that $\hat{M}$ is obtainable from $\hat{H}$ in such a way that $\nabla \cdot (\hat{H} + 4\pi\hat{M}) = \nabla \cdot \hat{M} = 0$ is guaranteed. We are asking for the steady-state fields in each cavity of an amplifier like that shown in Fig. 11-2, so they vary as $\epsilon^{j\omega_{1,2}t}$ with no exponential decay. The previous expressions for $\hat{M}_{\text{tot}}$ and $\hat{H}_{\text{tot}}$ are substituted into the equation of motion for the transverse component of $\hat{M}$,

$$\frac{d\hat{M}_t}{dt} = -\gamma(\hat{M} \times \hat{H})_t + \text{damping term}, \quad \gamma > 0 \quad (11-40)$$

in which the d.c. component of $\hat{H}$ is $\bar{H}_{\text{d.c.}} = \bar{H}_{z0} - N_z \bar{M}_{z0}$ and the exchange energy has negligible effect. The damping term is chosen phenomenologically so the transient transverse excitations would have a transient decay as $\epsilon^{-\lambda_s t}$. The x and y components of this equation at ω_1 obey the following relations, in which we introduce the parameters $\omega_0 = \gamma H_{\text{d.c.}} = \omega_p$, $\omega_M \triangleq 4\pi\gamma M_{z0}$, and $m_0 \triangleq M_p/M_{z0}$, the approximate precessional angle.

$$\begin{aligned} (j\omega_1 + \lambda_s)\hat{M}_{x1} + \omega_0\hat{M}_{y1} &= \frac{\omega_M}{4\pi} \hat{H}_{y1} - \frac{m_0\omega_M}{8\pi j} \hat{H}_{z2}^* \\ -\omega_0\hat{M}_{x1} + (j\omega_1 + \lambda_s)\hat{M}_{y1} &= -\frac{\omega_M}{4\pi} \hat{H}_{x1} + \frac{m_0\omega_M}{8\pi} \hat{H}_{z2}^* \end{aligned} \quad (11-41)$$

We have neglected the small $\hat{M}_{2z}^* \times \hat{H}_{p1}$ term. If the damping term in Eq. 11-40 were taken as

$$\frac{\alpha}{M_0} \left(\hat{M} \times \frac{d\hat{M}}{dt} \right)$$

instead, with $\alpha\omega_0 \equiv \lambda_s$, there would be minor changes in these relations.

We also need $\hat{M}_{z1}$. Since $|\hat{M}|^2 = |\bar{M}_{z0} + \hat{M}_{zp}|^2 + |\hat{M}_{tp}|^2$ remains

constant after turning on the ω_1 and ω_2 fields, we have $\hat{M}_{z1,z2}$ related to $\hat{M}_{x,y}$ as ($\omega_p = \omega_1 + \omega_2$),

$$\begin{aligned}\hat{M}_{z1} &= -\frac{1}{2}m_0(M_{x2}^* - jM_{y2}^*) \\ \hat{M}_{z2} &= -\frac{1}{2}m_0(M_{x1}^* - jM_{y1}^*)\end{aligned}\quad (11-42)$$

Then when we solve for $\hat{M}_{x1}$ and $\hat{M}_{y1}$ in Eq. 11-41 in terms of the $\hat{H}$ field components we obtain

$$\hat{M}_{x1} = \chi_1 \hat{H}_{x1} + j\kappa_1 \hat{H}_{y1} - \tau_1 \hat{H}_{z2}^* \quad (11-43a)$$

$$\hat{M}_{y1} = -j\kappa_1 \hat{H}_{x1} + \chi_1 \hat{H}_{y1} + j\tau_1 \hat{H}_{z2}^* \quad (11-43b)$$

$$\hat{M}_{z1} = -\tau_2^*(\hat{H}_{x2}^* - j\hat{H}_{y2}^*) \quad (11-43c)$$

in which the $\hat{H}_{z2}^*$ term is dropped from $\hat{M}_{z1}$ as being of order m_0^2 . The coefficients are

$$\begin{aligned}\chi_{1,2} &= \frac{\omega_0 \omega_M}{4\pi[(j\omega_{1,2} + \lambda_s)^2 + \omega_0^2]}, & \kappa_{1,2} &= \frac{\omega_M(\omega_{1,2} - j\lambda_s)}{4\pi[(j\omega_{1,2} + \lambda_s)^2 + \omega_0^2]} \\ \tau_{1,2} &= \frac{\omega_M m_0}{8\pi j(\lambda_s - j\omega_{2,1})}, & \omega_0 &\equiv \omega_p\end{aligned}\quad (11-43d)$$

The corresponding expressions for $\hat{M}_{x2}$, and so on, at ω_2 are obtained by interchanging subscripts 1 and 2.

We now wish to solve Maxwell's equations within a typical cavity of the cavity-chain amplifier with phase shift $\gamma_{1,2}$ at frequency $\omega_{1,2}$ across the coupling surfaces and also with wall and ferrite losses present. In cgs units these are, at frequency ω_1 ,

$$\begin{aligned}\nabla \times \hat{E}_1 &= -\frac{j\omega_1}{c}(\hat{H}_1 + 4\pi\hat{M}_1) + \text{wall-loss excitation} \\ \nabla \times \hat{H}_1 &= \epsilon_r \frac{j\omega_1}{c} \hat{E}_1 \\ \nabla \cdot \hat{E}_1 &= 0\end{aligned}\quad (11-44)$$

in which the fields are conveniently expanded with the *solenoidal* short-circuit resonant cavity mode of the m th passband within which ω_1 lies because this mode is excited on the coupling surfaces between cavities by tangential electric field which remains solenoidal in spite of the ferrite. Then $\hat{E}_1 = V_m(\omega_1)\hat{E}_m$ and $\hat{H}_1 = I_m(\omega_1)\hat{H}_m$, in which we understand that V_m and I_m vary at frequency ω_1 . The mode patterns are defined with respect to the Hermitian portions of χ_1 and κ_1 in Eqs. 11-43, as well as the relative permittivity ϵ_r , in the manner described at the opening of the preceding section. To evaluate V_m and I_m we dot-multiply Eq. 11-44a by

$\bar{H}_m^\dagger$ and Eq. 11-44b by $\bar{E}_m^\dagger$, integrate over the cavity volume including the sample, and follow the procedure on page 85. The cavity wall-loss term is given by Eq. 9-51 but we write it in terms of the short-circuit modes and divide it by $\mu_0 c$ for the proper cgs expression; then the cavity Q_0 parameter is introduced by way of Eq. 9-53. Evaluation of the χ and κ_1 loss terms proportional to λ_s shows that the former are a negligible correction to the wall loss if the sample Q is on the order of or greater than the cavity Q and the sample occupies a small fraction of the cavity volume. The small κ_1 loss terms nearly cancel. The final result obtained straightforwardly is

$$\begin{aligned} & \left[P_m^2 - k_1^2 + P_m M_{mm} (1 \mp \cosh \gamma_1) + \frac{j(1+j)k_1^2}{Q_{01}} \right] I_m(\omega_1) \\ &= -\frac{k_1^2 \omega_M m_0}{2\tau} \left[\frac{1}{\omega_2} \int_{\text{sample}} (\bar{H}_{xm}^* - j\bar{H}_{ym}^*) \bar{H}_{zn}^* dV \right. \\ & \quad \left. + \frac{1}{\omega_1} \int_{\text{sample}} \bar{H}_{zm}^* (\bar{H}_{xn}^* - j\bar{H}_{yn}^*) dV \right] I_n^*(\omega_2) \quad (11-45) \end{aligned}$$

This equation corresponds to Suhl's (13) with slight changes in notation⁴. The upper sign of $\mp$ is taken if the short-circuit mode of the m th passband resonates at frequency P_m corresponding to zero phase shift per cavity. The $1+j$ factor in the wall-loss term arises because the wall impedance is $Z_s = R_s(1+j)$ at microwave frequencies. Note that $k_1 = \omega_1/c$.

By interchanging subscripts 1 and 2 and m and n in Eq. 11-45 we obtain another relation between I_n and I_m^* . The two relations furnish a determinantal equation for γ_1 and the related $\gamma_2^* = \gamma_1 - j\varphi_p$ which can be cast into the form of Eq. 11-25 for the ferroelectric amplifier but with a new coupling coefficient w . We shall not write or solve this determinantal equation explicitly, but one fact is apparent from the form of Eq. 11-45: for a cavity unloaded Q_{01} of 10^3 or more the wall-loss will have a negligible effect upon the growing wave(s).

B. Semistatic Operation

Suhl has developed the general theory of electromagnetic, semistatic, and magnetostatic operation, but here we shall develop the equations only for a practical case of semistatic operation with ω_1 in the frequency range of the m.s. mode resonances—and far from a solenoidal cavity mode

⁴ Our V_n/P_n and $\bar{H}_n$ are equal to Suhl's a_n and $\bar{V}_n$, respectively, of his magnetic vector potential; our I_n is his A_n , the solenoidal magnetic field amplitude.

resonance—and ω_2 out of that range but in the narrow n th passband of the nonparametric structure.

To solve Eq. 11-44 and also guarantee zero divergence of $\hat{E}$,

$$\nabla \cdot (\hat{H}_1 + 4\pi\hat{M}_1) = 0 \quad (11-46)$$

we continue to represent total solenoidal $\hat{E}$ as $V_n(\omega_2)\hat{E}_n e^{j\omega_2 t} + \text{c.c.}$ at ω_2 but we include a set of m.s. modes in the expansion for $\hat{H}$:

$$\hat{H} = \sum_m s_m(\omega_1)\hat{h}_m' e^{j\omega_1 t} + \text{c.c.} + I_n(\omega_2)\hat{H}_n e^{j\omega_2 t} + \text{c.c.} \quad (11-47)$$

in which $\hat{h}_m' = \nabla\varphi_m'$ so that $\nabla \times \hat{h}_m' = 0$. These modified irrotational m.s. modes denoted by the primes are not orthogonal to but rather equivalent to the empty-cavity irrotational modes for the expansion of irrotational $\hat{H}$. We also expand the *transverse* component of $\hat{M}_t$ at ω_1 (since the m.s. modes are essentially unexcited at ω_2 by assumption) as

$$\hat{M}_t = \sum_m s_m(\omega_1)\bar{m}_{mt} e^{j\omega_1 t} + \text{c.c.} + \hat{M}_{2t} \quad (11-48)$$

Each $\bar{m}_{mt}$ pattern is related to φ_m' by the *definition* of a magnetostatic mode, namely,⁵

$$\begin{aligned} \hat{h}_m' &= \nabla\varphi_m' \\ \nabla \cdot \hat{h}_m' &= \nabla^2\varphi_m' = -4\pi\nabla_t \cdot \bar{m}_{mt} \\ j\omega_m' \bar{m}_{mt} &= -\gamma[(\bar{M}_{z0} + \bar{m}_{mt}) \times (\bar{H}_{\text{d.c.}} + \nabla\varphi_m')]_{mt} \\ \bar{m}_{mz} &= 0 \end{aligned} \quad (11-49)$$

The details of the many m.s. modes which satisfy these equations and have the proper continuities on the boundary of the ferrite sample have been discussed by Walker [9]. Now, we cannot satisfy Eq. 11-46 entirely with a set of these m.s. modes because only their transverse divergences are specified; Suhl introduces another potential φ'' which is related to $\hat{M}_{z1}$ as

$$\nabla^2\varphi'' = -4\pi\frac{\partial M_{z1}}{\partial z}, \quad \hat{h}'' = \nabla\varphi'' \quad (11-50)$$

so that Eq. 11-46 is satisfied identically in the following way:

$$\nabla \cdot \left(\sum_m s_m \hat{h}_m' + \nabla\varphi'' \right) = -4\pi\nabla_t \cdot \sum_m s_m \bar{m}_{mt} - 4\pi\frac{\partial \hat{M}_{z1}}{\partial z} \quad (11-51)$$

Equations 11-44 are satisfied by substituting these expansions for $\hat{E}_1$,

⁵ This notation agrees closely with Suhl's. The third equation is misprinted in Ref. [8].

$\hat{H}_1$, and $\hat{M}_1$ into Eq. 11-44a, including $\hat{h}'' = \nabla\varphi''$. We also satisfy Eq. 11-40 for $\hat{M}_1$ at ω_1 :

$$j\omega_1\hat{M}_{1t} = -\gamma[\hat{M}_{\text{tot}} \times (\bar{H}_{\text{d.c.}} + \sum_m s_m \bar{h}_m' \epsilon^{j\omega_1 t} + \text{c.c.} + \nabla\varphi'' + I_n \bar{H}_n \epsilon^{j\omega_2 t} + \text{c.c.} + \hat{H}_p)]_{\omega_1} + \text{dissipation term}, \quad (11-52)$$

where

$$\hat{M}_{\text{tot}} = \bar{M}_{z0} + \sum_m s_m \bar{m}_{mt} \epsilon^{j\omega_1 t} + \text{c.c.} + \hat{M}_z(\omega_1) + \hat{M}_2 + \hat{M}_p$$

At this point the problem is to obtain expressions for the mode amplitudes $V_n(\omega_2)$, $I_n(\omega_2)$, and the $s_m(\omega_1)$ in terms of each other, so as to form a determinantal equation for the γ_1 phase shifts of the various independent waves in the amplifier. These amplitude relations are developed with the aid of the orthogonality relations

$$\int_{\text{sample}} (\bar{m}_{xr} \bar{m}_{ys}^* - \bar{m}_{yr} \bar{m}_{xs}^*) dV = 0 \quad (\omega_r' \neq \omega_s') \quad (11-53)$$

The procedure for obtaining a relation between $s_r(\omega_1)$ and $I_n^*(\omega_2)$ starts with Eq. 11-52 from which we obtain two equations for the x and y components of the motion involving all the s_m amplitudes in terms of I_n , $\varphi' = \sum_m s_m \varphi_m'$, and φ'' . By multiplying the first equation by $\bar{m}_{yr}^*$ and the second by $\bar{m}_{xr}^*$, subtracting them, and integrating through the sample and cavity we isolate the s_r amplitude by the orthogonality relations. The result simplifies if $\bar{H}_n$ is only z directed to

$$K_r[j(\omega_1 - \omega_r') + \lambda_r]s_r(\omega_1) = -I_n^*(\omega_2) \frac{m_0 \omega_M}{8\pi} \int_{\text{sample}} \bar{H}_{zn}^* (\bar{m}_{xr}^* - j\bar{m}_{yr}^*) dV \quad (11-54)$$

with

$$K_r \triangleq \int_{\text{sample}} (\bar{m}_{xr} \bar{m}_{yr}^* - \bar{m}_{yr} \bar{m}_{xr}^*) dV \neq 0 \quad \text{and} \quad P_n \cong k_2$$

The mate of this equation obtains $I_n(\omega_2)$ in terms of all the m.s. mode amplitudes $s_m(\omega_1)$. By the same procedure used to obtain Eq. 11-45 we obtain now, through the first-order m_0 terms,

$$\begin{aligned} & \left[P_n^2 - k_2^2 + P_n M_{nn} (1 \mp \cosh \gamma_2^*) - \frac{j(1-j)k_2^2}{Q_{02}} \right] I_n^*(\omega_2) \\ & = -k_2^2 2\pi m_0 \sum_r s_r(\omega_1) \frac{1}{\tau} \left[\int_{\text{sample}} \bar{H}_{zn}^* (\bar{m}_{xr}^* - j\bar{m}_{yr}^*) dV \right]^* \end{aligned} \quad (11-55)$$

Here $k_2 = \omega_2/c$, the upper sign is taken if the n th short-circuit mode resonates at the nonparametric $\varphi = 0$ cutoff frequency, and $m_0 = M_p/M_{z0}$.

Equations 11-54 and 11-55 now furnish a determinantal equation for

γ_2^* and the related $\gamma_1 = \gamma_2^* + j\varphi_p$ of the waves in the cavity-chain ferromagnetic parametric amplifier. Walker's information indicates that the m.s. modes, although densely packed on the frequency scale, are not often sufficiently uniform to enable us to replace the sum in Eq. 11-55 by an integral and evaluate it by residue theory in the complex ω_r' plane.

11-5. The Cavity-Chain Maser

A. Introduction

The amplifier we shall describe utilizes the chain of cavities shown in Fig. 10-1, but without the electron beam. Rather, an active material within each cavity is maintained in a continuously excited state so that the molecules transfer power into the field by the stimulated emission of radiation [10, 11]. We describe the maser action under conditions similar to those of the ferromagnetic parametric amplifier of the preceding section because the operating equations are similar in the two devices. We should bear in mind, however, that the parametric principle differs fundamentally from the maser principle in that the former does not require a surplus of molecules in higher energy states in order to effect a conversion of power from one frequency to another [12]. We describe an amplifier in which the molecules or spins are continuously "pumped" by a coherent field at one frequency ω_p into excited states from which they radiate energy into the cavity fields which are resonant only at the signal frequency ω . To simplify the design the chain is presumed not to support a resonant field at the difference frequency $\omega_d = \omega_p - \omega$; however, molecules will experience this frequency because of the nonlinear equations of motion of the magnetization vector. Figure 11-8 shows the useful portion of the energy level diagram.

We shall discuss a continuously operated maser because it is preferred to the two-level pulsed amplifier for many applications and because the discussion of steady-state maser operating conditions is considerably simpler than that of transient operation. The ensuing treatment is rather esoteric because of the necessity of presenting a self-consistent quantum mechanical description of the radiation from molecules in quantized states into a quantized electromagnetic field. The signal field is so strongly perturbed by the molecules that we cannot rightly treat it on a semi-classical basis. The reader is presumed to have taken a first course in quantum mechanics and to be aware of the conceptual differences between the Schrödinger and Heisenberg descriptions of a quantum system. A

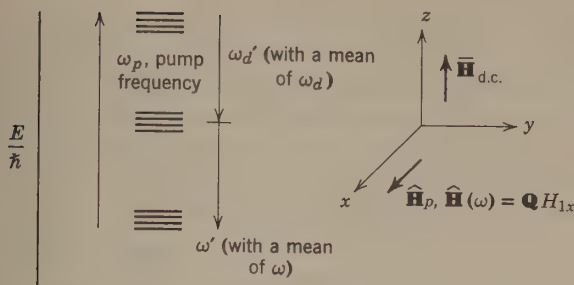


Fig. 11-8. Energy level diagram for the three-level maser material. The important molecular transitions are those from the upper group to the middle group through a mean energy difference of $\hbar\omega_d$ and from the middle to the lower group through a mean energy difference of $\hbar\omega$. All transitions for which $\omega_d' \simeq \omega_d$, $\omega' \simeq \omega$ and $\omega_d' + \omega' \simeq \omega_p$ are involved in the maser action.

beginning student will find this discussion helpful in understanding the philosophy which underlies these two formulations for we emphasize the formal and general aspects of the phenomena rather than the detailed nature of the field and molecular states. In brief, we point out the important aspects of the maser operation and outline how the reader would proceed, in principle, to calculate the steady-state expectation values of the fields and field energies in each cavity, given the equations of motion and the statistics established by the d.c. and "pump" fields.

Our model of the maser amplifier is specified as follows. The pumping field at frequency ω_p , greater than the signal frequency ω , is introduced either into the first cavity or through long narrow slots in the cavity walls; the cavities must be reasonably resonant at frequency ω_p . The signal frequency ω falls within a narrow passband so the fields are described accurately by the one short- and one open-circuit mode of that passband. If the maser pill in each cavity does not disturb the mirror symmetry about each cavity transverse bisecting plane, as in Fig. 11-2, for example, the resonant frequencies of these modes will still mark the cutoff frequencies of the passband. The cavities are not resonant at the difference frequency $\omega_d = \omega_p - \omega$. We assume magnetic dipole coupling between the molecules and the magnetic field, representing an energy $-\mathbf{M}_m \cdot \mathbf{H}$, $\mathbf{M}_m$ being the magnetic moment operator of molecule m and $\mathbf{H}$, the total magnetic field. All operators are written in boldface letters. The electric and magnetic field amplitudes at frequency ω will be quantized later; the pumping fields are treated classically, which implies a neglect of fluctuations transferred from the pumping field to the signal field. Both r.f. magnetic fields are assumed to lie in the x direction in each pill. The signal field at frequency ω absorbs the energy radiated by molecules making the

transition from any one of the second energy levels in Fig. 11-8 to a lower level through an energy difference of $\hbar\omega$. The homogeneous broadenings of the energy levels shown are due to irregularities in the material, spin-spin coupling, and so on. The interaction between pump and signal fields also causes molecules to make transitions from one of the upper energy levels to a lower level through a frequency difference of $\omega_d = \omega_p - \omega$. Later we point out how molecular transitions from a third to a second level through an energy difference of $\hbar\omega'_d \cong \hbar\omega_d$ and from a second to a first level through an energy difference of $\hbar\omega' \cong \hbar\omega$ can affect the maser action at frequencies ω_d and ω , provided $\omega'_d + \omega' = \omega_p$. We include the dipole-dipole coupling between molecules which is responsible for the spin-spin relaxation represented by the T_2 terms in Bloch's equations [13], but we neglect, for simplicity, the spin-lattice decay of energy usually specified by the time constant T_1 . The analysis may be modified to include T_1 coupling by analogy with our treatment of ohmic dissipation into the cavity walls and the load.

Of the two descriptions of quantum phenomena, the Schrödinger and the Heisenberg, the latter has two distinct advantages. The Heisenberg equations of motion for operators usually resemble the classical equations for the measured values of those operators. For example, we find that the equations of motion for dipole moment operator $\mathbf{M}_m$ look like Bloch's equations for the classical operator but are augmented by quantum terms. The second advantage accrues from the convenience with which we can include the effects of the hole coupling between cavities and the ohmic dissipation into the walls in the equations of motion for the electric and magnetic field operators. We develop Heisenberg equations for the steady-state field operators in each cavity, valid regardless of how the steady-state field is distributed among its states or how the molecules are distributed among *their* states. Naturally, we must specify these distributions for the entire field-molecular system in order to calculate the physical quantities of interest—the steady-state expectation (ensemble or measured) values of the field amplitudes and their associated energies—but the point is that the same operator equations are valid for all manner of correlated and uncorrelated states as represented by density matrices.

P. A. M. Dirac [14, Sections 27, 28, 33] explains just what the differences between the Schrödinger and Heisenberg descriptions are. In the former, the density operator representing the distribution of the wavefunction among the various stationary states of the system varies with time while the other operators such as $\mathbf{M}_m$ remain constant. In the Heisenberg formulation the density operator remains constant while all the other operators vary. The two formalisms are related by the fact that the physical quantities of interest—the expectation values of the operators,

which are the measured values at any given time—are independent of the formalism employed.

A comparison between the two descriptions is summarized as follows. The expectation value of an observable represented by Hermitian operator $\mathbf{O}$ is $\int \Psi^* \mathbf{O} \Psi d(\text{coordinates}) = \langle \Psi | \mathbf{O} | \Psi \rangle$ in Dirac's notation. The solution to Schrödinger's equation

$$\mathcal{H}\Psi = i\hbar\dot{\Psi}$$

is $\Psi(t) = \mathbf{U}(t)\Psi(0)$, where

$$\mathbf{U} = \exp \left[-\frac{i}{\hbar} \int_0^t \mathcal{H}(t') dt' \right]$$

$\mathcal{H}$ being the total Hamiltonian or energy operator and Ψ the total-system wavefunction. Now, if $\Psi(0)$ is expanded into system eigenfunctions ψ_s as $\Psi(0) = \sum a_s \psi_s$ we may show directly that the expectation value of $\mathbf{O}$ at any time t is also given by $\sum \langle \psi_s | \mathbf{U}\rho(0)\mathbf{U}^* \mathbf{O} | \psi_s \rangle$, where the elements of $\rho(0)$ are defined to be $\langle \psi_s | \rho(0) | \psi_{s'} \rangle = a_{s'}^* a_s$. Thus the expectation value of operator $\mathbf{O}$ may be calculated by the diagonal matrix elements of $\rho(t)\mathbf{O}$, where $\rho(t) = \mathbf{U}\rho(0)\mathbf{U}^*$ and ρ therefore satisfies the equation of motion

$$i\hbar\dot{\rho}(t) = -[\rho, \mathcal{H}] = \mathcal{H}\rho - \rho\mathcal{H}$$

in the *Schrödinger* formulation. But this expectation value is also identical to $\sum \langle \psi_s | \rho(0)\mathbf{U}^*\mathbf{O}\mathbf{U} | \psi_s \rangle$, in which ρ is treated as constant in time and $\mathbf{O}(t) = \mathbf{U}^*\mathbf{O}\mathbf{U}$, which means the operator satisfies the equation of motion

$$i\hbar\dot{\mathbf{O}} = [\mathbf{O}, \mathcal{H}]$$

in the *Heisenberg* formulation.

The effect of the pumping field is to add an extra energy term $-\mathbf{M} \cdot \hat{\mathbf{H}}_p(t)$ to the Hamiltonian, in which $\hat{\mathbf{H}}_p$ is treated as known and unperturbed by the fields at other frequencies. After the equation of motion is obtained for each operator varying *implicitly* with time the operator is resolved into various frequency components, the matrix elements of which can be related in principle to the (known) elements at the d.c. and pump frequencies.

B. Equations for the Cavity Field Operators

We first must quantize the amplitudes of the electric and magnetic fields in each cavity at or near frequency ω , regarded as time-varying operators in the Heisenberg formulation. The fields at or near the difference frequency ω_d are presumed to be negligible. In the cgs system we write [15]

$$\hat{\mathbf{E}} = -4\pi c \mathbf{P}(t) \bar{\mathbf{E}}_1, \quad \hat{\mathbf{H}} = \mathbf{Q} \nabla \times \bar{\mathbf{E}}_1 = \mathbf{Q} \bar{\mathbf{H}}_1 \quad (11-56)$$

with the operators in boldface letters. It is convenient to relate cavity mode pattern $\bar{H}_1$ to $\bar{E}_1$ differently from the way we did in other sections of the book. Here c is light velocity. The patterns may be conveniently normalized in each cavity as

$$\int_V \bar{E}_1^2 dV = \int_V \frac{(\nabla \times \bar{E}_1)^2}{(\omega_1/c)^2} dV = \int_V \frac{\bar{H}_1^2 dV}{(\omega_1/c)^2} = 1. \quad (11-57)$$

ω_1 being the resonant frequency of the cavity mode, very near the signal frequency ω . The instantaneous stored field energy in the cavity is, classically,

$$W_s = \frac{1}{8\pi} \int_V (|\bar{\mathbf{E}}|^2 + |\bar{\mathbf{H}}|^2) dV = 2\pi c^2 \mathbf{P}^2 + \frac{\omega_1^2}{8\pi c^2} \mathbf{Q}^2 \quad (11-58)$$

and we represent the field portion of the Hamiltonian by the latter expression in terms of operators.

The coupling between the field and the fluctuating wall loss mechanism is taken to be of the form $\mathbf{\Gamma} \cdot \mathbf{P} \bar{E}_1$, $\mathbf{\Gamma}$ being an abstract operator which will produce a term like $(\bar{n} \times \bar{E}_{\text{tan}})_{\text{walls}}$ in the first quantized Maxwell equation (refer to Eq. 9-42a). This term causes not only a slow decay of field energy but also creates fluctuations in the field amplitudes and field energies.

The energy associated with the molecule m in a magnetic field is taken to be magnetic dipole energy $-\mathbf{M}_m \cdot \bar{\mathbf{H}}$. Molecule m has a Hamiltonian $\mathcal{H}_m$ with the stationary-state energy levels E_{m3}, E_{m2}, E_{m1} , where $(E_{m3} - E_{m2})/\hbar$ lies close to ω_d and $(E_{m2} - E_{m1})/\hbar$ lies close to ω . Lastly, the Hamiltonian representing coupling energy between any two molecules m and n in a pill is taken to be of the classical form⁶

$$\mathcal{H}_{mn} = \frac{1}{r_{mn}^3} \mathbf{M}_m \cdot \mathbf{M}_n - \frac{3}{r_{mn}^5} (\mathbf{M}_m \cdot \mathbf{r}_{mn})(\mathbf{M}_n \cdot \mathbf{r}_{mn}) \quad (11-59)$$

$\mathbf{r}_{mn}$ being the vector connecting the two molecules.

The complete Hamiltonian for any one cavity of the chain, including the effect of the electromagnetic coupling to $\mathbf{P}_-$ on the $-z$ side (if fields exist there) and to $\mathbf{P}_+$ on the $+z$ side is

$$\begin{aligned} \mathcal{H}_{\text{tot}} = & 2\pi c^2 \mathbf{P}^2 + \frac{\omega_1^2}{8\pi c^2} \mathbf{Q}^2 + 2\pi c^2 m(\mathbf{P}^2 + \mathbf{P}\mathbf{P}_- + \mathbf{P}\mathbf{P}_+) + \mathbf{P} \int_{\text{wall}} \mathbf{\Gamma} \cdot \bar{\mathbf{E}}_1 dS \\ & + \mathcal{H}_d - \sum_m \int_V \mathbf{M}_m \cdot (\mathbf{Q} \bar{\mathbf{H}}_1 + \bar{\mathbf{H}}_p) dV + \sum_m \mathcal{H}_m + \frac{1}{2} \sum_{m, n \neq m} \mathcal{H}_{mn} \end{aligned} \quad (11-60)$$

$\mathcal{H}_{\text{tot}}$ would be independent of time were it not for the $\bar{H}_p$ term, and the

⁶ J. C. Slater [16, p. 191].

various operators are implicit functions of different frequencies. Here $\mathcal{H}_a$ is the energy operator of the wall loss mechanism, included only for formal reasons, and m is the normalized passband coupling coefficient of Eq. 10-11,

$$m = \frac{1}{\omega_1/c} \int_{M-} \bar{\mathbf{e}}_1 \times \bar{\mathbf{H}}_1 \cdot \bar{\mathbf{i}}_z dS \quad (11-61)$$

negative for a passband of even field symmetry (m does not include the factor τ because of the normalization in Eq. 11-57). The m terms are chosen in Eq. 11-60 so that the equations of motion for $\mathbf{P}$ and $\mathbf{Q}$ to follow will reduce to the correct classical amplitude equations for a passband in which the fields have *even* symmetry across each cavity. For the other symmetry $\mathbf{P}_+$ and $\mathbf{P}_-$ are negated. In the very first cavity we would omit the $\mathbf{P}_-$ term and substitute a driving term for the loop excitation in the form

$$\mathbf{QD}(t) \quad (11-62)$$

where $\mathbf{D}$ is determined by the loop geometry and generator current. In the last or load cavity we would omit the $\mathbf{P}_+$ term in Eq. 11-60 and simply let operator $\mathbf{\Gamma}$ in that cavity include the effect of dissipation into the load, since the formal quantum treatment of ohmic dissipation is only an approximate perturbation treatment anyway. Later we introduce the classical dissipation constant and quality factor Q of each cavity.

To obtain the equations of motion for $\mathbf{P}$ and $\mathbf{Q}$ in an intermediate cavity we must account for the uncertainty principle which states that both cannot be simultaneously measured to an arbitrary degree of accuracy. Mathematically we say $\mathbf{P}$ does not commute with $\mathbf{Q}$ and write

$$[\mathbf{Q}, \mathbf{P}] \triangleq \mathbf{QP} - \mathbf{PQ} = i\hbar \quad (11-63a)$$

from which it follows that

$$[\mathbf{Q}, \mathbf{P}^2] = 2i\hbar\mathbf{P} \quad (11-63b)$$

$$[\mathbf{P}, \mathbf{Q}^2] = -2i\hbar\mathbf{Q} \quad (11-63c)$$

Many of our subsequent expressions will involve *commutator brackets* such as $[\mathbf{Q}, \mathbf{P}]$. Now there is no inconsistency in allowing $\mathbf{P}$ and $\mathbf{Q}$ each to commute with $\mathbf{\Gamma}$, $\mathcal{H}_a$, and all $\mathbf{M}_m$, $\mathcal{H}_{mn}$ and $\mathcal{H}_m$. We can also say that $\mathbf{Q}$ effectively commutes with each $\mathbf{P}_\pm$ (but see Prob. 10!), as does $\mathbf{P}$. Then the Heisenberg equation of motion for operator $\mathbf{Q}$ is, according to the end of the previous subsection,

$$\begin{aligned} i\hbar\dot{\mathbf{Q}} &= [\mathbf{Q}, \mathcal{H}_{\text{tot}}] \\ &= \left[\mathbf{Q}, 2\pi c^2 \mathbf{P}^2 + 2\pi c^2 m(\mathbf{P}^2 + \mathbf{P}\mathbf{P}_- + \mathbf{P}\mathbf{P}_+) + \mathbf{P} \int_{\text{wall}} \mathbf{\Gamma} \cdot \bar{\mathbf{E}}_1 dS \right] \end{aligned} \quad (11-64a)$$

which is identically

$$\dot{\mathbf{Q}} = 4\pi c^2 \mathbf{P} + 4\pi c^2 m(\mathbf{P} + \frac{1}{2}\mathbf{P}_- + \frac{1}{2}\mathbf{P}_+) + \int_{\text{wall}} \mathbf{\Gamma} \cdot \bar{\mathbf{E}}_1 dS \quad (11-64b)$$

Analogously, from $i\hbar\dot{\mathbf{P}} = [\mathbf{P}, \mathcal{H}_{\text{tot}}]$ we obtain

$$\dot{\mathbf{P}} = -\frac{\omega_1^2}{4\pi c^2} \mathbf{Q} + \sum_m \int_V \mathbf{M}_m \cdot \bar{\mathbf{H}}_1 dV \quad (11-65)$$

As a check, make the substitutions

$$\begin{aligned} \frac{\omega_1}{c} \mathbf{Q} &= \left(\frac{\mu}{\varepsilon}\right)^{1/4} \left(\frac{2\pi\tau}{c}\right)^{1/2} i_1, & \dot{\mathbf{Q}} &= i\omega \mathbf{Q} \\ 4\pi c \mathbf{P} &= -\left(\frac{\varepsilon}{\mu}\right)^{1/4} \left(\frac{2\pi\tau}{c}\right)^{1/2} v_1, & \dot{\mathbf{P}} &= i\omega \mathbf{P} \end{aligned} \quad (11-66)$$

whereupon Eqs. 11-64 and 11-65 without the $\mathbf{\Gamma}$ and molecular terms go over into Maxwell's Eqs. 4-34 and 4-35, Eq. 11-58 goes over to the mks expression for stored energy, and the equations yield the correct single mode-pair dispersion relation (except for a factor of $k^2/p_1^2 \cong 1$ in several terms) for a passband of even field symmetry and phase shift φ between cavities.

Senitzky [17] looks at the equation of motion for $\mathbf{\Gamma}$ as driven by $\mathcal{H}_a$ and $\mathcal{H}_d$ as driven by the electric field through $\mathbf{\Gamma}$ and shows that if we assume slight perturbation of the loss mechanism by the field the net effect is the replacement of operator $\mathbf{\Gamma}(t)$ by the uncoupled operator $\mathbf{\Gamma}^{(0)}(t)$ —the statistics of which are known—plus a dissipation term proportional to $\mathbf{P}$. With $\dot{\mathbf{Q}} = i\omega \mathbf{Q}$ and $\dot{\mathbf{P}} = i\omega \mathbf{P}$, the basic operator equations for $\mathbf{P}$ and $\mathbf{Q}$ (which may be regarded as the real parts of complex vectors) are

$$i\omega \mathbf{Q}(\omega) = 4\pi c^2 \mathbf{P} + 4\pi c^2 m(\mathbf{P} + \frac{1}{2}\mathbf{P}_- + \frac{1}{2}\mathbf{P}_+) + \int_{\text{wall}} \mathbf{\Gamma}^{(0)} \cdot \bar{\mathbf{E}}_1 dS - \alpha \frac{4\pi c^2}{i\omega} \mathbf{P} \quad (11-67a)$$

$$i\omega \mathbf{P}(\omega) = -\frac{\omega_1^2}{4\pi c^2} \mathbf{Q} + \sum_m \int_V \mathbf{M}_m(\omega) \cdot \bar{\mathbf{H}}_1 dV \quad (11-67b)$$

Here α is related to the ordinary quality factor Q as $\alpha = \omega/Q$ in a cavity. Since we are primarily concerned with the field amplitudes and field energy we must now express operator $\mathbf{M}_m$ in terms of those amplitudes.

C. Equations for the Components of $\mathbf{M}_m$

We have assumed, for simplicity, that both the “pump” and the “signal” magnetic fields lie in the x direction in each maser pill, perpendicular to

the axial d.c. magnetic field. Unfortunately, in order to calculate $\Sigma \mathbf{M}_m(\omega)$ for Eq. 11-67b we must also calculate $\mathbf{M}_{my}$ and $\mathbf{M}_{mz}$, as well as the elements of $\mathcal{H}_m$.

The equation of motion for $\mathbf{M}_m$ considered as a vector at a given frequency depends on the fact that it commutes with all other $\mathbf{M}_n$ and $\mathcal{H}_n$, and $\mathcal{H}_{nn'}$ for $n, n' \neq m$, as well as with $\mathbf{\Gamma}$ and $\mathcal{H}_d$; but not with itself, $\mathcal{H}_m$, $\mathcal{H}_{mn}$, and $\mathcal{H}_{nm}$. Thus, at a point in the material we have the equation of motion

$$i\hbar \dot{\mathbf{M}}_m = \left[\mathbf{M}_m, -\mathbf{M}_m \cdot \mathbf{H}_{\text{tot}} + \mathcal{H}_m + \frac{1}{2} \sum_{n \neq m} \mathcal{H}_{mn} \right] \quad (11-68)$$

From the commutation relation for angular momentum⁷ $\mathbf{S}: \mathbf{S} \times \mathbf{S} = i\mathbf{S}$ and the relation of $\mathbf{M}$ to $\mathbf{S}$ by the gyromagnetic factor γ given by $\mathbf{M} = \gamma\hbar\mathbf{S}$ ($\gamma < 0$ for electrons) we have the vector relation

$$\mathbf{M} \times \mathbf{M} = i\hbar\gamma\mathbf{M} \quad (11-69a)$$

which means

$$[\mathbf{M}_x, \mathbf{M}_y] = i\hbar\gamma\mathbf{M}_z \quad (11-69b)$$

and so on. The $\mathcal{H}_{mn}$ portion of this commutator represents a mutual effect of all the dipole moments on each other; the rate of change of one $\mathbf{M}_n$ operator due to this dipole-dipole coupling tends to be compensated by an opposite rate of change in another such operator. Therefore we would expect little or no effect on the *net* moment operator $\Sigma \mathbf{M}_m(t)$. And, in fact, we can show that the first portion of $\mathcal{H}_{mn}$ given by Eq. 11-59 has zero effect on the net moment operator, for

$$\sum_{m, n \neq m} \frac{1}{r_{mn}^3} [\mathbf{M}_m, \mathbf{M}_m \cdot \mathbf{M}_n] \equiv 0 \quad (11-70)$$

because the m th term exactly cancels the n th term when we employ Eq. 11-69. However, the second or r_{mn}^{-5} portion of $\mathcal{H}_{mn}$ will generally contribute to $\Sigma \mathbf{M}_m$ but the effect will be the smaller the more isotropic the spacial distribution of molecules. Let us assume a reasonably isotropic material and neglect this term.

For a calculation of the net operator $\Sigma \mathbf{M}_m$, then, we may neglect the last term in Eq. 11-68 and write the equations of motion for the three components of $\mathbf{M}_m$ as

$$i\hbar \dot{\mathbf{M}}_{mx} = [\mathbf{M}_{mx}, -\mathbf{M}_m \cdot \mathbf{H}_{\text{tot}} + \mathcal{H}_m] = +i\hbar\gamma\mathbf{M}_{my}\mathbf{H}_{\text{tot } z} + [\mathbf{M}_{mx}, \mathcal{H}_m] \quad (11-71a)$$

$$i\hbar \dot{\mathbf{M}}_{my} = +i\hbar\gamma\mathbf{M}_{mz}\mathbf{H}_{\text{tot } x} - i\hbar\gamma\mathbf{M}_{mx}\mathbf{H}_{\text{tot } z} + [\mathbf{M}_{my}, \mathcal{H}_m] \quad (11-71b)$$

$$i\hbar \dot{\mathbf{M}}_{mz} = -i\hbar\gamma\mathbf{M}_{my}\mathbf{H}_{\text{tot } x} + [\mathbf{M}_{mz}, \mathcal{H}_m] \quad (11-71c)$$

⁷ L. I. Schiff [18, p. 141].

where there is no magnetic field in the y direction. These formally correspond to Bloch's celebrated equations of motion [13] (but without the T_1 and T_2 dissipation terms, the latter of which is introduced next) which were originally written assuming the moment operator to commute with the molecular Hamiltonian $\mathcal{H}_m$. In reality, $\mathcal{H}_m$ does not commute with $\mathbf{M}_m$, and it is necessary to consider its equation of motion:

$$i\hbar\dot{\mathcal{H}}_m = \left[\mathcal{H}_m, -\mathbf{M}_{mx}\hat{\mathbf{H}}_{\text{tot}x} - \mathbf{M}_{mz}\hat{\mathbf{H}}_{\text{tot}z} + \frac{1}{2} \sum_{n \neq m} \mathcal{H}_{mn} \right] \quad (11-72)$$

In this equation we cannot neglect the $\mathcal{H}_{mn}$ term for reasons of isotropy; we must include it as an effect which tends to equalize the energy distribution among the various molecules. Since only those molecules with transition frequency $\omega_m \equiv \omega$ can radiate directly into the field, the $\mathcal{H}_{mn}$ effect corresponds to the "decay" of energy given by the T_2 terms in Bloch's equations. However, Eq. 11-72 enables us to study this energy exchange among the molecules in finer detail than Bloch's equations allow.

Equations 11-71 and 11-72 contain a tremendous amount of implicit information about the zero-order operators defined with the d.c. and pumping fields present and the first-order operators which result when the signal frequency is applied, both for those operators at the steady-state frequencies ω and $\omega_d = \omega_p - \omega$ and for the many operators at the various frequencies $\omega' \cong \omega$ and related $\omega'_d = \omega_p - \omega' \cong \omega_d$. Let us consider first the zero-order operators. All of them at frequencies ω and ω_d are not only related to each other but are also connected to operators at other frequencies through the $\mathcal{H}_{mn}(t)$ term in Eq. 11-72. Thus it is necessary, in principle, to consider the zero-order matrix elements of the molecular operators at a typical frequency $\omega'_d \cong \omega_d$ and related $\omega' \cong \omega$. Equations 11-71 and 11-72, of course, also describe these operators, where the $\mathcal{H}_{mn}$ term again furnishes time-average coupling between these operators and the ω , ω_d , and ω_p operators. All these equations will yield—subject to whatever assumptions prove convenient and realistic—the matrix elements of all the zero-order components of $\mathbf{M}_m$ and $\mathcal{H}_m$. The *first-order* operators are established by the signal-frequency portion of $\hat{\mathbf{H}}_{\text{tot}}$. This field at frequency ω creates first-order operators not only at frequency ω but also at ω_d , and these may be related to the zero-order operators by Eqs. 11-71 and 11-72 if we neglect the products of first-order terms. What is more, the $\mathcal{H}_{mn}$ coupling in Eq. 11-72 will allow first-order components at nearby frequencies $\omega' \cong \omega$ and $\omega'_d \cong \omega_d$ to be maintained, and these will produce, through the $\mathcal{H}_{mn}$ coupling, contributions to the first-order operators at frequencies ω and ω_d . So the situation is complicated in principle, and we usually make approximations to obtain a convenient

solution. For example, the gross effect of the $\mathcal{H}_{mn}$ coupling upon the operators at frequencies ω and ω_a is often represented by a simple T_2 decay term in the Bloch equations for the first-order portions of $\Sigma \mathbf{M}_{mx}$ and $\Sigma \mathbf{M}_{my}$.

As an example of how we use these equations to calculate matrix elements of zero-order operators with the d.c. and pump fields present, consider Eqs. 11-71 and 11-72 for a particular molecule m and ignore the $\mathcal{H}_{mn}$ coupling. Let the three eigenstates for this molecule have energies $E_{m3,2,1}$ such that $(E_{m3} - E_{m2})/\hbar = \omega_a'$, equal or nearly equal to ω_a and $(E_{m2} - E_{m1})/\hbar = \omega'$, equal or nearly equal to ω . To calculate the expectation value of any operator before turning on $\hat{\mathbf{H}}_x(\omega)$ we need matrix elements such as $\langle \psi_{mi} | \mathbf{M}_{mx}^{(0)} | \psi_{mj} \rangle$ for example, with $i, j = 1, 2, 3$. These can be arranged in matrix form as

$$\begin{pmatrix} M_{mx, 11} & M_{mx, 12} & \cdots \\ M_{mx, 21} & \cdots & \\ \vdots & & \\ \vdots & & \\ \vdots & & M_{mx, 33} \end{pmatrix} \quad (11-73)$$

and each matrix element may have more than one frequency component. When we substitute this and similar matrices for the other operators into Eqs. 11-71 and 11-72 (with $\hat{\mathbf{H}}_{\text{tot}} = \hat{\mathbf{H}}_{px}$ in the x direction and $\hat{\mathbf{H}}_{\text{d.c.}}$ in the z direction) and compare frequency terms we find that only d.c. and $2\omega_p$ frequency components are present. The elements of $\mathbf{M}_{mx}^{(0)}$ and $\mathbf{M}_{my}^{(0)}$ contain only ω_p terms, $\mathbf{M}_{mz}^{(0)}$ contains d.c. and $2\omega_p$ terms in each element, and $\mathcal{H}_m^{(0)}$ contains only d.c. terms except for the off-diagonal elements which contain both d.c. and $2\omega_p$ terms in general. The ω' and ω_a' , as well as other frequency components only appear in the *first-order* matrix elements established by the magnetic field at frequency ω .

In principle we may straightforwardly calculate first the zero-order matrix elements of these operators, then the first-order elements in terms of the known zero-order ones. Then the first-order component $\mathbf{M}_{mx}(\omega)$ may be expressed in terms of $\hat{\mathbf{H}}_x(\omega) = \mathbf{Q}\hat{H}_1$ for Eq. 11-67b in each cavity. We can then solve a number of simultaneous equations for the matrix elements of $\mathbf{P}$ and $\mathbf{Q}$ in each cavity, as related to the matrix elements of $\mathbf{P}_{\pm}$ and $\mathbf{Q}_{\pm}$ in the neighboring cavities and to the drive $\mathbf{D}$ in the first cavity. These matrix elements contain terms representing fluctuations due to the loss mechanism operator $\mathbf{\Gamma}^{(0)}(t)$, which includes the useful load in the last cavity, as averaged over the states of the loss mechanism. Finally, we can determine the expectation value of the time-average field energy and

power flow into the load of the last cavity (dissipation), the latter being

$$P_{\text{av, load}} = \frac{1}{2} \alpha \omega \operatorname{Re} i \sum_s \langle \psi_s | \rho(\omega) \mathbf{P}(\omega) \mathbf{Q}^*(\omega) | \psi_s \rangle \quad (11-74)$$

in complex notation, according to the end of Subsection A. The power flow of course depends on what steady-state density matrix for ρ is assumed to represent the molecules and the field; it depends upon the strength of the pumping field and the temperature.

Problems

1. Include the backward circuit modes at frequencies ω_1 and ω_2 in the Tien analysis according to the remark following Eq. 11-13.
2. What happens to the development of Section 11-3 if the pump phase shift is complex?
3. Prove from the Manley-Rowe relations that if four frequencies are present, ω_p , $\omega_1 < \omega_p$, $\omega_p - \omega_1$, and $\omega_p + \omega_1$, power gain to the frequency $\omega_p + \omega_1$, can occur at the expense of only pump power provided $\omega_p < \omega_p + \omega_1 < 2\omega_p$.
4. Show by sketches that the condition of $\hat{E}$ being a single-valued function of $\hat{P}$ need not always imply that the voltage between two points is a single-valued function of electric flux along the same path between the two points.
5. Prove $\epsilon_p(-1, 1; +\omega_p) = \epsilon_p^\dagger(0, 1; -\omega_p)$ component by component utilizing the Manley-Rowe relations.
6. Could we obtain Eqs. 11-21 through 11-23 from a variational principle for Maxwell's equations with many frequencies present? If so, write down the principle, verify its variational properties, expand the trial fields with the resonant cavity modes, and obtain the equations.
7. Verify the fact that the group velocity in the cavity chain of Fig. 11-4, with the diodes present but unpumped, will be positive in the range $0 < \beta_0 L < \pi$. Refer to Table 4-1 on p. 103.
8. Equations 11-36 and 11-37 furnish two conditions in the first cavity of a parametric amplifier, which must be satisfied at frequencies ω_1 and ω_2 , and also two conditions in the output cavity at these frequencies. Hence the amplitudes of the four independent waves are determined. Let $M_{mm}/p_m = M_{nn}/p_n = -0.1$ and choose operating conditions so that only one growing wave exists (Table 11-1). Let the output loads be matched according to Eq. 11-38. Evaluate the relative amplitudes of all four waves. What difficulty occurs if two growing waves are present?
9. With one ferrite post in each coupling slot between cylindrical cavities, page 386, estimate the eccentricity of the polarization of $\hat{H}$ for a periodic phase shift of $\pi/2$. Let the long slot dimension be much shorter than half a free-space wavelength. Estimate the $\hat{h}_z$ pattern of the open-circuit mode on the slot as equal in magnitude to the short-circuit mode pattern $\hat{H}_0$.
10. Show that an alternative statement which also implies Eqs. 11-64 and 11-65 is: $\mathbf{Q}$ does not commute with $\mathbf{P}_\pm$ but rather $[\mathbf{P}_\pm, \mathbf{Q}] = \pm i\hbar$, a being any

number. Justify the $\pm$ signs by the directions of the mode patterns $\vec{e}$ and $\vec{H}$ on the coupling holes of the cavity within which the quantum amplitudes of the electric and magnetic fields are $\mathbf{P}$ and $\mathbf{Q}$.

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Chapter XII

Aspects of Slow Wave Structure Synthesis

12-1. Synopsis

The choice of a particular slow wave structure for microwave work is dictated not so much by a specified impedance and bandwidth as by the optimum bandwidth obtainable for a minimum allowable impedance or else the maximum impedance attainable for a specified bandwidth. Therefore it is advantageous to set up computer programs and compute the ω - β curves and their associated impedances for a wide variety of helices and a wide range of parameters of each. We illustrate the steps in such a procedure with respect to a particular contrawound helix appropriate for high-power beam work. We first list the symmetry properties of the structure and deduce from them the most general expressions for the fields of the waves with different transverse symmetries. The field expressions for a wave of a given symmetry would then be substituted into a variational expression for ω and the relative amplitudes determined by the variational procedure so as to make ω stationary. Finally a computer could be programmed to solve the large determinantal equation for $\omega(\beta_0)$ by an iterative process, whereupon the value of ω determines not only a point on the dispersion curve but also the relative amplitudes of the components in the wave and hence its impedance. In order to judge the efficiency of a given helix we outline a procedure for calculating a so-called ideal interaction parameter and impedance Z_I which are nearly the maximums attainable for that structure. A sheath-helix has an efficiency Z/Z_I on the order of 0.95 for the wave with predominant $n = 0$ mode component.

Upper- and lower-bound expressions for $\omega(\beta_0)$ on the first dispersion

curve at a given β_0 are available. Their properties may be verified with respect to a Hermitian operator $\mathfrak{L}$, for which an infinity of "eigenfrequencies" and associated "eigenfields" must exist, satisfying the boundary conditions with the stipulated periodic phase shift. Each "eigenfield" is simply a wave at a frequency on one of the ω - β curves. In order to guarantee the bound properties we must have an infinity of unbounded "eigenfrequencies" in a closed structure. But since the expressions remain upper bound and lower bound regardless of how large the radius of the outer conductor may be, we expect to have useful bounds in the limit of an *open* structure. The upper-bound expression is, in fact, a very tight upper bound on certain unifilar and ring-plane helices which were analyzed in Chapter 6. We show how the earlier variational Eq. 6-45 is an upper-bound expression for $\omega(\beta_0)$ of a helix when the trial fields are defined for continuity between inner and outer regions and on the metal surfaces. It turns out the lower-bound expression is not useful when the trial fields are defined in this way. These upper- and lower-bound expressions are *not valid* for a cavity chain (for which they are simply variational without bound properties) in which the trial fields are written in standing wave form and cannot be represented by a linear combination of the forward-traveling "eigenfields." The proof of boundedness breaks down because the backward-traveling "eigenfields" are not orthogonal to the forward-traveling ones.

A perturbation formula for $\omega(\beta)$ is developed for the change in frequency of a general helix at fixed β due to a geometric perturbation at the surface bounding the inner and outer regions. The usefulness of this formula is verified on a contrawound helix, for which we assume the simplest trial fields.

We close the chapter with some discussion of how we may shift an ω - φ curve of a cavity chain by inserting resonant-slot or resonant-loop coupling, the passband of which lies just above or just below the "cavity" passband. A series of normalized curves shows the resonant-slot coupling to be somewhat more efficient than the loop coupling for the same numerical values of cross-coupling coefficients.

12-2. Machine Computation of the Characteristics of a Given Helix

In this section we adopt the viewpoint that the synthesis problem is best solved by computing the ω - β and impedance characteristics of a wide variety of helices which are likely candidates for microwave frequency work, so that we may select a helix of suitable phase velocity, group

velocity, and impedance merely by consulting standard tables and charts. The existence of rapid computers with large storage make this procedure eminently practical and most economical in the long run. This viewpoint is based upon the fact that we rarely ask for a slow wave structure of prescribed impedance and operating bandwidth. Rather, we may ask for that slow wave structure of optimum impedance within a specified frequency range in which the phase velocity is correct and the group velocity is essentially constant. Or we may want that structure of optimum operating bandwidth which is compatible with a minimum tolerable impedance. Rather than calculate laboriously by hand a number of necessarily approximate helix characteristics it is advisable to develop appropriate computer programs for the various helices and ladder structures.

We illustrate the procedure to follow in setting up a computer program for a given helix by using a certain contrawound helix as an example. This example is sufficiently nontrivial to illustrate some peculiarities in the trial fields which may arise in the course of developing them to be compatible with the structural symmetry.

There are three steps in the process of developing a computer program for the dispersion and impedance characteristics of such a helix. In the first step we list the symmetry properties of the structure and deduce from them the general expressions for the field components in a given wave. Often a structure will have a set of ω - β curves belonging to the various modes or mode-harmonics which comprise each wave of one transverse symmetry and another set of curves for each wave of another symmetry as, for example, the curves for the $(++)$ and $(+-)$ waves on the ring-plane helix (page 181). Of course, all the components of a given wave have the same symmetry properties. For the modified contrawound helix of Fig. 12-1 the symmetry operations are as follows.

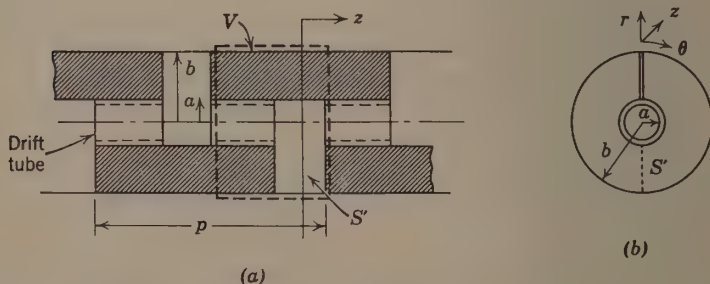


Fig. 12-1. Modified contrawound helix for high-power electron beam work. V denotes the volume of a unit cell $p/2$ long, as enclosed by the dashed lines. S' denotes the enclosed open surface across which some of the trial field components are discontinuous.

A. Reflection through a Vertical Plane Which Represents the Surfaces $\theta = 0$ and π

This means the geometry we see at the first point (r, θ, z) looking in the $-\theta$ direction is identical to the geometry we see at the point $(r, -\theta, z)$ looking in the $+\theta$ direction. Such a symmetry always implies two classes of waves:

1. The class for which

$$\begin{pmatrix} \hat{E}_z \\ \hat{E}_r \\ \hat{H}_\theta \end{pmatrix} \text{ at } (r, \theta, z) = + \begin{pmatrix} \hat{E}_z \\ \hat{E}_r \\ \hat{H}_\theta \end{pmatrix} \text{ at } (r, -\theta, z) \quad (12-1a)$$

and $\hat{H}_z$, $\hat{H}_r$, and $\hat{E}_\theta$ change sign.

2. The class for which

$$\begin{pmatrix} \hat{H}_z \\ \hat{H}_r \\ \hat{E}_\theta \end{pmatrix} \text{ at } (r, \theta, z) = + \begin{pmatrix} \hat{H}_z \\ \hat{H}_r \\ \hat{E}_\theta \end{pmatrix} \text{ at } (r, -\theta, z) \quad (12-1b)$$

and $\hat{E}_z$, $\hat{E}_r$, and $\hat{H}_\theta$ change sign.

B. Translation along the z Axis by $\pm p$, p Being the Pitch

Such a symmetry always implies an infinite set of harmonics in each wave, each with a factor

$$\exp \left[-j \left(\beta_0 + \frac{2\pi\ell'}{p} \right) \right], \quad \ell' \text{ integral}$$

(In a moment we change ℓ' to $\ell + 2m$ to accord with the Chodorow-Chu notation [1].)

C. Symmetry with Respect to a Rotation-Translation from (r, θ, z) to $(r, \theta + \pi, z \pm p/2)$

This means the geometry we observe from these two points is identical. Such a symmetry implies that the total field changes only by a phase shift from the first point to the second. The phase factor must be $e^{\mp j\beta_0 p/2}$

because two such transformations have the effect of advancement along the z axis by $\pm p$ and a phase shift by $\mp \beta_0 p$ according to property B .

D. Reflection through the Transverse $z = 0$ Plane

This symmetry operation, analogous to that of property A , implies some of the field components in a particular wave vary as $\sin(2\pi\ell'z/p)$ and the remaining components vary as $\cos(2\pi\ell'z/p)$ within a periodic cell. If such standing wave components are employed for the field expansions, they must all be phase-shifted by $e^{\mp j\beta_0 p}$ in the neighboring periodic cells in order to satisfy property B . This representation is unnatural for the relatively open contrawound helix, so we shall ignore this symmetry operation D and write the field components in traveling wave form.

The most general expression for $\hat{E}_z$ consistent with all these symmetry operations is

$$\begin{aligned} \hat{E}_z(r, \theta, z) &= \sum_{\ell} \sum_m A_{\ell m} R_{\ell}(\gamma_{\ell m} r) e^{j\ell\theta - j\beta_{\ell m} z} \\ \beta_{\ell m} &= \beta_{00} + \frac{2\pi}{p}(\ell + 2m), \quad \ell, m \text{ integral} \end{aligned} \quad (12-2)$$

Fundamental propagation constant β_{00} bears the double subscript to denote $\ell = m = 0$. At a given point r the set of $e^{j\ell\theta}$ functions is complete for expanding $\hat{E}_z$ in the θ direction and the z harmonics are complete for $\hat{E}_z$ along the z axis. Each ℓ' of symmetry operation B equals $\ell + 2m$ to accord with symmetry operation C . Because operation C is implied in Eq. 12-2 the unit cell may be regarded as the volume V of length $p/2$ in Fig. 12-1. Waves with the symmetry property $A1$ would consist of linear combinations of the functions in Eq. 12-2, each of which varies as $\cos \ell\theta$; waves of the symmetry $A2$ would have $\hat{E}_z$ components varying as $\sin \ell\theta$. By substituting $\hat{E}_z$ into the wave equation and separating the Fourier components we find R_{ℓ} satisfies the equation for the modified Bessel function (refer to Eq. 1-5) of argument $\gamma_{\ell m} r$, where

$$\gamma_{\ell m}^2 = \beta_{\ell m}^2 - k^2 \quad (12-3)$$

In the region $r \leq a$, $R_{\ell} \equiv I_{\ell}$, the modified Bessel function of the first kind, whereas in the outer $r \geq a$ region R_{ℓ} is a linear combination of I_{ℓ} and K_{ℓ} functions which is zero at $r = b$. The expression for $\hat{H}_z$ is, of course, similar except for amplitudes differing from the $A_{\ell m}$ and for the θ symmetry dual to that for $\hat{E}_z$.

In view of these statements the complete fields of a wave with symmetry

$A1$, for which $\hat{E}_z \neq 0$ on the axis, are obtained from these expressions for $\hat{E}_z$ and $\hat{H}_z$ within the inner portion of the unit cell V of Fig. 12-1:

$$\begin{aligned}\hat{e}_z &= \sum_{\substack{\ell > 0, \\ \text{all } m}} A_{\ell m} I_{\ell}(\gamma_{\ell m} r) \cos \ell \theta \epsilon^{-i\beta \ell m z} \\ \hat{h}_z &= \sum_{\substack{\ell > 0, \\ \text{all } m}} B_{\ell m} I_{\ell}(\gamma_{\ell m} r) \sin \ell \theta \epsilon^{-i\beta \ell m z} \quad (r \leq a)\end{aligned}\quad (12-4)$$

In the outer $a \leq r \leq b$ region $\hat{E}_z$ and $\hat{H}_z$ (capital letters for the outer region) must have the same symmetries with $\hat{E}_z = 0$ on the fin at $\theta = 0$. Now it is natural to try to specify $\sin \ell \theta$ variations of the ℓ th components of $\hat{E}_z$, where ℓ is half-integral ($\frac{1}{2}, \frac{3}{2}, \dots$). However, even though the components of Eqs. 12-2 and 12-4 will still satisfy the wave equation, we cannot satisfy both symmetry conditions B and C . Hence, we represent the fields as we did for the $(++)$ wave of the ring-plane helix, namely as

$$\begin{aligned}\hat{E}_z &= \sum_{\substack{\ell > 0, \\ \text{all } m}} A_{\ell m}' R_{\ell}(\gamma_{\ell m} r) \sin \ell \theta \epsilon^{-i\beta \ell m z} \\ \hat{H}_z &= \sum_{\substack{\ell > 0, \\ \text{all } m}} B_{\ell m}' R_{\ell}(\gamma_{\ell m} r) \cos \ell \theta \epsilon^{-i\beta \ell m z} \quad (a \leq r \leq b, 0 \leq \theta \leq \pi)\end{aligned}\quad (12-5)$$

In the remaining region $\pi \leq \theta \leq 2\pi$ the field components are the negatives of those evaluated by Eq. 12-5 so they will have the proper symmetries. As we said before, R_{ℓ} is a linear combination of I_{ℓ} and K_{ℓ} functions adjusted so $R_{\ell}(\gamma_{\ell m} b) = 0$.

The remaining field components of the ℓm th mode-harmonic are obtained from Eqs. 12-4 and 12-5 in the usual way,

$$\begin{aligned}\hat{e}_r &= \frac{1}{\gamma^2} \left(j\beta \frac{\partial \hat{e}_z}{\partial r} + \frac{j\omega\mu}{r} \frac{\partial \hat{h}_z}{\partial \theta} \right), & \hat{h}_r &= \frac{1}{\gamma^2} \left(-\frac{j\omega\epsilon}{r} \frac{\partial \hat{e}_z}{\partial \theta} + j\beta \frac{\partial \hat{h}_z}{\partial r} \right) \\ \hat{e}_{\theta} &= \frac{1}{\gamma^2} \left(j\beta \frac{\partial \hat{e}_z}{\partial \theta} - j\omega\mu \frac{\partial \hat{h}_z}{\partial r} \right), & \hat{h}_{\theta} &= \frac{1}{\gamma^2} \left(j\omega\epsilon \frac{\partial \hat{e}_z}{\partial r} + j\beta \frac{\partial \hat{h}_z}{\partial \theta} \right)\end{aligned}\quad (12-6)$$

(Subscript ℓm has been omitted from $\hat{e}_r$, $\hat{h}_z$, and so on in these expressions.) This completes the construction of the fields for a wave which has the symmetry property $A1$. For the overwhelming majority of helical structures there are no additional complications; however, certain varieties of cavity chains may have a host of symmetries. For these it may prove useful to classify the modes by a more formal procedure, such as that afforded by group theory [2].

We have just completed the first step in the development of a computer program for the characteristics of the waves with symmetry properties specified by $A1$, B , and C , and we go on now to the second step. We insert the field expressions into a variational expression for $\omega(\beta_{00})$ and set the change of ω , $\delta\omega$, equal to zero for every variation of one of the

trial fields. We must take care to use a variational expression which constrains the fields to be continuous everywhere. Thus, although Eq. 6-45 tends to constrain the proper boundary conditions on the drift tubes and continuities across the open surface at $r = a$, it does not constrain continuity of $\hat{H}_z$, $\hat{H}_r$, and $\hat{E}_\theta$ across surface S' in Fig. 12-1. We may easily prove that the following terms must be added to the right side of Eq. 6-45:

$$\begin{aligned} & -\frac{1}{2} \int_{S'} [\hat{E}_-^*(+) + \hat{E}_-^*(-)] \times [\hat{H}_+(+) - \hat{H}_+(-)] \cdot \hat{i}_\theta dS \\ & -\frac{1}{2} \int_{S'} [\hat{E}_+^*(+) - \hat{E}_+^*(-)] \times [\hat{H}_-^*(+) + \hat{H}_-^*(-)] \cdot \hat{i}_\theta dS \end{aligned} \quad (12-7)$$

in which $(+)$ denotes a field on the $+\theta$ side of S' and $(-)$, a field on the $-\theta$ side. Now the fact is that trial field E_z is identically zero on each side of S' and so is E_r by Eq. 12-6. Therefore these S' integrals happen to be identically zero even though the trial $\hat{H}_z$ and $\hat{H}_r$ are discontinuous across S' ! This curious and undesirable situation has the following explanation. The correct $\hat{E}_z$ field is finite on S' , yet we are representing it by mode-harmonics which are each identically zero on S' . Therefore the expansion will never converge to $\hat{E}_z$ on S' and will converge slowly to $\hat{E}_z$ near S' . At the same time we are representing $\hat{H}_z$ by mode-harmonics each of which changes sign across S' , whereas the true $\hat{H}_z$ field is exactly zero on S' . Therefore the correct expression for $\hat{H}_z$ on S' is the average of the expressions for it on either side, namely zero, and either expansion converges poorly to $\hat{H}_z$ near S' . Similar remarks hold for the other field components. These errors are tolerable when we realize the fact that the fields are well represented at all points not in the immediate vicinity of S' by the predominant mode-harmonics of the appropriate symmetry.

Although the fields are not well represented near and on surface S' of Fig. 12-1 we can substitute their expansions into variational Eq. 6-45 with the area integrals changed to volume integrals, whereby the latter are zero because each mode-harmonic satisfies Maxwell's equations identically. The result of the variational procedure is a set of homogeneous equations for the $A_{\ell m}$ and other amplitudes, the determinant of which set equal to zero yields the value of $\omega(\beta_{00})$ for the wave of the prescribed transverse symmetry. The determinantal equation is best solved by trial-and-error by an iteration technique which starts with an appropriate value of ω because the arguments of the Bessel functions include $\gamma_{\ell m} = (\beta_{\ell m}^2 - k^2)^{1/2}$ and $k = \omega\sqrt{\mu\epsilon}$ is being sought.

For any particular computation it is important to know which mode-harmonics will contribute predominantly to the fields so only they will be

included. For example, suppose we wish to compute a point on the ω - β curve in the fundamental region

$$-0.5 \cot \psi \leq \beta_{00} a \leq 0.5 \cot \psi \quad (\cot \psi \triangleq 2\pi a/p)$$

for a wave in which we specify the $\ell = L$ mode to be predominant. If this is the case, there must be a β_{LM} in the $\beta_{\ell m}$ series which is $\approx k$. Let us renumber the mode-harmonics such that β_{LM} is the new β_{00} and the other propagation constants are designated as

$$\beta_{\ell' m'} = \beta_{00} + \frac{2\pi}{p} [(\ell' - L) + 2(m' - M)],$$

$$\ell' = \dots, L - 1, L, L + 1, \dots$$

$$m' = \dots, M - 1, M, M + 1, \dots$$

The important mode-harmonics are those for which the $\gamma_{\ell' m'}$ are small and the $\beta_{\ell' m'} \approx k$. In a practical case we might take those harmonics specified by $\ell' = L - 1, m' = M + 2, M + 1, M$; $\ell' = L, m' = M + 1, M, M - 1$; and $\ell' = L + 1, m' = M - 2, M - 1, M$, for a total of nine components in the inner region and nine in the outer region. A total of $9 \times 4 = 36$ amplitudes would be involved in the variational procedure, unless some explicit field continuities are specified, and this would lead to a 36×36 determinantal equation for $\omega(\beta_{LM})$. A modern digital computer can easily handle a determinantal equation of this size.

In the third step of the process we plot the value of $\omega(\beta_{LM})$ computed and also insert it into the homogeneous equations for the mode amplitudes, thereby determining their relative values. Then we can compute the relative stored energy

$$W_s = \frac{1}{4} \int_V (\epsilon |\hat{E}^2| + \mu |\hat{H}^2|) dV = \frac{1}{2} \int_V \epsilon |\hat{E}^2| dV$$

in the unit cell. The interaction parameter $|\hat{E}_{\alpha, \ell m}^2|/W_s$ may then be plotted as a point, where $\hat{E}_{\alpha, \ell m}$ is the field component in the α th direction of the ℓm th mode-harmonic utilized for interaction purposes. Alternatively, $|\hat{E}_{\alpha}^2|/W_s$ may be plotted. By repeating the calculation over the fundamental range of β_{LM} we can plot the ω - β curve in this range and then extend it along the β axis so as to represent the other mode-harmonics in this wave.

12-3. Ideal Interaction Parameter and Impedance Z_I for a Helix or Cavity Chain

To judge the efficiency of a given helix and also provide a means of comparing various helices we now formulate a method of estimating the

minimum value of stored energy W_s . The ideal interaction parameter will be defined as $|\hat{E}_\alpha|^2/W_{s,\min}$, where $\hat{E}_\alpha$ is the electric field in the α direction. $\hat{E}_\alpha$ is appropriately composed of just a single mode-harmonic in a helical structure, whereas in a cavity chain it is represented by one or more resonant cavity modes. The ideal impedance is

$$Z_I = \frac{|\hat{E}_\alpha|^2}{2\beta_n^2 W_{s,\min} v_g}$$

where β_n is the propagation constant of the n th mode-harmonic which is principally involved in the interaction (β_n corresponds to a flock of $\beta_{\ell m}$ such that $n = \ell + 2m$ on a contrawound helix) and v_g is the actual group velocity at the mean frequency in the range of interest. These parameters are defined for a helix model of a physical helix which has infinitely thin metal surfaces. We cannot prove that the ideal parameters are always larger than the measured ones, but they may be regarded as upper bounds for practical purposes and discussions. Indeed, we have not found a true upper-bound expression for the impedance of a helix. These ideal parameters are nearly the maximum possible values for a given helix because of the fact developed in Chapter 6, namely, that the fields in a helix at a particular point on the ω - β curves are predominantly described by one or two modes or mode-harmonics of the appropriate symmetry both inside and outside the helix radius. In ladder structures several mode-harmonics also suffice for each region, although the functions in one region may differ considerably in form from those of another region. As an example, the waves with predominant $n \neq 0$ and $n = 0$ mode components in Section 6-3 were each described by just a few modes, as were the $(++)$ and $(+-)$ waves of the ring-plane helix in Section 6-4. The predominant mode-harmonics are those with the smallest radial propagation constants.

In order to calculate the ideal parameters we propose to minimize the expression for W_s written as twice the electric field energy in terms of these predominant modes or mode-harmonics, with the proviso that one of the amplitudes of the reference field in $\hat{E}_\alpha$ is held fixed. The amplitudes of the modes in each helix region are to be related to those of the neighboring regions by equating tangential electric fields on common boundaries in the best way possible, that is, in a variational sense on the boundary between the two regions.

Let us apply the procedure to the especially simple unifilar helix carrying the wave of predominant $n = 0$ mode. Inside the helix $\hat{e}_z$ will have the reference amplitude of 1. The fields within the helix are represented by the $n = 0$ sheath mode of Eqs. 2-1 and 2-2, and the outer fields by the $n = 0$ mode of Eqs. 2-3 and 2-4. Clearly the stored energy inside and outside

the helix does not involve cross products of the amplitudes, so the minimum energy is attained by neglecting the TE portion of the field. The fields are taken to be

$$\begin{aligned} \hat{e}_z &= B_3 I_0(\gamma_0 r) & \hat{E}_z &= B_4 K_0(\gamma_0 r) \\ r \leq a, \quad \hat{e}_r &= B_3 \frac{j\beta_0}{\gamma_0} I_1 & r \geq a, \quad \hat{E}_r &= -B_4 \frac{j\beta_0}{\gamma_0} K_1 \\ \hat{h}_\theta &= B_3 \frac{j\omega\epsilon}{\gamma_0} I_1 & \hat{H}_\theta &= -B_4 \frac{j\omega\epsilon}{\gamma_0} K_1 \\ B_3 &= 1 \end{aligned} \quad (12-8)$$

Here the $\epsilon^{-j\beta_0 z}$ variation is understood. We match $\hat{e}_{\tan} = \hat{e}_z$ to $\hat{E}_z$ at $r = a$ by making $B_4 = I_0(\gamma_0 a)/K_0(\gamma_0 a)$. Then the total minimum stored energy $W_{s,\min}$ per unit axial length is easily calculated as

$$W_{s,\min} = 2 \cdot \frac{1}{4} \epsilon \left(\int_0^a e_z^2 \cdot 2\pi r \, dr + \int_a^\infty E_z^2 \cdot 2\pi r \, dr \right) \quad (12-9)$$

With the aid of the integral table on page 176 and the properties of the Bessel functions given by Pierce¹ we obtain

$$W_{s,\min} = \pi \epsilon \left(\frac{\beta_0}{\gamma_0} \right)^4 \frac{I_0}{K_0} \left[1 + \frac{k^2}{2\beta_0^2} \gamma_0 a \left(\frac{I_1}{I_0} - \frac{K_1}{K_0} \right) \right] \quad (12-10)$$

with $\gamma_0 a$ as the Bessel function argument. The ideal interaction parameter of the unifilar wave with predominant $n = 0$ mode is then given by $|\hat{e}_z^2|/W_{s,\min} = I_0^2(\gamma_0 r)/W_{s,\min}$.

If we compare the impedance Z of the $n = 0$ mode on a sheath helix with the ideal value Z_I calculated by Eq. 12-10 for the parameters $v_g/v_p = v_g\beta_0/\omega = 0.70$ and $\gamma_0 a = 1$, we find the sheath impedance is 95% as large as the ideal value. This is very reasonable considering that the electric energy in the E_θ field of the sheath helix is negligible for these parameters, so that we expect the ideal impedance to be only slightly higher.

By analogy the ideal impedance of the cavity chain at a frequency in the n th passband would be given by the short- or open-circuit mode of that passband, depending on whether the operating frequency is nearer the resonant frequency of the former or the latter mode. In the former case the ideal impedance is obtained from Eq. 4-27 as

$$Z_I = \frac{\bar{E}_{zn}^2 \cdot 2L^2 P_n}{\omega \epsilon (-M_{nn}) \tau \varphi^2 \sin \varphi} \quad \text{ohms} \quad (12-11)$$

subscript n denoting the parameters of the short-circuit mode of the n th passband assumed narrow compared to the width of the adjacent stop-band(s), L the periodic length, τ the cavity volume, and $\varphi = \beta_0 L$ the

¹ J. R. Pierce [3, p. 224].

periodic phase shift. If the operating frequency lies nearer p_n , the resonant frequency of the n th open-circuit mode $\bar{E}_{zn}^2$ is replaced by $\bar{e}_{zn}^2$.

Each helix has its own ideal interaction parameter and impedance which may be calculated in the manner outlined (see Prob. 2).

12-4. Upper and Lower Bounds to $\omega(\beta_0)$

If we do not have a computer and programmer available to perform an accurate computation of the ω - β curves for a given helix we can utilize expressions which overestimate or underestimate the value of $\omega(\beta_0)$ for each value of β_0 on the first dispersion curve. The expressions we shall develop may, in principle, be used to obtain bounds on the higher-frequency ω - β curves but only if the trial fields are constructed to be orthogonal to the fields which exist at all the lower frequencies on the various curves. The expressions are valid for all kinds of helices and the ladder structures but they are *not* valid for cavity chains in which the trial fields are represented by resonant cavity modes or by standing modes in one region of the periodic cell and traveling modes in another region of the cell. We shall give the reason for these statements in due course. First, we want to develop the bounds for the first ω - β_0 curve of an arbitrary open structure, be it a multifilar helix, ring-plane helix, or some variety of ladder structure.

We first develop an upper-bound expression for $\omega(\beta_0)$ on the first dispersion curve of a helix in terms of trial fields which are *continuous* throughout each periodic cell. The lower-bound expression follows naturally. Then we show how this upper-bound expression becomes variational Eq. 6-45 when the *discontinuous* trial fields are redefined so as to have appropriate continuities.

It is convenient to define a column vector $\mathfrak{F}$ for the entire field.

$$\mathfrak{F} = \text{col}(\mathbf{E}_r, \mathbf{E}_\theta, \mathbf{E}_z, \mathbf{H}_r, \mathbf{H}_\theta, \mathbf{H}_z) \quad (12-12)$$

(the field in boldface represents trial field both inside and outside the helix radius), as well as a 6×6 operator matrix $\mathfrak{L}$ as

$$\mathfrak{L} = j \begin{bmatrix} 0 & -\mathbf{L} \\ \mathbf{L} & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial r} \\ -\frac{1}{r} \frac{\partial}{\partial \theta} & \frac{1}{r} \frac{\partial}{\partial r} r & 0 \end{pmatrix} \quad (12-13)$$

The operator $\mathfrak{L}$ applied to $\mathfrak{F}$ will generate the $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{H}$ terms in Maxwell's equations, which are equivalent to the single equation

$$\mathfrak{L}\mathfrak{F} = \omega\mathfrak{G}\mathfrak{F} \quad (12-14)$$

in which $\mathfrak{G}$ is the diagonal matrix

$$\mathfrak{G} = \text{diag} (\epsilon_0\epsilon_0\epsilon_0\mu_0\mu_0\mu_0) \quad (12-15)$$

We now declare that the following is an upper-bound expression for $\omega(\beta_0)$,

$$\omega_u = \frac{\int_V \mathfrak{F}^\dagger \mathfrak{L}\mathfrak{F} dV}{\int_V \mathfrak{F}^\dagger \mathfrak{G}\mathfrak{F} dV} \quad (u \text{ for upper bound, } \dagger \text{ for transpose conjugate}) \quad (12-16)$$

V being the periodic volume of the helix. The proof may be carried out by analogy with the proof starting on page 158 that Eq. 6-8 is an upper bound for k^2 satisfying the Helmholtz equation. The proof depends, however, upon the existence of a complete set of "eigenfields," the n th one of which satisfies the equation

$$\mathfrak{L}\mathfrak{F}_n = \omega_n\mathfrak{G}\mathfrak{F}_n, \quad n = 1, 2, \dots \quad (12-17)$$

Each of these "eigenfields" is an independent wave consisting of a proper linear combination of modes or mode-harmonics satisfying all the boundary conditions at frequency ω_n on the n th ω - β curve, with the specified phase shift per period of $\beta_0 p$. The various ω_n are found at the intersections with all the dispersion curves of a line parallel to the ω axis through β_0 . We must now examine the conditions under which we have a complete set.

In order that the "eigenfields" for operator $\mathfrak{L}$ form a complete set it is sufficient that²

1. $\mathfrak{L}$ is Hermitian, which implies for any field $\mathfrak{F}$ that

$$\int_V \mathfrak{F}^\dagger \mathfrak{L}\mathfrak{F} dV$$

is real. This condition is satisfied by Eq. 12-13.

2. $\mathfrak{L}$ is positive-definite for the trial $\mathfrak{F}$ of interest, so that

$$\int_V \mathfrak{F}^\dagger \mathfrak{L}\mathfrak{F} dV > 0$$

This condition must be verified directly; for all trial fields of physical interest the integral will be positive and will give a positive value of ω_u in Eq. 12-16, the denominator of which is always positive.

² P. M. Morse and H. Feshbach [4, p. 776].

3. Equation 12-17 for the "eigenfield" $\mathfrak{F}_n$ corresponds to a variational principle. The principle here is simply that the ratio in Eq. 12-16 is a minimum for $\mathfrak{F} = \mathfrak{F}_1$; again a minimum for $\mathfrak{F}$ orthogonal to $\mathfrak{F}_1$, in which case the ratio is ω_2 and the new $\mathfrak{F}$ is $\mathfrak{F}_2$; and so on.

With these conditions satisfied the proof is straightforward that an arbitrary field $\mathfrak{F}$ which exhibits the specified symmetry and has a periodic phase shift $\beta_0 p$ will be replaced at every point in V by a sum

$$\mathfrak{F} = \sum_1^N c_m \mathfrak{F}_m$$

which converges to $\mathfrak{F}$ as $N \rightarrow \infty$. Therefore the set of $\mathfrak{F}_n$'s is complete. *However*, there is the additional requirement that the set of eigenfrequencies ω_n must not terminate, and this condition is violated on an open structure for which the ω_n 's converge to $\omega = \beta_0/\sqrt{\mu\epsilon}$ on the velocity-of-light line. Therefore Eq. 12-16 is strictly an upper bound only for a *closed helix or ladder structure, surrounded by a perfect conductor*. The outer conductor can have as large a radius as desired; it need only be present to guarantee an infinite set of "eigenfields" (waves) which is complete.

Since Eq. 12-16 is an upper bound regardless of the radius of the outer conductor, and the tangential electric field on it due to the ordinary helix fields approaches zero as its radius approaches infinity we expect ω_u to be an upper bound when the outer conductor is ignored. Another reason for this behavior is the fact that the effect of the outer conductor on the first few dispersion curves of interest occurs primarily in the forbidden regions, more so as the conductor moves out to infinity. Thus it is not surprising that calculations based on Eq. 6-45 have indeed yielded upper bounds in Figs. 6-5 and 6-6.

To obtain a lower-bound expression to $\omega(\beta_0)$ we employ another functional which has the same dimensions as ω_u , namely

$$\omega_a = \frac{\int_V \mathfrak{F}^\dagger \mathfrak{L} \mathfrak{G}^{-1} \mathfrak{L} \mathfrak{F} dV}{\int_V \mathfrak{F}^\dagger \mathfrak{L} \mathfrak{F} dV} \quad (12-18)$$

and state without proof that the following quantity is a lower-bound expression for ω under the same conditions as ω_u is an upper bound:

$$\omega_\ell = \omega_u \left(1 - \frac{\omega_a - \omega_u}{\omega_2 - \omega_u} \right) \quad (\ell \text{ for lower bound}) \quad (12-19)$$

in which ω_2 is the second "eigenfrequency" corresponding to the specified β_0 , presumed known accurately on the second ω - β curve. This is just the

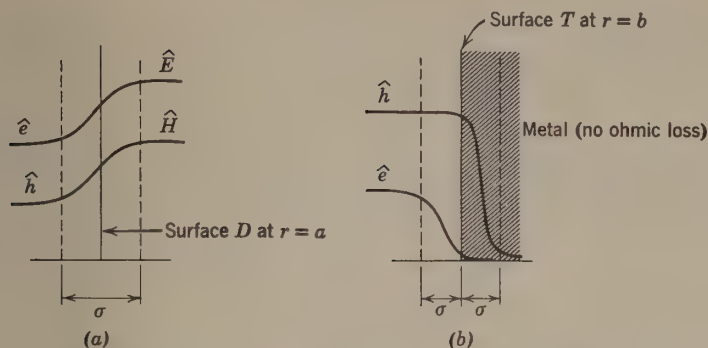


Fig. 12-2. Illustration of the defined continuities of the trial fields, (a) across open surface D and (b) at metal surface T . σ is a parameter which can be made as small as desired.

Morse and Feshbach equation 110,³ for which a proof of the lower-bound nature can be carried out straightforwardly without invoking square roots of operators.

We now show how the trial fields which were discontinuous between inner and outer helix regions in Chapter 6 (see Eqs. 6-51 and 6-52, for example) may be redefined to be continuous with respect to a “smoothing parameter” σ in such a way that, as $\sigma \rightarrow 0$ upper-bound Eq. 12-16 becomes precisely the variational Eq. 6-45. In other words, we will show that Eq. 6-45 is an upper-bound variational expression for $\omega(\beta_0)$ on the first dispersion curve of an arbitrary helix or ladder structure, enclosed by a perfect conductor of arbitrary radius. After this discussion we shall observe the care we must take to specify an appropriate trial field F .

Consider for definiteness either a unifilar or a ring-plane helix discussed in Chapter 6. Figure 12-2 illustrates the continuity of the trial fields which we wish to establish across “discontinuity” surface D at $r = a$ where the inner $\hat{e}$ and $\hat{h}$ trial fields must be “blended” into the outer $\hat{E}$ and $\hat{H}$ fields. Shown also is a convenient way to define the $\hat{e}$ and $\hat{h}$ fields on a metal surface where they would not otherwise be correct. The justification for these defined trial field behaviors resides solely in the fact that as $\sigma \rightarrow 0$ variational Eq. 12-16 will become precisely Eq. 6-45, which we know is variational for discontinuous trial fields. The indicated continuities may be attained by employing a smoothing function s which becomes a Dirac delta function as $\sigma \rightarrow 0$,

$$s(r) = \frac{1}{\sigma\sqrt{2\pi}} e^{-r^2/2\sigma^2} \quad (12-20)$$

³ P. M. Morse and H. Feshbach [4, p. 1144].

Let the surface D be at $r = a$; then a desirable gate function $G_D(r)$ which approaches a unit step at $r = a$ as $\sigma \rightarrow 0$ is

$$G_D(r) = \int_{-\infty}^r s(r' - a) dr' \quad (12-21)$$

The continuous fields $\mathbf{E}$ and $\mathbf{H}$ which change over from $\hat{e}$ and $\hat{h}$ inside the helix to $\hat{E}$ and $\hat{H}$ outside across $r \cong a$, as shown in Fig. 12-2, are

$$\begin{aligned} \mathbf{E} &= \hat{e}[1 - G_D(r)] + G_D(r)\hat{E} \\ \mathbf{H} &= \hat{h}[1 - G_D(r)] + G_D(r)\hat{H} \end{aligned} \quad (12-22)$$

Let metal surface T be at $r = b$; then $\mathbf{E}$ and $\mathbf{H}$ in the $r \leq b$ region can be specified in terms of a G_{Te} which approaches a unit step at $b - \sigma/2$ as $\sigma \rightarrow 0$ and a G_{Th} which approaches a unit step at $b + \sigma/2$:

$$\begin{aligned} G_{Te}(r) &= \int_{-\infty}^r s\left(r' - b + \frac{\sigma}{2}\right) dr' \\ G_{Th}(r) &= \int_{-\infty}^r s\left(r' - b - \frac{\sigma}{2}\right) dr' \end{aligned} \quad (12-23)$$

so that $\mathbf{E}$ and $\mathbf{H}$ are, according to the figure, in the $r \leq b$ region just

$$\begin{aligned} \mathbf{E} &= (1 - G_{Te})\hat{e} \\ \mathbf{H} &= (1 - G_{Th})\hat{h} \end{aligned} \quad (12-24)$$

$\mathbf{E}$ and $\mathbf{H}$ on the outer metal surface are defined analogously in terms of $\hat{E}$ and $\hat{H}$.

Strictly speaking, we should multiply these trial fields by angularly varying gate functions which "blend" $\mathbf{E}$ and $\mathbf{H}$ on surface D into the corresponding fields on the metal surfaces. However, a little thought will show that the new

$$\iint \mathfrak{F}^\dagger \left(\frac{\partial}{\partial \theta} \right) \mathfrak{F} dr d\theta$$

term would approach zero as $\sigma \rightarrow 0$.

By substituting these trial fields into $\mathfrak{F}$ of Eq. 12-12 and performing the integration in Eq. 12-16 with the aid of the integrals

$$\int G_D^2(r) dr \cong \sigma \quad (12-25a)$$

$$\int G_D \left(\frac{\partial}{\partial r} \right) G_D dr = \frac{1}{2} \quad (12-25b)$$

we may verify the fact that variational Eq. 6-45 emerges in the limit as $\sigma \rightarrow 0$. This verifies the upper-bound nature of Eq. 6-45 for $\omega_1(\beta_0)$ on the

first dispersion curve. It follows that once the associated fields $\mathbf{E}_1$ and $\mathbf{H}_1$ are known, a trial field $\mathfrak{F}$ can be constructed orthogonal to $\mathfrak{F}_1$, whereupon Eq. 12-16 or 6-45 would yield an upperbound to $\omega_s(\beta_0)$ on the second dispersion curve. This, needless to say, is a tedious procedure because $\mathfrak{F}_1$ must be known accurately if we are to have any confidence in the result.

Figure 6-5 and particularly Fig. 6-6 show how close the upper-bound calculations can be to the measured values. In the first approximation, the small finite radial thickness of the rings in the ring-plane helix will not affect the dispersion curve plotted with respect to the normalized parameters ka and $\beta_0 a$. Note on Fig. 6-6 that the curve for the $(+ -)$ ring helix wave is the first of a set of curves for waves of this particular symmetry while the waves of $(+ +)$ symmetry are treated disjointly. A valuable point may be made by reference to Fig. 6-4 in connection with the fact that the first variational calculation for a unifilar helix of infinitely thin radial tape thickness did not furnish a true upper bound! This happened because we imposed overly strict *explicit* continuity conditions on the trial fields (see Eq. 6-54) in order to simplify the calculation. However, the second variational calculation based upon only the explicit continuity of $\hat{e}_{||} = \hat{E}_{||}$ at $r = a$ yielded a $ka - \beta_0 a$ curve which is compatible with measurements made on a similar unifilar round wire helix. The lesson here seems to be this: It is not sufficient for the trial field to have continuity everywhere and the specified phase shift per period, it must also not be constrained in any manner incompatible with its (unknown!) behavior with frequency. The implications of this remark remain to be explored further.

Equation 12-19 does not yield a useful lower bound when we employ the trial function for $\mathfrak{F}$ defined for continuity as shown by Fig. 12-2, for a very good reason. The $\mathfrak{L}\mathfrak{G}^{-1}\mathfrak{L}$ operator in the numerator includes the $\partial^2/\partial r^2$ operator, so that there are terms of the form

$$\int_{a-\sigma/2}^{a+\sigma/2} G_D \left(\frac{\partial^2}{\partial r^2} \right) G_D dr = - \int \left(\frac{\partial G_D}{\partial r} \right)^2 dr \rightarrow \infty \quad \text{as} \quad \frac{1}{\sigma} \quad (12-26)$$

To avoid the difficulty G_D could be redefined as the double integral of the s function which is a ramp function beginning on the D or T surface, but this is not convenient for the integrations over the volumes.

In closing this section we must point out the fact that Eqs. 12-16 and 12-19 cease to represent upper and lower bounds to the first $\omega - \beta$ curve of a slow wave structure when the trial fields include standing wave (resonant mode) components. This is true even though variational Eq. 6-77 for the cavity chain can be regarded as a limiting form of Eq. 12-16 when the trial fields are redefined for continuity across the coupling surfaces. Evidence of this was first mentioned on page 165 in connection with the

calculation of the first dispersion curve of a periodic lumped-loaded transmission line by variational equations which stem from Eq. 12-16. Additional evidence is furnished by the work on the ridged waveguide in Section 6-6, for which Fig. 6-10 certainly does not show upper bounds. Equation 12-16 and its limiting forms, Eq. 6-45 or Eq. 6-77, cease to be upper-bound expressions when both $\mathbf{E}$ and $\mathbf{H}$ in the trial fields contain standing wave components, as in Eqs. 6-69 and 6-70 in a cavity of the ridged waveguide. The proof of upper boundedness breaks down unless the trial fields can be expanded in a (complete) set of traveling wave eigenfields, one belonging to each dispersion curve at the given fundamental propagation constant β_0 . Now it is certainly true that standing wave components of *either* $\mathbf{E}$ or $\mathbf{H}$ could be represented by these eigenfields, but eigenfields propagating in *both* directions as $e^{-j\beta_0 z}$ and $e^{j\beta_0 z}$ are required to represent standing wave components of *both* $\mathbf{E}$ and $\mathbf{H}$. The backward eigenfields as solutions to Maxwell's equations at the various points on the dispersion curves corresponding to $-\beta_0$ instead of β_0 are not orthogonal to the forward-traveling eigenfields, and so the demonstration of upper boundedness of Eq. 12-19 fails just as it would fail for variational Eq. 6-22 appropriate for the Helmholtz equation. And so it is not surprising to learn that the variational Eqs. 6-79 and 6-87 for $k^2(\varphi)$ of the cavity chains, in terms of either $\mathbf{E}$ or $\mathbf{H}$ field alone, are not upper or lower bound when the fields are expanded with the resonant cavity modes

12-5. A Perturbation Formula for $\omega(\beta)$ of a General Helix

In this section we develop a perturbation formula which predicts the change in frequency at fixed periodic phase shift due to a change in the geometry of a helix near the boundary between inner and outer regions. The formula enables us to estimate the effect of changing the tape width in a multifilar helix, for example, or the width of the rings in a ring helix. The geometric changes we have in mind are depicted in Fig. 12-3.

To develop the formula, assume in the unperturbed structure that we have already chosen the amplitudes of a convenient set of trial fields ($\hat{e}_t + \Delta\hat{e}$), and so on, so as to satisfy variational Eq. 6-45 for a given β_0 . If the helix has finite periodicity, the equation is written for a periodic volume. Subscript t denotes the true field and $\Delta\hat{e}$ is an unknown error in that field. The variational expression gives us not the true ω but rather $\omega + \Delta\omega$, where $\Delta\omega$ is an error in ω which we must be careful to distinguish from $\delta\omega$, the change in the true frequency due to the perturbation. Now we substitute these *same trial fields* into the same Eq. 6-45 for the

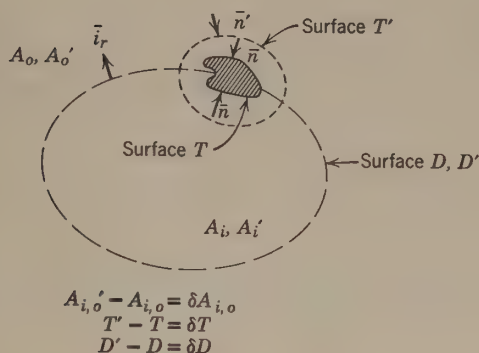


Fig. 12-3. The cross section of a general unifilar helix before and after a perturbation. The prime denotes a quantity after a perturbation which enlarges the wire size.

perturbed helix, for which the correct frequency is now $\omega' + \Delta'\omega$ at the same β_0 , where $\Delta'\omega \approx \Delta\omega$ is the new unknown error in the true frequency ω' . We now subtract the two variational statements and collect the terms. Error terms such as

$$\begin{aligned} \Delta'\omega \int_{A_{i'}} (\hat{e}_i^* + \Delta\hat{e}^*) \cdot \varepsilon(\hat{e}_i + \Delta\hat{e}) dA \\ - \Delta\omega \int_{A_i} (\hat{e}_i^* + \Delta\hat{e}^*) \cdot \varepsilon(\hat{e}_i + \Delta\hat{e}) dA \end{aligned} \quad (12-27a)$$

which, due to the variational nature of the expression, are nearly

$$\omega' \int_{A_{i'}} \varepsilon(\Delta'\hat{e})^2 dA - \omega \int_{A_i} \varepsilon(\Delta\hat{e})^2 dA \approx (\omega' - \omega) \int_{A_i} \varepsilon(\Delta\hat{e})^2 dA \quad (12-27b)$$

will be present, but they are two orders of magnitude smaller than the important terms, such as

$$(\omega' - \omega) \int_{A_i} \hat{e}^* \cdot \varepsilon\hat{e} dA \quad (12-27c)$$

The result is a perturbation formula which is readily verified to be

$$\begin{aligned} j(\omega' - \omega) \left[\int_{A_{i'}} (\varepsilon |\hat{e}|^2 + \mu |\hat{h}|^2) dA + \int_{A_{D'}} (\varepsilon |\hat{E}|^2 + \mu |\hat{H}|^2) dA \right] \\ = \int_{\delta T} (\hat{e} \times \hat{h}^* + \hat{E} \times \hat{H}^*) \cdot \vec{n}' dC \\ + \frac{1}{2} \int_{\delta D} (\hat{e}^* + \hat{E}^*) \times (\hat{h} - \hat{H}) \cdot \vec{i}_r dC \\ + \frac{1}{2} \int_{\delta D} (\hat{e} - \hat{E}) \times (\hat{h}^* + \hat{H}^*) \cdot \vec{i}_r dC \end{aligned} \quad (12-28)$$

with ω' equal to the new frequency at the given β_0 after the perturbation. Here δT and δD are the increases in the T and D areas, so if one of the areas decreases a minus sign is prefixed to the integral. The $\pm$ subscripts are omitted in Eq. 12-28 because there is only one set of trial fields in the absence of ohmic loss and energy sources. If the helix has periodicity L , $A'_{i,0}$ becomes $V'_{i,0}$, the inner or outer volume of the periodic cell, and surfaces D and T of course extend throughout this cell.

Table 12-1. Verification of the Perturbation Formula on a Contrawound Helix of Parameters $\delta/p = \frac{1}{4}$, $2\pi a/p = 4.4^*$

$\beta_0 a$	ka	$\delta(ka)/ka$, Calc.	$\delta(ka)/ka$, Meas.
0.465	0.254	0.054	0.071
1.07	0.385	0.070	0.092
2.15	0.490	0.059	0.084

* The angle χ in Fig. 6-8b was changed from 42° to 68° .

In order to verify the usefulness of Eq. 12-28 we considered the perturbation of a contrawound helix described by Birdsall and Everhart [5]. Specifically the angle χ shown in our Fig. 6-8b was changed from 42° to 68° on a helix with the parameters $\delta/p = \frac{1}{4}$ (rings as wide as the spacings between them) and $2\pi a/p = 4.4$. Without a knowledge of a compact dispersion relation and accurate field expressions we simply took the velocity-frequency curve (Fig. 12 of Ref. [5]) as known before the perturbation and estimated the fields by the TM mode given by Eq. 12-8, with B_4 set equal to 1. The fundamental $\ell = 0$ mode component of TE field is absent in the wave of even transverse E_z symmetry according to Eq. 12-4. We set $\delta\omega = 0$ with respect to δB_3^* in the variational Eq. 6-45 and obtained this expression for B_3 :

$$B_3 = \frac{1}{2I_0 I_1 (T/D)} \left[\frac{1}{\gamma_a} - \sqrt{\left(\frac{1}{\gamma_a}\right)^2 - 4I_0 I_1 K_0 K_1 \left(\frac{T}{D}\right)^2} \right] \quad (12-29)$$

T/D being the ratio of metal surface to open surface over the entire periodic cell $0 \leq z \leq p$. The argument of the Bessel functions is $\gamma_0 a$. As $\gamma \rightarrow 0$, $B_3 \rightarrow K_0(T/D)$, so $(\hat{e}_z/\hat{E}_z)_{r=a} \rightarrow T/D$ and therefore behaves qualitatively as in the sheath helix. At large frequencies the square root would go imaginary, in which case we dropped it and let $B_3 \rightarrow [2\gamma_0 a I_0 I_1 (T/D)]^{-1}$, whereupon $(\hat{e}_z/\hat{E}_z)_{r=a} \rightarrow D/T$, constant, which is qualitatively the sheath helix behavior. All the necessary information is now available to calculate $(\omega' - \omega) = \delta\omega$ and $\delta(ka)$ for the perturbation in angle χ by Eq. 12-28. Table 12-1 shows how close the calculated $\delta(ka)$ is to the measured

value over a wide range of $\beta_0 a$. The calculated perturbations are about 75% of the measured ones, which is reasonable agreement considering the rather crude nature of the TM trial field.

12-6. Resonant-Slot and Resonant-Loop Dispersion Curve Shaping in a Cavity Chain

The fundamental theorem for a narrow passband of any slow wave structure with mirror symmetry about a transverse plane was quoted on page 102 as

Given a passband for a symmetrical periodic structure [one that supports resonant modes of either even or odd symmetry] which is narrow (on a frequency scale) compared to its two adjacent stopbands, then its $\omega^2 - \beta_0^2$ curve ($\beta_0 = \varphi/L$) is independent of the nature of the coupling and is a $1 \pm \cos \varphi$ function between the cutoff points.

The remarkable aspect of the theorem is its disregard of the precise nature of the coupling between cavities. In this section we examine the effect on the dispersion curve of resonant slots or loops which augment the coupling and have their resonant frequencies (at which the open- and short-circuit "coupling" modes resonate) very near one of the cutoff frequencies of the passband under discussion.

Let us speak first about resonant-slot coupling. Equation 7-26 was the variationally derived dispersion relation for the $\omega - \varphi$ curve between the resonant frequencies of a pair of open- and short-circuit resonant "cavity" modes and also for the curve between the nearby resonances of a pair of "slot" modes. The "slot" passband is usually very narrow compared to the separation between the two passbands, so we can make the approximation

$$\frac{1}{2}p_s^2(1 + \cos \varphi) + \frac{1}{2}p_1^2(1 - \cos \varphi) \cong p_1^2 \quad (12-30)$$

But then the $p_1^2 - k^2$ factor in Eq. 7-26b cancels, which means that only the coupling to, and not the frequencies of, the resonant "slot" modes affects the "cavity" dispersion curve in the first approximation. The dispersion relation can be written in normalized form as

$$\begin{aligned} \frac{1}{2}\Pi_c^2(1 + \cos \varphi) + \frac{1}{2}(1 - \cos \varphi) - m_s \sin^2 \frac{\varphi}{2} \\ = \kappa^2 \left(1 - m_s \sin^2 \frac{\varphi}{2} \right); \quad \Pi_c = \frac{P_c}{p_c}, \quad \kappa = \frac{k}{p_c} \end{aligned} \quad (12-31)$$

Here Π_c is the normalized short-circuit mode cutoff frequency, κ is the

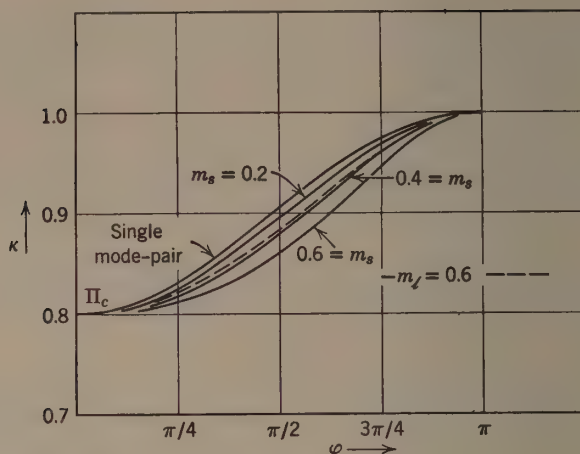


Fig. 12-4. Perturbation of the normalized dispersion curve of a cavity chain due to the presence of a resonant-slot or resonant-loop passband. Π_c is the relative zero phase shift cutoff frequency. m_s and m_l are the crosscoupling coefficients between the modes of the "cavity" passband and the "slot" or "loop" passband, respectively.

normalized operating frequency, and m_s is the crosscoupling coefficient between passbands, as defined in Eq. 7-26.

As an example of the influence of the resonant-slot modes upon the "cavity" dispersion curve, assume the "slot" passband lies just above the p_c or π cutoff frequency in Fig. 12-4 and that $\Pi_c = 0.8$, which means about a 20% passband width. The curves for various values of m_s are shown, neglecting the effect of the change in slot area upon the π cutoff frequency (it will have no effect on the zero phase shift cutoff frequency, P_c). In practice the p_c point of the first passband in a chain of cylindrical cavities will fall somewhat as the slots are widened or increased in number, as may be verified by ordinary considerations of the effect on the passband parameter M , such as given on page 104. Indeed, we see that the linear portion of the passband is effectively shifted to the right on the φ axis. Earlier, on page 381, we pointed out how such an effect can increase the directivity of a cavity-chain parametric amplifier.

We can also account for the effect of S-shaped loops on the "cavity" passband, for which the determinantal equation for the two passbands was Eq. 7-30. According to the discussion on page 232, Eq. 7-30 is accurate for coupling by just a few loop-pairs between cavities but gradually becomes inaccurate as the number of loop-pairs is increased. Let us use Eq. 7-30 as a first approximation and assume the "loop" passband is narrow in the sense that

$$\frac{1}{2}P_l^2(1 + \cos \varphi) + \frac{1}{2}P_l^2(1 - \cos \varphi) \cong P_l^2 \quad (12-32)$$

whereupon the $P_\ell^2 - k^2$ factor cancels and we can write the dispersion relation in the form

$$\frac{1}{2}\Pi_c^2(1 + \cos \varphi) + \frac{1}{2}(1 - \cos \varphi) - m_\ell \Pi_c^2 \cos^2 \frac{\varphi}{2} = \kappa^2 \left(1 - m_\ell \cos^2 \frac{\varphi}{2}\right) \quad (12-33)$$

in which m_ℓ is the crosscoupling coefficient between the modes of the two passbands, as defined in Eq. 7-30. For an S-shaped "loop" passband above the "cavity" passband, $m_\ell < 0$ (see page 232) and we can calculate the perturbed ω - φ curves for the parameters of Fig. 12-4 and compare the effect of the loops with that of the slots. Upon doing so we find that the loop coupling is less efficient, for as $|m_\ell|$ is increased the ω - φ curve drops less and less rapidly. The $m_\ell = -0.6$ curve nearly coincides with the $m_s = 0.4$ curve. We might say the slot coupling is a more efficient means of perturbing the ω - φ curve than is the loop coupling with respect to the magnitude of m_s and m_ℓ . In this connection the efficiency of a large number of slots in the cloverleaf cavity chain (Section 7-7) is greater than the efficiency of the ring of loops in the centipede chain (Section 7-6) in the sense that, for comparable passband widths, the area of the slots is a much smaller fraction of the total cross-sectional area than is the coupling area on the loop midplanes.

Other sets of curves similar to Fig. 12-4 may be constructed for different values of Π_c and a "slot" or "loop" passband either above the higher cutoff frequency of the "cavity" passband or below the lower cutoff frequency. In the latter case, m_s is a negative number (refer to discussion on page 224) or m_ℓ is positive and the single mode-pair dispersion curve is pushed upward.

Problems

1. Verify the statement on page 411 to the effect that the field expansion in terms of half-integral values of ℓ does not satisfy the symmetry conditions.
2. Obtain expressions for the ideal impedance of the $(++)$ and $(+-)$ ring-plane helix waves at $r = a-$, $\theta = \pi/2$, using the trial fields of Section 6-4. Take the dispersion curves of Fig. 6-6 as known and determine the relative amplitudes of the $\hat{e}_z$ trial fields to match $\hat{E}_z$ in a variational manner on the $r = a$ surface. Compare the answer for the $(++)$ wave with the impedance curve on Fig. 6-6.
3. Show that the proof of the upper-bound nature of Eq. 6-8 for the eigenvalues of the Helmholtz equation breaks down if the ψ_n are not orthogonal. Similarly, Eq. 12-16 is not upper bound if the $\hat{F}_n$ eigenfields are not orthogonal.
4. Apply Eq. 12-19 to a wave on the sheath helix and show that it does yield

an impractical lower bound. Let smoothing parameter σ be one-tenth the helix radius.

5. Prove that Eq. 12-19 is actually a lower-bound expression for the frequency of a closed helix at a given fundamental propagation constant.
6. Use the perturbation formula 12-28 to determine the effect of changing the tape thickness of a unifilar tape helix described in Section 6-3D. Study the wave with predominant $n = 0$ mode.
7. Show that for a radial perturbation of the ring thickness in the ring-plane helix the change in frequency is given by $(\omega' - \omega)/\omega = \Delta(W_m - W_e)/(W_m + W_e)$ where $\Delta W_m(\Delta W_e)$ is the magnetic (electric) energy displaced.
8. If the passband associated with strong fields around the resonant slots lies just above the π cutoff frequency of the passband of interest, Fig. 12-4, the finite width of the "slot" passband cannot be neglected. Let $P_c/p_c = 0.8$ of the "cavity" passband as in Section 12-6, but $p_1/p_c = 1$ and $p_s/p_c = 1.05$ for the "slot" passband, and use Eq. 7-26 instead of Eq. 12-31 to modify the curves of Fig. 12-4. p_c , the π -cutoff frequency of the "cavity" passband, is p_2 in Eq. 7-26.

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Appendix A

Proofs Based upon the Variational Expressions

A-1. The Power Flow-Stored Energy-Group Velocity Relation. General Expression for Power Flow in a Cavity Chain

We can show very easily how variational Eq. 6-45 implies the correct relationship among power flow, stored energy, and group velocity for a "cold" helical structure. Let us write the trial fields as

$$\hat{e}_-^* = e_-^*(x, y, z)\epsilon^{j\beta_0 z}, \quad \hat{h}_+ = h_+(x, y, z)\epsilon^{-j\beta_0 z}$$

and so on, so that they conform to the symmetry of the helix. Now we change ω to $\omega + \delta\omega$, $\hat{e}_-$ to $\hat{e}_- + \delta\hat{e}_-$, and so on; also β_0 to $\beta_0 + \delta\beta_0$. We know from the variational properties that the linear $\delta\hat{e}_-$ and other changes in the fields do not create any linear changes in ω at fixed β_0 . Therefore, the $\delta\omega$ term must be due entirely to the change of β_0 . Since $\nabla \times = (\nabla_T + \hat{i}_z \partial/\partial z) \times$ we have, from Eq. 6-45,

$$\begin{aligned} 0 = & \int_{A_i} [\hat{e}_-^* \cdot \hat{i}_z(-j\delta\beta_0) \times \hat{h}_+ - \hat{h}_-^* \cdot \hat{i}_z(-j\delta\beta_0) \times \hat{e}_+] dA \\ & + \int_{A_o} [\hat{E}_-^* \cdot \hat{i}_z(-j\delta\beta_0) \times \hat{H}_+ - \hat{H}_-^* \cdot \hat{i}_z(-j\delta\beta_0) \times \hat{E}_+] dA \\ & - j\delta\omega \left[\int_{A_i} (\hat{e}_-^* \cdot \epsilon \hat{e}_+ + \hat{h}_-^* \cdot \mu \hat{h}_+) dA \right. \\ & \left. + \int_{A_o} (\hat{E}_-^* \cdot \epsilon \hat{E}_+ + \hat{H}_-^* \cdot \mu \hat{H}_+) dA \right] \quad (\text{A-1}) \end{aligned}$$

If the helix has periodicity L in the axial direction, as a ring-plane helix for example, the integrals would be taken over a cross-sectional slab of

thickness L . But this does not change our essential result to follow. Upon rearranging the order of dot- and cross-multiplication in the first four integrals, we divide by 4 and take the real terms to obtain

$$\begin{aligned} \frac{1}{4} \operatorname{Re} \left[\int_{A_i} (\hat{e}_-^* \times \hat{h}_+ + \hat{e}_+ \times \hat{h}_-^*) \cdot \hat{i}_z dA + \int_{A_o} \text{term} \right] \\ = \frac{\delta\omega}{\delta\beta_0} \frac{1}{4} \operatorname{Re} \left[\int_{A_i} (\hat{e}_-^* \cdot \varepsilon \hat{e}_+ + \hat{h}_-^* \cdot \mu \hat{h}_+) dA + \int_{A_o} \text{term} \right] \quad (\text{A-2}) \end{aligned}$$

The minus field solution is fully equivalent to the plus solution in a lossless structure not interacting with a beam, and so we have proven the relation $P_{av} = W_s v_g$ of Section 1-4.

Now, Eq. 6-79 is a variational expression for k^2 of a periodic cavity chain formulated in terms of electric field alone. It, too, should imply the same relation, but in terms of power flow across a coupling surface between any two neighboring cavities. In this expression we expand $\hat{E}_-$ with the open-circuit cavity modes (denoted by $\hat{e}_-$) and $\hat{E}_+$ with the short-circuit modes. The open-circuit modes comprising $\hat{e}_-$ in the cavity of reference phase zero do not explicitly involve phase shift φ ; nor does the expansion for $\hat{E}_+$. On the boundary between cavity (0) and cavity (+1) the phase shift is explicit in the expressions

$$\begin{aligned} \hat{E}_-^{(+)} &= \hat{e}_{-M}^{(+)} \epsilon^{-j\varphi}, & \hat{E}_+^{(+)} &= \hat{e}_{+M}^{(+)} \epsilon^{-j\varphi} \\ \hat{E}_-^{(-)} &= \hat{e}_{-M}^{(-)}, & \hat{E}_+^{(-)} &= \hat{e}_{+M}^{(-)} \end{aligned} \quad (\text{A-3})$$

(M denotes mode expansion; only one discontinuity surface per periodic cell is assumed). In cavity (+1) the expansions are identical to those in cavity (0) except for the explicit factor $\epsilon^{-j\varphi}$.

Since Eq. 6-79 is variational, a linear change in ω by $\delta\omega$ is not created by changes in the trial fields at fixed φ , but only by a change of φ , $\delta\varphi$. These two variations are related in Eq. 6-79 as

$$\begin{aligned} 0 &= -2k \delta k \int_V \hat{e}_-^* \cdot \hat{E}_+ dV \\ &+ \frac{1}{2} \int_D j \delta \varphi \hat{e}_{-M}^{(+)*} \epsilon^{j\varphi} \cdot \hat{i}_z \times (\nabla \times \hat{E}_{+M}^{(+)} \epsilon^{-j\varphi} - \nabla \times \hat{E}_{+M}^{(-)}) dS \\ &+ \frac{1}{2} \int_D (\hat{e}_{-M}^{(+)*} \epsilon^{j\varphi} + \hat{e}_{-M}^{(-)*}) \cdot \hat{i}_z \times (-j \delta \varphi) \nabla \times \hat{E}_{+M}^{(+)} \epsilon^{-j\varphi} dS \quad (\text{A-4}) \end{aligned}$$

Two of the integrands cancel in these $\int_D$ integrals. With $k = \omega \sqrt{\mu \epsilon}$ we

divide both sides by $4\omega\mu$, take the real terms and have (with $\varphi = \beta_0 L$)

$$\frac{\delta\omega}{\delta\beta_0} \frac{W_s}{L} = \text{Re} \frac{j}{8\omega\mu} \int_D (\hat{e}_{-M}^{(+)*} \epsilon^{j\varphi} \times \nabla \times \hat{E}_{+M}^{(-)} + \hat{e}_{-M}^{(-)*} \times \nabla \times E_{+M}^{(+)} \epsilon^{-j\varphi}) \cdot \hat{i}_z dS$$

$$= P_{\text{av}} \quad (\text{A-5})$$

The right side of Eq. A-5 must be the time-average power flow, P_{av} .

In the single mode-pair case, $\hat{e}_{-}^{(-)} = v_n \hat{e}_n^{(-)}$, $\hat{e}_{-}^{(+)} = v_n \epsilon^{-j\varphi} \hat{e}_n^{(+)}$, and similarly for $\hat{E}_{+}^{(+)}$, whereupon we obtain for P_{av}

$$\mp \frac{\tau}{4\omega\mu} P_n V_n M_{nn} v_n^* \sin \varphi \quad \begin{array}{l} \text{upper sign, even-mode pair} \\ \text{lower sign, odd mode-pair} \end{array} \quad (\text{A-6})$$

which agrees with direct evaluation of the Poynting flow in Section 4-4. In terms of all the solenoidal modes (irrotational modes are not included in the electric fields) the integral over surface D is written

$$\int_D \left[\left(\sum_n v_n^* \epsilon^{j\varphi} \hat{e}_n^{(+)} \right) \times \left(\sum_m P_m V_m \hat{H}_m^{(-)} \right) + \left(\sum v_n \hat{e}_n^{(-)} \right) \right. \\ \left. \times \left(\sum P_m V_m \epsilon^{-j\varphi} \hat{H}_m^{(+)} \right) \right] \cdot \hat{i}_z dS \quad (\text{A-7})$$

This may be cast into matrix form by defining these matrices:

$$\mathbf{M}_e = \begin{pmatrix} M_{11} & 0 & M_{13} & \cdots \\ 0 & 0 & 0 & \cdots \\ M_{31} & 0 & M_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathbf{M}_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & M_{22} & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (\text{A-8})$$

$$\mathbf{M}_{e0} = \begin{pmatrix} 0 & +M_{12} & 0 & \cdots \\ -M_{21} & 0 & -M_{23} & \cdots \\ 0 & +M_{32} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$\mathbf{V} = \text{column } (V_1, V_2, V_3, \dots)$

$\mathbf{v} = \text{column } (v_1, v_2, \dots)$

$\mathbf{P} = \text{diagonal } (P_1, P_2, \dots)$

with

$$M_{ij} = \frac{1}{\tau} \int_D \hat{e}_j^{(+1)} \times \hat{H}_i^{(+1)} \cdot \hat{i}_z dS$$

the usual definition. The lowest-frequency passband is invariably characterized by an even mode-pair, so M_{11} is in the upper left-hand corner of matrix $\mathbf{M}_e$. The passband parameters and mode amplitudes are arranged in the order of even mode, odd mode, even mode, . . . , although their order on the frequency scale may be somewhat different.

In terms of these matrices we can write the general power flow expression proportional to the quantity A-7 as

$$P_{av} = \frac{\tau}{4\omega\mu} \operatorname{Re} [-\mathbf{V}^t \mathbf{P} (\mathbf{M}_e - \mathbf{M}_o) \mathbf{v}^* \sin \varphi + j \mathbf{V}^t \mathbf{P} \mathbf{M}_{e0} \mathbf{v}^* \cos \varphi] \\ t, \text{ matrix transpose} \quad (\text{A-9})$$

Note that $\mathbf{M}_e$ couples together only even modes and $\mathbf{M}_o$ couples only odd ones. Equation A-9 agrees exactly with the power flow expressions as based upon some rather involved properties of the mode expansions.

A-2. Derivation of the Pierce Equation for a Weak Beam Interacting with One Mode of a Slow Wave Structure

Our definition of the adjective "weak" will be apparent shortly. Let us write the variational Eq. 6-45, augmented by the terms (a) and (b) on page 252 in the form

$$D_1(\gamma, \Gamma) + \int_{A_i} \hat{e}_-^* \cdot \mathbf{J}_+ dA = D_2(\gamma) \quad (\text{A-10})$$

D_1 includes the area integrals in terms of the unconjugated minus trial fields which propagate as $\epsilon^{+\Gamma z}$, and the plus fields which propagate as $\epsilon^{-\Gamma z}$ in the presence of the beam. Surface D_2 contains the integrals on the discontinuity surface between inner and outer helix regions and the tape (or wire) surfaces. If the helix has axial period p , then the integrals are taken within a cross-sectional slab p axial units thick. Our final answer, in terms of the interaction between one circuit mode and the fast and slow beam modes, remains the same.

We agree to use circuit trial fields which satisfy Maxwell's equations *identically* in the presence of the beam, in the sense specified by these equations for the n th such mode:

$$\nabla_T \times \bar{e}_{n+} - \Gamma \bar{i}_z \times \bar{e}_{n+} = -j\omega\mu \bar{h}_{n+} \\ \nabla_T \times \bar{h}_{n+} - \Gamma \bar{i}_z \times \bar{h}_{n+} = j\omega\epsilon \bar{e}_{n+} \quad (\text{no } J \text{ term}) \quad (\text{A-11})$$

at every point of the inner region A_i . Equation 8-18 furnishes an example of such trial fields for the inner and outer helix regions. Then $D_1(\gamma, \Gamma) = 0$. Propagation constant Γ is related to radial constant γ as $-\Gamma^2 - \gamma^2 = k^2$.

Now in the process of determining the mode amplitudes so as to make Γ an extremum $D_2(\gamma)$ evaluates in such a way that

$$D_2(\gamma) \rightarrow D_2(\gamma_1) = 0 \quad \text{as} \quad \mathcal{J}_+ \rightarrow 0 \quad (\text{A-12})$$

because the "cold" circuit determinantal equation for $\Gamma_1 = j\beta_1$ reduces to just $D_2(\gamma_1) = 0$. This being the case, we can evaluate $D_2(\gamma)$ in the presence of the beam as

$$D_2(\gamma(\Gamma)) \cong \left. \frac{\partial D_2(\Gamma)}{\partial \Gamma} \right|_{\Gamma=\Gamma_1} \cdot (\Gamma - \Gamma_1) \quad (\text{A-13})$$

which is sufficiently accurate if the beam is weak and

$$\left. \frac{\frac{1}{2} \partial^2 D_2}{\partial \Gamma^2} \right|_{\Gamma_1} (\Gamma - \Gamma_1)^2$$

is negligible. We should add that both D_1 and D_2 are written with the amplitude of $\hat{e}_z$ interacting with the beam as fixed reference. It now remains to relate this derivative of D_2 to that of D_1 and thence to the "cold" circuit power flow.

Since $\partial D_2(\Gamma)/\partial \Gamma|_{\Gamma_1}$ is analytic as a function of $\Gamma = \alpha + j\beta$ and the mode amplitudes, it can be evaluated without the beam in the "cold" helix as

$$\frac{\partial D_2(\Gamma_1)}{\partial \Gamma_1} = \frac{\partial D_1(\Gamma_1)}{\partial \Gamma_1}, \quad \omega \text{ fixed} \quad (\text{A-14})$$

To evaluate the latter derivative we have only to let the fields, $\Gamma_1 = j\beta_1$, and ω vary in Eq. 6-45 by $\delta \hat{e}_-$, and so on, $\delta \beta_1$, and $\delta \omega$. The field variations at fixed β_1 all cancel out and create no $\delta \omega$. Therefore the terms proportional to $\delta \beta_1$ are equated to the terms proportional to $\delta \omega$ as

$$0 = \delta D_1|_{\delta \beta_1} - j \delta \omega \left(\int_{A_i} \hat{e}_-^* \cdot \epsilon \hat{e}_+ dA + \dots \right) \quad (\text{A-15})$$

that is,

$$\frac{\partial D_1}{\partial(j\beta_1)} = \frac{\delta \omega}{\delta \beta_1} \bigg|_0 4W_s \quad \text{at the reference } e_z \text{ amplitude}$$

Since $P_{av,c} = W_s(\partial \omega / \partial \beta_1)$ in the "cold" circuit we have

$$\left(\frac{\partial D_1}{\partial \Gamma_1} \right)_{\Gamma_1} = 4P_{av,c} \quad (\text{A-16})$$

The net result of Eqs. A-10, A-13, A-14, and A-16 is

$$4P_{av,c}(\Gamma - \Gamma_1) = \int_{A_i} \hat{e}_-^* \cdot \mathcal{J}_+ dA \quad (\text{A-17})$$

In the Pierce notation $\Gamma - \Gamma_1 = -\beta_e C_P(\delta_P + j b_P)$.

In Eq. A-17, $\hat{J}_+ = a_1 \hat{J}_1 + a_2 \hat{J}_2$ ($\hat{J}_2$ may be taken $= \hat{J}_1$) and a_1 and a_2 were determined by Eqs. 8-10 and 8-12 to be

$$a_1 = \frac{\int_A \hat{e}_c \cdot \hat{J}_1 dA}{(j\beta_s - \Gamma)4P_{av,1}}, \quad a_2 = \frac{\int_A \hat{e}_c \cdot \hat{J}_1 dA}{(j\beta_f - \Gamma)4P_{av,2}} \quad (\text{A-18})$$

$$\beta_s = \beta_e + \beta_q, \quad \beta_f = \beta_e - \beta_q$$

with neglect of the electromagnetic crosspower between each beam mode and the circuit field. It is now a straightforward task to insert Eq. A-18 into Eq. A-17, use the relation $P_{av,2} = -P_{av,1}$ between the power flows in the beam modes, together with the definitions of the Pierce parameters in Eq. 8-14, and thereby obtain the Pierce Eq. 8-15 for incremental propagation constant δ_P .

Appendix B

Measurement of Tube and Cavity Parameters

B-1. Perturbation Formula for Electric Field

To measure the magnitude of the electric field which can act upon an electron beam at a given point we can insert small objects and measure the perturbation in the resonant frequency of a section of the structure shorted at its symmetry planes.

Maier and Slater [1] quote a number of formulas for the perfectly conducting prolate spheroid (needle) of semimajor axis a and semiminor axis b . If the electric field $\hat{E}_0$ is parallel to axis a , then the resonant frequency of the cavity into which the spheroid is inserted changes from ω_0 to ω given by

$$(\hat{E}_0 \parallel a) \quad \frac{\omega^2 - \omega_0^2}{\omega_0^2} = -4\pi a^3 |\hat{E}_0|^2 F_1 \cong \frac{2(\omega - \omega_0)}{\omega_0}$$

where

$$\int_{\text{cav}} |\hat{E}_0|^2 dV = 1 \quad (\text{B-1})$$

$$F_1 = \frac{\frac{2}{3}(1 - \beta_M^2)^{3/2}}{\ln \left(\frac{1 + \sqrt{1 - \beta_M^2}}{1 - \sqrt{1 - \beta_M^2}} \right) - 2\sqrt{1 - \beta_M^2}}, \quad \beta_M = \frac{b}{a} \quad (M \text{ for Maier})$$

If, instead, $\hat{E}_0$ is perpendicular to semimajor axis a the formula is

$$(\hat{E}_0 \perp a) \quad \frac{\omega^2 - \omega_0^2}{\omega_0^2} = -4\pi a^3 |\hat{E}_0|^2 F_2 \quad (\text{B-2})$$

where

$$F_2 = \frac{\frac{4}{3}(1 - \beta_M^2)^{3/2}}{2 \frac{\sqrt{1 - \beta_M^2}}{\beta_M^2} - \ln \left(\frac{1 + \sqrt{1 - \beta_M^2}}{1 - \sqrt{1 - \beta_M^2}} \right)}$$

If $\beta_M < 0.2$, then $F_2 < 0.10F_1$. If the prolate spheroid becomes a sphere of radius a , Eq. B-1 applies with $F_1 = 1$.

If a small cylinder of length $2a$ and radius b is used in place of the prolate spheroid, we can correct the formulas by multiplying by the factor $\frac{3}{2}$, which is the ratio of the cylinder volume to the spheroid volume.

We may not be able to avoid measuring magnetic field along with electric field at a point. A perfectly conducting sphere placed at a point where the magnetic field is $\hat{H}_0$ will create a change of resonant frequency given by

$$\frac{\omega^2 - \omega_0^2}{\omega_0^2} = 2\pi a^3 |\hat{H}_0|^2, \quad \int_{\text{cav}} |\hat{H}_0|^2 dv = 1 \quad (\text{B-3})$$

so that the net change of resonant frequency due to displacement of both $\hat{E}_0$ and $\hat{H}_0$ by that sphere is

$$\frac{\omega^2 - \omega_0^2}{\omega_0^2} = 2\pi a^3 (|\hat{H}_0|^2 - 2|\hat{E}_0|^2) \quad (\text{B-4})$$

Other formulas for the change in resonant frequency when a small oblate spheroid is inserted into a magnetic field, and when circular disks are inserted into electric and magnetic fields, are listed in Ref. [1].

When a small dielectric sphere of volume V and dielectric constant ϵ_1 is inserted into an electric field $\hat{E}_0$ of a resonant cavity the frequency changes from ω_0 to ω given by [2]

$$\frac{\omega - \omega_0}{\omega_0} = -\frac{3}{2} V \frac{\epsilon_e |\hat{E}_0|^2}{\int_{\text{cav}} \epsilon_0 |\hat{E}_0|^2 dV}, \quad \epsilon_e = \frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0} \epsilon_0 \quad (\text{B-5})$$

B-2. Definition of the Cavity Parameter R_{sh}/Q

This parameter specifies essentially the electric field across a cavity gap in terms of the stored energy. It becomes less meaningful physically and more a defined parameter as the gap widens. Across a narrow gap the equivalent circuit voltage is

$$V = \int_{\text{gap}} \hat{E} \cdot d\ell$$

Since $P_{av} = |V^2|/2R_{sh}$ and $Q = \omega_0 W_s/P_{av}$, Ginzton [3, p. 440] defines

$$\frac{R_{sh}}{Q} = \frac{\left| \int_{gap} \hat{E} \cdot d\ell \right|^2}{(2\omega_0 W_s)} \quad (B-6)$$

On the other hand, Chodorow *defined* voltage V for a ring-bar helix cavity of 2π electronic radians in length as

$$V^2 = L \int_0^L |\hat{e}_z(z)|^2 dz$$

(see page 344).

B-3. Helix Impedance Based upon Perturbation of Electric Field

This section treats only the impedance measured along the axis, as due to the nonangularly varying modes in the helix wave of interest. The off-axis impedance is best measured directly by an electron beam (Section B-4).

Ideally, we should short the output load of a finite section of "cold" helix so as to set up a standing wave such that $e_z = e_m \sin \beta_0 z$, where $z = 0$ at the load end of the helix and β_0 is the (known) propagation constant of the predominant $n = 0$ mode of the wave being investigated. Jaynes [4] and Kino [5] have obtained the following formula for the impedance which we can modify for this problem:

$$Z = \frac{|\hat{E}_z(0)|^2}{2\beta_0^2 P_{av}} = \frac{2}{\omega\beta(\epsilon' - 1)\epsilon_0\pi b^2} \left(\frac{\Delta\beta}{\beta} \right) \quad (B-7)$$

Here a dielectric rod of small radius b and relative dielectric constant ϵ' is assumed to be immersed in a *uniform* axial electric field. The magnitude of the perturbed $\hat{E}_z$ field is taken equal to the unperturbed field through the rod. Here $\Delta\beta$ is the change in propagation constant induced. Now if the field is sinusoidally varying with a wavelength much greater than the rod radius we can replace the numerator in good approximation by

$$\int_{-L}^L e_m^2 \sin^2 \beta_0 z dz$$

and also replace P_{av} by $W_s v_g$, where the components of W_s vary as $\sin^2 \beta_0 z$ or $\cos^2 \beta_0 z$ and $v_g = \Delta\omega/\Delta\beta$ on the dispersion curve. Now define $\Delta\omega$ as that frequency change necessary to correct for $\Delta\beta$ and restore the standing wave pattern before insertion of the rod. The $\int \sin^2 \beta_0 z dz$ factor in the numerator cancels the corresponding factors in the denominator *provided*

the tube is an integral number of half wavelengths long and so we can write

$$\begin{aligned} \frac{|\hat{E}_z(0)|^2}{W_s} &= -\frac{4}{(\epsilon' - 1)\epsilon_0\pi b^2} \left(\frac{\Delta\omega}{\omega} \right) \\ &= -\frac{480c}{(\epsilon' - 1)b^2} \left(\frac{\Delta f}{f} \right), \quad \text{mks units} \end{aligned} \quad (\text{B-8})$$

c being light velocity. This expression is independent of the precise length of the helix.

We can correct the formulas B-7 and B-8 by inserting into the denominators a factor which Lagerstrom [6] calls p_e :

$$p_e = \frac{I_0^2(\gamma_0 b)}{(\epsilon' - 1)U' + 1/V'} \quad (\text{B-9})$$

where

$$U' = \frac{2}{(\gamma_0 b)^2 I_0(\gamma_0 b) K_0(\gamma_0 b)}, \quad V' = \frac{2I_1(\gamma_0 b)}{\gamma_0 b I_0(\gamma_0 b)} \cong 1$$

Use of the p_e factor still neglects the $\hat{H}_z^2$ component of stored energy displaced by the rod; this error diminishes rapidly with decreasing rod radius.

Suppose, now, a highly conductive rod of radius b is substituted for the dielectric rod. A formula of Kino [5] can be cast into the form

$$\begin{aligned} \Delta\beta &= \frac{\omega\epsilon_0\pi b^2 I_0(\gamma_0 b)}{(\gamma_0 b)^2 K_0(\gamma_0 b)} \cdot \frac{|\hat{E}_z(0)|^2}{2P_{av}} F_K \\ F_K &= 1 - \frac{I_1^2 K_0}{I_0 I_1 K_1} \cdot \frac{\mu_0 |\hat{H}_z(0)|^2}{\epsilon_0 |\hat{E}_z(0)|^2} \quad (K \text{ for Kino}) \end{aligned} \quad (\text{B-10})$$

According to the preceding remarks the interaction parameter can be converted into

$$\frac{|\hat{E}_z(0)|^2}{W_s} = \frac{2(\gamma_0 b)^2}{\epsilon_0\pi b^2} \frac{K_0(\gamma_0 b)}{I_0(\gamma_0 b) F_K} \left(\frac{\Delta f}{f} \right) \quad (\text{B-11})$$

Observe that, as rod radius $b \rightarrow 0$, $K_0 \rightarrow \infty$ as $\ln(\gamma_0 b)$ but $\Delta f/f$ goes to zero much faster. It may be to our advantage to use a rod of such size that the second term of F_K cannot be neglected.

These formulas give us impedance $Z = |E_z(0)|^2 / 2\beta_0^2 P_{av}$, since $P_{av} = W_s v_g$ and v_g is presumed known from the dispersion curve. Then the Pierce gain parameter C_P may be calculated as

$$C_P^3 = \frac{Z |I_0|}{4U_0} \quad (\text{B-12})$$

U_0 being the d.c. beam voltage and I_0 the d.c. beam current. A relativistic correction factor R may be appended to I_0 according to Eq. D-12.

B-4. Helix Gain Parameter C_P from the Kompfner Dip Condition

Perhaps the best way to measure the impedance either on or off the axis of a helix is to insert an electron beam of small cross-sectional area and observe the behavior of the circuit field. Louisell [7, pp. 80–89] has discussed this matter completely and we shall only summarize the results here.

1. Beam Interacting with One Mode of a Wave with Forward Circuit Power Flow (Positive Group Velocity)

We adjust the d.c. beam velocity at the frequency of interest until the r.f. circuit voltage at the output end “dips” to a minimum. If the Pierce space charge parameter $(QC)_P$ is ≥ 0.8 (with $\beta_o/\beta_e = C_P\sqrt{4(QC)_P}$) we can obtain a very accurate expression for impedance Z by assuming only the fast beam mode interacts with the circuit mode. Then the Pierce synchronism parameter b_P has the value

$$b_P \cong -\sqrt{4(QC)_P} \quad (\text{B-13})$$

so that if a standing wave of n quarter-wavelengths exists on the helix,

$$(C_P N)_{\text{dip}} = \sqrt{2} [4(QC)_P]^{1/4} \frac{2n+1}{4}, \quad n = 0, 1, \dots \quad (\text{B-14})$$

Here N is the electronic tube length, $\beta_e L/2\pi$.

If $(QC)_P \ll 0.8$ or a more accurate expression for C_P is desired, we can use a curve of $(C_P N)_{\text{dip}}$ vs. $(QC)_P$ given by Louisell which accounts for the presence of the slow space charge mode. By estimating C_P we can determine $(QC)_P$ and hence the value of $(C_P N)_{\text{dip}}$ from this curve, correcting the initial estimate if necessary.

2. Beam Interacting with One Mode of a Wave with Backward Circuit Power Flow

Now the beam velocity and current are adjusted so a given length of helix just barely oscillates at the frequency of interest. If $(QC)_P \geq 0.8$ the slow beam mode is nearly in synchronism with the “cold” circuit:

$$b_P \cong +\sqrt{4(QC)_P} \quad (\text{B-15})$$

It is rather easy to show from Eq. 8-12 that the three independent waves are exactly those waves of part 1 except that b_P changes sign, δ_P changes sign, and the fast and slow beam mode amplitudes (but not the power flows $P_{av,1}$ and $P_{av,2}$) are interchanged. Therefore we can say immediately that, for $(QC)_P \geq 0.8$, Eq. B-14 again determines $(C_P N)_{osc}$; for $(QC)_P \leq 0.8$ we can use Louisell's curve to determine C_P self-consistently as before.

B-5. C_P for the Cavity Chain

For the cavity chain operated at a frequency within a rather narrow passband, C_P is defined in terms of the modes of that passband by Eq. 8-37, in which the effective electric field acting on the beam across a cavity is

$$E_{eff} = \frac{1}{L} \left| \int_0^L \epsilon^{-j\beta_e z} E_z dz \right| \quad (B-16)$$

If the operating frequency lies close to the resonant frequency of the "straight-through" short-circuit mode, $\bar{E}_z$ may be evaluated by either rod or small-object perturbation. $\beta_e L$ is nearly zero and there is no difference between $\bar{E}_z$ and E_{eff} in Eq. B-8. If the operating frequency falls near the center of the passband, E_z is best determined at each point along the axis of one cavity by a small-object perturbation, whereupon E_{eff} is obtained from Eq. B-16. If the operating frequency falls near the first open-circuit mode cutoff frequency, $E_z \cong e_m \sin(\pi z/L)$. A rod measurement by Eq. B-8 of N cavities shorted at the first and last midplanes will yield

$$\frac{N \int_0^L e_m^2 \sin^2(\pi z/L) dz}{NW_s} = \frac{Le_m^2/2}{W_s}$$

provided the electric field pattern is normalized to the cavity volume. On the other hand the Pierce parameter involves E_{eff}^2 :

$$E_{eff}^2 = \frac{1}{L^2} \left| \int_0^L \epsilon^{-j\beta_e z} e_m \sin \frac{\pi z}{L} dz \right|^2 \cong \frac{e_m^2}{4} \quad (\beta_e L \cong \pi)$$

Therefore the value of $|\hat{E}_z(0)|^2/W_s$ obtained by a rod measurement of $\Delta f/f$ in Eq. B-8 must be multiplied by the factor 0.5 to obtain $E_{eff}^2/(W_s/L)$ for the cavity chain.

References

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Appendix C

Dispersion and $\overline{E_z(0)^2}/W_s$ in the Cloverleaf Slow Wave Structure

C-1. Dispersion Relation

Within reference cavity (0) of Fig. 7-17c the π mode patterns are defined by

$$\begin{aligned} \nabla \times \bar{e}_\pi &= p_\pi \bar{h}_\pi & \bar{e}_\pi \times \bar{n} &= 0 \text{ on outer slot surfaces} \\ \nabla \times \bar{h}_\pi &= p_\pi \bar{e}_\pi & \bar{e}_\pi \cdot \bar{n} &= 0 \text{ on inner slot surfaces and center-} \\ & & & \text{holes lying in the centerhole region} \end{aligned} \quad (C-1)$$

$$\int_{\text{cav}} \bar{e}_\pi^2 dV = \int_{\text{cav}} \bar{h}_\pi^2 dV = \tau, \quad \text{cavity volume}$$

With the fields represented by $\hat{e} = v_\pi \bar{e}_\pi$ and $\hat{h} = i_\pi \bar{h}_\pi$ in Maxwell's Eq. 4-33 we obtain, by the procedure of Section 4-6, these expressions for v_π and i_π :

$$p_\pi v_\pi = -\frac{1}{\tau} \int_{S_\pm} \hat{E} \times \bar{h}_\pi \cdot \bar{n} dS - j\omega\mu i_\pi \quad (C-2a)$$

$$p_\pi i_\pi = +\frac{1}{\tau} \int_{CS_\pm} \bar{e}_\pi \times \hat{H} \cdot \bar{n} dS + j\omega\epsilon v_\pi \quad (C-2b)$$

in which $CS_\pm$ denotes the inner slot and centerhole surfaces belonging to the centerhole region, and $S_\pm$ denotes the outer slot surfaces where $\hat{E}$ is given by

$$\hat{E}_{\text{tan}} \text{ on } (S_\pm) = v_{s-} \bar{e}_{s-} \text{ on } (S-); \quad = v_{s+} \bar{e}_{s+} \text{ on } (S+) \quad (C-3)$$

in terms of the "slot" modes. We distinguish the two slots opposite each other in the lower sketch *d* of Fig. 7-17 because the periodicity operation

involves both a translation and a rotation by $\pm 45^\circ$. $\vec{H}$ on $(CS\pm)$ is given, as usual, by the "cavity" $\varphi = 0$ mode as

$$\vec{H}_{\text{tan on } (CS\pm)} \cong I_0(\epsilon^{\mp j\varphi} + 1)\vec{H}_0^{(0)} \quad (\text{C-4})$$

because $\vec{H}_0(x, y)$ is primarily in the same angular direction on both sides of each centerhole region $(CS\pm)$ bounding cavity (0).

If we substitute Eqs. C-4 and C-3 into C-2 and define

$$M_{cs} = \frac{1}{\tau} \int_{\text{center region}} \vec{e}_\pi^{(0)} \times \vec{H}_0 \cdot \hat{i}_z dS > 0 \quad (\text{C-5})$$

$$M_s \text{ (one slot)} = \frac{1}{\tau} \int_{S\pm} \vec{e}_{s\pm} \times \vec{h}_\pi^{(0)} \cdot \hat{i}_z dS > 0, \text{ for either sign choice}$$

(where the first integral may be extended over the whole slot plane and the second one over the whole length of one slot because $\vec{H}_0 \times \hat{i}_z = 0$ is zero on the outer slot surfaces and $\vec{h}_\pi \times \hat{i}_z = 0$ on each inner slot surface), then with the relations

$$\left. \begin{aligned} \vec{e}_{s-}(x, y) &= \vec{e}_{s+}(x, y) && \text{in Fig. 7-17d} \\ \vec{h}_\pi^{(0)}(x, y) \text{ on } (S-) &= \vec{h}_\pi^{(0)}(x, y) \text{ on } (S+) && \\ &\text{along the long axes} && \\ \vec{e}_\pi^{(0)}(x, y) &= -\vec{e}_\pi^{(0)}(x, y) && \\ \text{on one center region} &\text{on the center region} && \\ &\text{a period away.} && \end{aligned} \right\} \text{Fig. 7-17c} \quad (\text{C-6})$$

we obtain Eq. C-2 in the form

$$p_\pi v_\pi = (v_{s-} - v_{s+})M_s S - j\omega\mu i_\pi \quad (\text{C-7a})$$

$$p_\pi i_\pi = -M_{cs}(1 + \cos \varphi)I_0 + j\omega\epsilon v_\pi \quad (\text{C-7b})$$

S being the number of slots in a slot plane. In the single-passband approximation the magnetic field expressions $i_\pi \vec{h}_\pi$ and $I_0 \vec{H}_0$ are considered equivalent within the cavity proper and so $i_\pi = U_c I_0$. Then Eqs. C-7 combine to yield a single equation for i_π :

$$[(p_\pi^2 - k^2)U_c + p_\pi M_{cs}(1 + \cos \varphi)]i_\pi = j\omega\epsilon M_s S(v_{s-} - v_{s+}) \quad (\text{C-8})$$

The fact that the total slot excitation in a slot plane is S times the excitation of one slot may be verified from the relation between v_{s-} and v_{s+} of the slot opposite, derived next.

Within the $(S-)$ slot region shown bounded by magnetic shorting planes (B_1) and (B_2) in Fig. 7-17d we define the $(S-)$ "slot" mode as

$$\begin{aligned} \nabla \times \vec{e}_{s-} &= p_s \vec{h}_{s-} \\ \nabla \times \vec{h}_{s-} &= p_s \vec{e}_{s-} \end{aligned}$$

$$\vec{e}_{s-} \cdot \vec{n} = 0 \text{ on surfaces } (B_1) \text{ and } (B_2) \text{ of Fig. 7-17d;} \quad (\text{C-9})$$

$$\int_{\text{cell}} \vec{e}_{s-}^2 dV = \int_{\text{cell}} \vec{h}_{s-}^2 dV = \tau, \quad \text{also the slot cell volume}$$

Instead of Eq. C-2 we now obtain for v_{s-} and i_{s-} ,

$$\begin{aligned} p_s v_{s-} &= -j\omega\mu i_{s-} \\ p_s i_{s-} &= \frac{1}{\tau} \int_{B1, B2} \bar{e}_{s-} \times \hat{H} \cdot \bar{n} dS + j\omega\epsilon v_{s-} \end{aligned} \quad (\text{C-10})$$

$\hat{H}$ is given by the "cavity" $\varphi = \pi$ mode on the cavity midplanes (B_1) and (B_2):

$$\hat{H}_{\tan} \text{ on } (B_1) = i_\pi \epsilon^{j\varphi} \hat{h}_\pi^{(-1)}; \quad \text{on } (B_2), = i_\pi \hat{h}_\pi^{(0)} \quad (\text{C-11})$$

Since Fig. 7-17c shows $\hat{h}_\pi^{(-1)}(x, y)|_{B1} = -\hat{h}_\pi^{(0)}(x, y)|_{B2}$ we obtain, from Eqs. C-10 and C-11,

$$v_{s-} = \frac{-j\omega\mu m_s i_\pi (1 + \epsilon^{j\varphi})}{(p_s^2 - k^2)} \quad (\text{C-12a})$$

$$m_s = \frac{1}{\tau} \int_{B2} \bar{e}_{s-} \times \hat{h}_\pi^{(0)} \cdot \bar{i}_s dS > 0 \quad (\text{C-12b})$$

Now if we determine amplitudes v_{s+} and i_{s+} of the ($S+$) "slot" mode excited on planes (B_2) and (B_3) we find

$$v_{s+} = -v_{s-} \epsilon^{-j\varphi} \quad (\text{C-13})$$

the extra minus sign arising from the rotational effect. It follows that two *adjacent* slots in the same plane excite the cavity with opposed electric fields, but since the π mode pattern $\hat{h}_\pi$ along the slot axis also reverses direction both slots have twice the effect of one slot.

Now when we substitute Eqs. C-12 and C-13 into C-8 we obtain dispersion relation 7-34 of the text, for which the effective cavity-slot coupling coefficient m_{cs}^2 was defined as proportional to $M_s m_s$.

C-2. Interaction Parameter, $\frac{1}{L} \int_0^L \frac{|\hat{E}_z(0)|^2 dz}{W_s}$

At frequencies near the π cutoff where the structure is operated as a beam amplifier, axial electric field pattern $\hat{E}_z$ is given by the cavity π mode as $v_\pi e_{\pi z} \cong v_\pi e_m \sin(\pi z/L)$, where v_π phase shifts from cavity to cavity. We choose to measure total time-average stored energy as LP_{av}/v_π , P_{av} being the total time-average power flow given by Eq. 7-35 and $v_\pi = L\partial\omega/\partial\varphi$ as obtained from dispersion Eq. 7-34. After we relate amplitudes v_{s-} and i_π to v_π by means of Eqs. C-12 and C-7, with $I_0 \cong i_\pi$, we can put

the interaction parameter into the form

Interaction parameter near π

$$\propto \left[1 + \frac{\sigma p_s^2}{p_s^2 - k^2} (1 + \cos \varphi) \right]^{-1} \cdot \frac{\sigma k^2 - p_\pi M_{cs}}{\sigma k^2 - M_{cs} [p_\pi + M_{cs} (1 + \cos \varphi)]} \quad (\text{C-14})$$

$$\sigma \triangleq \frac{2M_s m_s S}{p_s^2 - k^2} = \frac{m_{cs}^2}{p_s^2 - k^2}$$

This expression was plotted in Fig. 7-19 with the π point "fit" to the measured parameter. The values of M_{cs} and m_{cs}^2 were chosen so that the calculated and experimental ω - φ curves of Fig. 7-18 agree at the $\varphi = 0$ and $\pi/2$ points.

Appendix D

Development of the Small-Signal Beam Equations

We shall write the beam equations in transmission line form so as to include relativistic corrections to the Chu kinetic voltage and the characteristic a.c. admittance, Y_0 .

The laboratory coordinate system sees each electron with a velocity $\hat{u}(z, t) = \bar{u}_0(z) + \text{Re} [\hat{v}_1(z) \epsilon^{j\omega t}]$. The force equation is

$$\frac{d}{dt}(\hat{m}u) = -e\hat{E}_z, \quad e > 0 \quad (\text{D-1})$$

We let

$$\begin{aligned} \hat{m}(z, t) &= m_0(z) + \text{Re} [\hat{m}_1(z) \epsilon^{j\omega t}] \\ \hat{E}(z, t) &= \bar{E}_0(z) + \text{Re} [\hat{E}_1(z) \epsilon^{j\omega t}] \end{aligned} \quad (\text{D-2})$$

with $\hat{m}_1$, $\hat{v}_1$, and $\hat{E}_1$ complex amplitudes. Only $\text{Re} [\hat{m}_1 \epsilon^{j\omega t}]$ has physical meaning. Amplitude $\hat{m}_1$ is related to $\hat{v}_1$ by differentiation:

$$\hat{m} = m_{00} \left(1 - \frac{\hat{u}^2}{c^2}\right)^{-1/2}, \quad m_{00}, \text{ the rest or proper electron mass} \quad (\text{D-3a})$$

$$\hat{m}_1 = m_{00} \hat{v}_1 \frac{\partial}{\partial u} \left(1 - \frac{\hat{u}^2}{c^2}\right)^{-1/2} \Big|_{\hat{u}=\bar{u}_0} \quad (\text{D-3b})$$

Substitution of Eq. D-2 into D-1 and separation of zero-order from first-order a.c. terms, with neglect of products of a.c. terms, yields

$$\begin{aligned} \left[j\beta_e(z) + \frac{\partial}{\partial z} \right] \left(-\frac{m_0}{e} u_0 \hat{v}_1 - \frac{\hat{m}_1}{e} u_0^2 \right) \\ + \frac{\hat{m}_1}{e} u_0 \frac{\partial u_0}{\partial z} - \hat{v}_1 \frac{u_0}{e} \frac{\partial m_0}{\partial z} = \hat{E}_z \end{aligned} \quad (\text{D-4})$$

where $\hat{E}_z$ includes any space charge field, of course. Now, because of Eq. D-3b and the fact that

$$\frac{\partial m_0}{\partial z} = m_{00} \frac{\partial}{\partial u} \left(1 - \frac{u^2}{c^2} \right)^{-1/2} \bigg|_{u=u_0} \cdot \frac{\partial u_0}{\partial z} \quad (\text{D-5})$$

the last two terms on the left side of Eq. D-4 cancel. We then define the relativistic Chu kinetic voltage in parentheses as

$$\hat{V}_k = -\frac{m_0}{e} u_0 \hat{v}_1 \left(1 + \frac{u_0^2/c^2}{1 - u_0^2/c^2} \right) = -\frac{m_\ell}{e} u_0 \hat{v}_1 \quad (\text{D-6})$$

with

$$m_\ell = m_{00} \left(1 - \frac{u_0^2}{c^2} \right)^{-3/2}, \quad \text{the longitudinal mass}$$

$$m_{00}, \quad \text{the rest or proper electron mass}$$

Our first relativistic beam equation therefore has the usual form with the kinetic voltage defined in Eq. D-6,

$$\left(j\beta_e + \frac{\partial}{\partial z} \right) \hat{V}_k = \hat{E}_z \quad (\text{D-7})$$

In deriving the second beam equation from the current and continuity expressions

$$\hat{J} = \rho_0 \hat{v}_1 + \rho_1 \bar{u}_0 \quad (\text{D-8})$$

$$\frac{\partial \hat{J}}{\partial z} = -j\omega \rho_1 \quad (\text{D-9})$$

we eliminate ρ_1 and reach the form

$$\left[j\beta_e(z) + \frac{\partial}{\partial z} \right] \hat{J} = j\omega \epsilon \beta_p^2 \left(1 - \frac{u_0^2}{c^2} \right) \hat{V}_k$$

with

$$\beta_p^2 = \left| \frac{e\rho_0}{m_0 \epsilon u_0^2} \right|$$

which is one way of writing the space charge factor multiplying $\hat{V}_k$ in this beam equation. We can write it another way by introducing the energy conservation relation

$$m_{00}c^2 + eU_0 = m_0c^2 = m_{00}c^2 \left(1 - \frac{u_0^2}{c^2} \right)^{-1/2} \quad (\text{D-10})$$

so that an alternate way of writing the differential equation for $\hat{J}$ is

$$\left(j\beta_e + \frac{\partial}{\partial z} \right) \hat{J} = \frac{j\beta_e |J_0|}{2U_0} R \hat{V}_k \quad (\text{D-11})$$

with R , the relativistic reduction factor, given by

$$R = \frac{2U_0}{U_0 + m_{00}c^2/e} \cdot \frac{1 - u_0^2/c^2}{u_0^2/c^2} \quad (\text{D-12})$$

and

$$\frac{m_{00}c^2}{e} = 505 \text{ kv}$$

For a 60 kv beam, $R = 0.885$.

Our second beam equation is Eq. D-11, in which admittance Y_0 can be considered reduced by factor R :

$$\left(j\beta_e + \frac{\partial}{\partial z}\right)\hat{J} = jY_0 R \beta_q \hat{V}_k \quad (\text{D-13})$$

with

$$Y_0, \text{ a.c. beam admittance, } = \frac{1}{2} \frac{|J_0|}{u_0} \frac{\omega}{\omega_q}$$

If we solve Eqs. D-7 and D-13 for the "free" space charge modes which can exist if the space charge $\hat{E}_z$ is proportional to $\hat{J}$ at every point, we find that the constant of proportionality is such that

$$\hat{E}_z (\text{space charge}) = j \frac{Z_0}{R} \beta_q \hat{J}, \quad Z_0 = Y_0^{-1} \quad (\text{D-14})$$

the propagation constants being $\beta_e \pm \beta_q$.

In these equations β_q is the reduced plasma propagation constant in the laboratory coordinate system. To obtain the relativistic correction to it we can start with the ordinary reduction factor ω_q'/ω_p' as evaluated in the frame of reference moving with velocity u_0 . In this frame $\omega_p' = |e\rho_0|/m_{00}\epsilon_0|^{1/2}$. The actual plasma resonant frequency, ω_q' , is related to ω_q in the laboratory coordinate system as

$$\omega_q = \omega_q' \left(1 - \frac{u_0^2}{c^2}\right)^{+1/2} \quad (\text{D-15})$$

while $\omega_p' = \omega_p$ because both ρ_0' and mass $m_0' = m_{00}$ transform the same way. Therefore β_q in the laboratory system is found to be

$$\begin{aligned} \beta_q &= \frac{\omega_q}{u_0}, & \omega_q &= \frac{\omega_q'}{\omega_p'} \omega_p \left(1 - \frac{u_0^2}{c^2}\right)^{+1/2} \\ & & \omega_p &= \left| \frac{e\rho_0}{m_0\epsilon_0} \right|^{1/2} \end{aligned} \quad (\text{D-16})$$

and the relativistic effect is evident.

By treating the beam as a medium supporting elastic forces we find upon transforming the longitudinal equation of motion from the rest frame of a unit volume of the beam to the laboratory frame, that the elastic or space charge stored energy contributes to the mass. However, the contribution is extremely small, being about a fraction

$$\frac{1}{2} \left(\frac{v_1}{c}\right)^2 \left(1 - \frac{u_0^2}{c^2}\right)^{-1/2}, \quad v_1, \text{ a.c. velocity} \quad (\text{D-17})$$

of the mass and is therefore neglected in the force equation.

Appendix E

Solution of the Beam and Cavity Equations

We shall obtain equations for the circuit electric field amplitude v and the kinetic voltage $\hat{V}_k(0)$ and current density $\hat{J}(0)$ entering the cavity of reference phase zero. The basic equations, in terms of the one open-circuit solenoidal cavity mode, are

$$[p^2 - k^2 + pM(1 \pm \cosh \gamma)]v = -\frac{j\omega\mu}{\tau} \int_V \hat{J} \cdot \bar{e}_z dA dz \quad (\text{E-1a})$$

$$\left(j\beta_e + \frac{\partial}{\partial z}\right) \hat{V}_k(z) = v\bar{e}_z + jZ_0\beta_q \hat{J}(z) \quad (\text{E-1b})$$

$$\left(j\beta_e + \frac{\partial}{\partial z}\right) \hat{J}(z) = jY_0\beta_q \hat{V}_k(z) \quad 0 \leq z \leq L \quad (\text{E-1c})$$

in which we select the upper sign prefixing $\cosh \gamma$ for a passband of even e_z symmetry. The last term of Eq. E-1b is the space charge electric field, with $Z_0 = Y_0^{-1}$ a positive number.

E-1. Uniform Field Pattern $\bar{e}_z$

If $\hat{V}_k(0)$ and $\hat{J}(0)$ are the beam excitations entering the cavity at $z = 0$, the solutions to Eqs. E-1b and E-1c are

$$\begin{aligned} \hat{V}_k(z) = & \left[\left(\hat{V}_k(0) + \frac{jve_z\beta_e}{\beta_e^2 - \beta_q^2} \right) \cos \beta_q z \right. \\ & \left. + jZ_0 \left(\hat{J}(0) + jY_0 \frac{\beta_q}{\beta_e^2 - \beta_q^2} ve_z \right) \sin \beta_q z \right] e^{-j\beta_e z} - \frac{jve_z\beta_e}{\beta_e^2 - \beta_q^2} \quad (\text{E-2a}) \end{aligned}$$

$$\begin{aligned} \hat{J}(z) = & \left\{ jY_0 \left(\hat{V}_k(0) + \frac{jve_z\beta_e}{\beta_e^2 - \beta_q^2} \right) \sin \beta_q z \right. \\ & \left. + \left[\hat{J}(0) + jY_0 \frac{\beta_q}{\beta_e^2 - \beta_q^2} ve_z \right] \cos \beta_q z \right\} \epsilon^{-j\beta_e z} - \frac{jY_0\beta_q ve_z}{\beta_e^2 - \beta_q^2} \quad (\text{E-2b}) \end{aligned}$$

For periodic systems,

$$\hat{V}_k(L) = \hat{V}_k(0)\epsilon^{-\gamma}, \quad \hat{J}(L) = \hat{J}(0)\epsilon^{-\gamma}, \quad (\text{E-3})$$

with which Eqs. E-2 multiplied by $\epsilon^{j\theta_e}$ read, at $z = L$,

$$\begin{aligned} \hat{V}_k(0)(\epsilon^{-\gamma+j\theta_e} - \cos \theta_q) - jZ_0\hat{J}(0) \sin \theta_q \\ = +v \frac{je_z\beta_e}{\beta_e^2 - \beta_q^2} \left(\cos \theta_q + j \frac{\theta_q}{\theta_e} \sin \theta_q - \epsilon^{j\theta_e} \right), \quad (\text{E-4a}) \end{aligned}$$

$$\begin{aligned} -\hat{V}_k(0)j \sin \theta_q + Z_0\hat{J}(0)(\epsilon^{-\gamma+j\theta_e} - \cos \theta_q) \\ = +v \frac{je_z\beta_e}{\beta_e^2 - \beta_q^2} \left[j \sin \theta_q + \frac{\theta_q}{\theta_e} (\cos \theta_q - \epsilon^{j\theta_e}) \right] \quad (\text{E-4b}) \end{aligned}$$

The solutions to these can be abbreviated as

$$\hat{V}_k(0) = Av, \quad \hat{J}(0) = Bv \quad (\text{E-5})$$

where

$$\begin{aligned} A = \frac{e_z\beta_e}{j(\beta_e^2 - \beta_q^2)} \left[\left(\epsilon^{j\theta_e} - \cos \theta_q - j \frac{\theta_q}{\theta_e} \sin \theta_q \right) (\epsilon^{-\gamma+j\theta_e} - \cos \theta_q) \right. \\ \left. + j \sin \theta_q \frac{\theta_q}{\theta_e} \left(\epsilon^{j\theta_e} - j \frac{\theta_e}{\theta_q} \sin \theta_q - \cos \theta_q \right) \right] \|D\|^{-1} \quad (\text{E-6a}) \end{aligned}$$

$$\begin{aligned} B = \frac{e_z\beta_e Y_0}{j(\beta_e^2 - \beta_q^2)} \left[(\epsilon^{-\gamma+j\theta_e} - \cos \theta_q) \frac{\theta_q}{\theta_e} \left(\epsilon^{j\theta_e} - j \frac{\theta_e}{\theta_q} \sin \theta_q - \cos \theta_q \right) \right. \\ \left. + j \sin \theta_q \left(\epsilon^{j\theta_e} - \cos \theta_q - j \frac{\theta_q}{\theta_e} \sin \theta_q \right) \right] \|D\|^{-1} \quad (\text{E-6b}) \end{aligned}$$

$$\|D\| = (\epsilon^{-\gamma+j\theta_e} - \cos \theta_q)^2 + \sin^2 \theta_q \quad (\text{E-6c})$$

When we substitute Eq. E-2b into the right side of Eq. E-1a, integrate, and use Eq. E-3, we obtain, after making the approximation $(\beta_q/\beta_e)^2 \ll 1$,

$$\begin{aligned} [p^2 - k^2 + pM(1 + \cosh \gamma) + \xi_u]v + \hat{V}_k(0) \frac{j\omega\mu e_z A_b Y_0}{\beta_e^2 \tau} \\ \cdot (j\beta_q \cos \theta_q - \beta_e \sin \theta_q - j\beta_q \epsilon^{j\theta_e}) \epsilon^{-j\theta_e} - \hat{J}(0) \frac{j\omega\mu e_z A_b}{\beta_e^2 \tau} \\ \cdot (\beta_q \sin \theta_q - j\beta_e \cos \theta_q + j\beta_e \epsilon^{j\theta_e}) \epsilon^{-j\theta_e} = 0 \quad (\text{E-7}) \end{aligned}$$

in which

$$\xi_u \triangleq p^2 C_B^2 \epsilon^{-j\theta_e} \left[\theta_q \epsilon^{j\theta_e} + \left(\frac{\theta_q}{\theta_e} \right)^2 \left(\sin \theta_q - j \frac{\theta_e}{\theta_q} \cos \theta_q \right) + 2j \frac{\theta_q}{\theta_e} \epsilon^{j\theta_e} - \frac{1}{\theta_e} (j\theta_q \cos \theta_q - \theta_e \sin \theta_q) \right] \quad (\text{E-8a})$$

and

$$C_B^2 = \frac{\omega \mu e_z^2 A_b Y_0}{p^2 \beta_e^2 \tau}, \quad A_b \text{ being the beam area} \quad (\text{E-8b})$$

Equations E-4 or E-5 and E-7 furnish a set of three equations for the three unknowns v , $\hat{V}_k(0)$, and $\hat{J}(0)$; the determinant of the matrix of their coefficients set equal to zero gives us the determinantal equation for the allowed values of γ :

$$\begin{aligned} & [p^2 - k^2 + pM(1 + \cosh \gamma) + \xi_u](\epsilon^{-\gamma+j\theta_e} - \epsilon^{j\theta_q})(\epsilon^{-\gamma+j\theta_e} - \epsilon^{-j\theta_q}) \\ & + 2C_B^2 p^2 (\epsilon^{-\gamma+j\theta_e} - \cos \theta_q) \left[j \sin \theta_q + \frac{\theta_q}{\theta_e} (\cos \theta_q - \epsilon^{j\theta_e}) \right] \\ & \times \left[\frac{\theta_q}{\theta_e} \sin \theta_q - j(\cos \theta_q - \epsilon^{j\theta_e}) \right] \epsilon^{-j\theta_e} + C_B^2 p^2 \sin \theta_q \epsilon^{-j\theta_e} \\ & \times \left\{ (\cos \theta_q - \epsilon^{j\theta_e} + j \frac{\theta_q}{\theta_e} \sin \theta_q)^2 + \left[j \sin \theta_q + \frac{\theta_q}{\theta_e} (\cos \theta_q - \epsilon^{j\theta_e}) \right]^2 \right\} = 0 \end{aligned} \quad (\text{E-9})$$

This equation was solved on the computer as a function of C_B , θ_q/θ_e , θ_e , and M/p .

E-2. Sinusoidal Field Pattern $\bar{e}_z = e_m \sin(\pi z/L)$

The solutions to Eqs. E-1b, c are now

$$\begin{aligned} \hat{V}_k(z) = & \left\{ \hat{V}_k(0) + \frac{1}{2} v e_m \left[\frac{\beta_e + \pi/L}{(\beta_e + \pi/L)^2 - \beta_q^2} - \frac{\beta_e - \pi/L}{(\beta_e - \pi/L)^2 - \beta_q^2} \right] \right\} \\ & \times \cos \beta_q z \epsilon^{-j\beta_e z} \\ & + \left\{ j Z_0 \hat{J}(0) + j \frac{v e_m \beta_q}{2} \left[\frac{1}{(\beta_e + \pi/L)^2 - \beta_q^2} - \frac{1}{(\beta_e - \pi/L)^2 - \beta_q^2} \right] \right\} \\ & \times \sin \beta_q z \epsilon^{-j\beta_e z} \\ & + \frac{1}{2} v e_m \left[- \frac{\beta_e + \pi/L}{(\beta_e + \pi/L)^2 - \beta_q^2} \epsilon^{j\pi z/L} + \frac{\beta_e - \pi/L}{(\beta_e - \pi/L)^2 - \beta_q^2} \epsilon^{-j\pi z/L} \right] \end{aligned} \quad (\text{E-10a})$$

$$\begin{aligned}
\hat{J}(z) = & \left\{ \hat{J}(0) + \frac{1}{2} v e_m Y_0 \beta_q \left[\frac{1}{(\beta_e + \pi/L)^2 - \beta_q^2} - \frac{1}{(\beta_e - \pi/L)^2 - \beta_q^2} \right] \right\} \\
& \times \cos \beta_q z \epsilon^{-j\beta_e z} \\
& + j Y_0 \left\{ \hat{V}_k(0) + \frac{1}{2} v e_m \left[\frac{\beta_e + \pi/L}{(\beta_e + \pi/L)^2 - \beta_q^2} - \frac{\beta_e - \pi/L}{(\beta_e - \pi/L)^2 - \beta_q^2} \right] \right\} \\
& \times \sin \beta_q z \epsilon^{-j\beta_e z} \\
& - \frac{1}{2} v e_m Y_0 \beta_q \left[\frac{1}{(\beta_e + \pi/L)^2 - \beta_q^2} \epsilon^{j\pi z/L} - \frac{1}{(\beta_e - \pi/L)^2 - \beta_q^2} \epsilon^{-j\pi z/L} \right]
\end{aligned} \tag{E-10b}$$

in which we have made no assumptions about θ_q/θ_e .

With the abbreviations

$$\mathcal{A} = \frac{\theta_e + \pi}{(\theta_e + \pi)^2 - \theta_q^2} - \frac{\theta_e - \pi}{(\theta_e - \pi)^2 - \theta_q^2} \tag{E-11a}$$

$$\mathcal{B} = \frac{1}{(\theta_e + \pi)^2 - \theta_q^2} - \frac{1}{(\theta_e - \pi)^2 - \theta_q^2} \tag{E-11b}$$

we can evaluate $\hat{V}_k(L)$ and $\hat{J}(L)$, use Eq. E-3, and write the equations for $\hat{V}_k(0)$ and $\hat{J}(0)$ in the form

$$\begin{aligned}
\hat{V}_k(0)(\cos \theta_q \epsilon^{-j\theta_e} - \epsilon^{-\gamma}) + j Z_0 \hat{J}(0) \sin \theta_q \epsilon^{-j\theta_e} \\
= -\frac{1}{2} v e_m L [\mathcal{A}(\cos \theta_q \epsilon^{-j\theta_e} + 1) + j \theta_q \mathcal{B} \sin \theta_q \epsilon^{-j\theta_e}]
\end{aligned} \tag{E-12a}$$

$$\begin{aligned}
\hat{V}_k(0) j \sin \theta_q \epsilon^{-j\theta_e} + Z_0 \hat{J}(0)(\cos \theta_q \epsilon^{-j\theta_e} - \epsilon^{-\gamma}) \\
= -\frac{1}{2} v e_m L [\theta_q \mathcal{B}(\cos \theta_q \epsilon^{-j\theta_e} + 1) + j \mathcal{A} \sin \theta_q \epsilon^{-j\theta_e}]
\end{aligned} \tag{E-12b}$$

When we integrate the right side of Eq. E-1a with the use of Eq. E-10b and also employ Eq. E-3 we obtain the following circuit equation:

$$\begin{aligned}
[p^2 - k^2 + pM(1 + \cosh \gamma) + \xi_s]v - \hat{V}_k(0) \frac{j\omega \mu e_m A_b Y_0 \pi L}{2\tau} \\
\times \left[\frac{1}{(\theta_e - \theta_q)^2 - \pi^2} (\epsilon^{-j(\theta_e - \theta_q)} + 1) - \frac{1}{(\theta_e + \theta_q)^2 - \pi^2} (\epsilon^{-j(\theta_e + \theta_q)} + 1) \right] \\
- \hat{J}(0) \frac{j\omega \mu e_m A_b \pi L}{2\tau} \left[\frac{1}{(\theta_e - \theta_q)^2 - \pi^2} (\epsilon^{-j(\theta_e - \theta_q)} + 1) \right. \\
\left. + \frac{1}{(\theta_e + \theta_q)^2 - \pi^2} (\epsilon^{-j(\theta_e + \theta_q)} + 1) \right] = 0
\end{aligned} \tag{E-13}$$

in which

$$\begin{aligned} \xi_s = & -jp^2 C_{BS}^2 \frac{\theta_e^2 \pi}{4} \left\{ \left[\frac{\theta_q \mathcal{B} + \mathcal{A}}{(\theta_e - \theta_q)^2 - \pi^2} (\epsilon^{-j(\theta_e - \theta_q)} + 1) \right. \right. \\ & + \left. \frac{\theta_q \mathcal{B} - \mathcal{A}}{(\theta_e + \theta_q)^2 - \pi^2} (\epsilon^{-j(\theta_e + \theta_q)} + 1) \right] \\ & + \left. j \frac{\theta_q}{\pi} \left[\frac{1}{(\theta_e + \pi)^2 - \theta_q^2} + \frac{1}{(\theta_e - \pi)^2 - \theta_q^2} \right] \right\} \quad (E-14a) \end{aligned}$$

and C_{BS}^2 is now defined as

$$C_{BS}^2 = \frac{\omega \mu e_m^2 A_b Y_0}{p^2 \beta_e^2 \tau} \text{ (dimensionless),} \quad A_b \text{ being the beam area} \quad (E-14b)$$

Equations E-12 and E-13 are three equations in the three unknowns v , $\hat{V}_k(0)$, and $\hat{J}(0)$; the determinant of their matrix of coefficients set equal to zero gives the following determinantal equation for the allowed γ :

$$\begin{aligned} [p^2 - k^2 + pM(1 + \cosh \gamma) + \xi_s][(\cos \theta_q \epsilon^{-j\theta_e} - \epsilon^{-\gamma})^2 + \sin^2 \theta_q \epsilon^{-2j\theta_e}] \\ + j \frac{C_{BS}^2 p^2 \theta_e^2 \pi}{4} (\cos \theta_q \epsilon^{-j\theta_e} - \epsilon^{-\gamma})(X_1 X_2 + X_3 X_4) \\ + \frac{C_{BS}^2 p^2 \theta_e^2 \pi}{4} \sin \theta_q \epsilon^{-j\theta_e} (X_1 X_3 + X_2 X_4) = 0 \quad (E-15) \end{aligned}$$

Here

$$\begin{aligned} X_1 &= \mathcal{A}(\cos \theta_q \epsilon^{-j\theta_e} + 1) + j\theta_q \mathcal{B} \sin \theta_q \epsilon^{-j\theta_e} \\ X_2 &= -\mathcal{D}(\cos \theta_q \epsilon^{-j\theta_e} + 1) + j \sin \theta_q \mathcal{E} \epsilon^{-j\theta_e} \\ X_3 &= \mathcal{E}(\epsilon^{-j\theta_e} \cos \theta_q + 1) - j \sin \theta_q \mathcal{D} \epsilon^{-j\theta_e} \\ X_4 &= \theta_q \mathcal{B}(\cos \theta_q \epsilon^{-j\theta_e} + 1) + j\mathcal{A} \sin \theta_q \epsilon^{-j\theta_e} \\ \mathcal{D} &= \frac{1}{(\theta_e + \theta_q)^2 - \pi^2} - \frac{1}{(\theta_e - \theta_q)^2 - \pi^2} \\ \mathcal{E} &= \frac{1}{(\theta_e + \theta_q)^2 - \pi^2} + \frac{1}{(\theta_e - \theta_q)^2 - \pi^2} \end{aligned}$$

This determinantal equation remains well behaved even if $\theta_e \pm \theta_q = \pi$.

Index

- Admittance, beam, 248
 external load, 311, 312, 316, 329, 345
 match to growing wave, 311, 312, 316
 normalized load, 312, 316
 rectangular cavity, 132
 relativistic correction to beam, 448
 waveguide mode, TE, 28; TM, 28
- Aikin, A. W., 436
- Allen, M. A., 208, 214
- Amplifier, beam, 270, 271, 275, 323, 326
 maser, 394
 parametric, ferroelectric, 368, 377; ferromagnetic, 387
- Amplitude modes, in parametric amplifier, 370
 irrotational, 129, 131
 solenoidal, 123, 129, 131
- Anderson, L. B., 352
- Angular momentum, commutation relation, 401
- Aperture, dipole moments, 78, 79, 86
 electrically shorted, 30
 equivalent circuit elements, 32, 81
 ferrite post, 386
 magnetically shorted, 32, 81, 82
 susceptance, TE mode, 89; TM mode, 89
- Attenuation in cavity chain, 100, 319
- Babinet's principle, 75
- Backward wave excitation, beam amplifier, 312
 parametric amplifier, 381, 383
- Backward wave reflection in beam tube, 312-315
- Bandwidth, beam amplifier models, 349, 350
 frequency, 345
 parametric amplifier, 375
 voltage, 284, 291
- Beam equations, 246, 252, 276, 446
- Bers, A., 352
- Bessel function, modified, 12, 42, 192; table of integrals of, 176
 ordinary, 192
- Bethe, H. A., 65
- Bethe theory, 65, 294
- Bevensee, R. M., 102, 137, 147-149, 161, 221, 259, 341, 411
- Birdsall, C. K., 181, 189, 251, 424
- Bisecting plane, transverse, 92, 122
- Bloch, F., 396, 402
- Bloch equations of motion for magnetization vector, 402
- Bloembergen, N., 394
- Brewer, G. R., 251
- Brillouin, L., 14
- Brillouin zone, 14
- Broadening, homogeneous, 396

- Buncher power, 348
 table for beam amplifier models, 349
 Butcher, P. N., 51
- Campbell, G. A., 19
 Capacitance, slot, 210
 Carlike, R. N., 385
- Cavities, amplifier input, 329, 355, 356
 amplifier intermediate, 328, 355, 356
 amplifier output, 328, 357
 buncher, 347
 input in parametric amplifier, 383
 optimized output, 335, 347
 two coupled, 83
 waveguide section, 132
- Cavity chain, centerhole-coupled, 13, 24, 72, 95, 96, 103
 centipede, 227, 236, 340
 cloverleaf, 112, 238, 340
 heavily coupled, 34
 lightly coupled, 34
 nonsymmetric, 118 (Prob. 6)
 resonant-slot coupled, 103–105, 138, 207, 215; successive slots rotated, 216
 S-loop coupled, 229
 symmetry, 24, 102, 122, 199
 U-loop coupled, 227
- Cell, "cavity," 105, 221
 "loop," 230
 "slot," 105, 221
- Centipede chain, dispersion, 237
 interaction parameter, 237, 242
 modes, 230
 schematic, 236
- cgs system of units, 71
 cgs-mks conversion, 71
 Charge, magnetic, 66, 68
- Chodorow, M., 115, 188, 189, 238, 343, 409
- Chu, E. L., 188–190, 409
 Chu-Schwinger method, 73, 191
- Circuit, amplitudes, open-circuit mode, 113, 114; short-circuit mode, 112, 113
 aperture in waveguide, 32, 116
 effect of beam, 113, 115
 equivalent, 32
- Circuit elements, aperture in waveguide, 32, 81
 cavity chain, 112, 114
- Circuit equations, cavity chain and beam, 258, 260, 266, 276, 321
 parametric amplifier, ferroelectric, 368; ferromagnetic, 389
- Cloverleaf chain, dispersion, 240
 interaction parameter, 241, 444
 modes, 238, 239
 schematic, 238
- Collin, R. E., 78
- Commutator bracket, 399
- Completeness of field expansion, 124, 134, 165, 417
- Condon, E. V., 24
- Continuity of fields across round hole, 66, 74, 77
- Continuity of trial fields, cavity chains, 196, 201, 202, 419
 helices, 168, 176, 179, 419
- Coupled-coaxial-cavity amplifier, parametric, 384
- Coupling, beam-circuit, 112
 between cavities, 122
 "cavity" to "loop" modes, 232
 "cavity" to "slot" modes, 224
 field to wall loss, 398
 hole input and output, 317
 loop input, 302
 loop output, 311, 314, 328
 secular or time average, 387
 symmetrical, 122
 T_1 or molecular-lattice, 396
 T_2 or molecular-molecular, 396
- Coupling coefficient, C_B (uniform E_z), 109, 110, 277
 C_{B8} (sinusoidal E_z), 109, 110
 C_P (effective E_z), 261, 276, 440
 C_P measured by Kompfner dip, 439
 C_P (Pierce), 276, 438
 corrections for other modes, 263, 266
 parametric, 371, 373
 resonant-slot coupled chain, 212
- Coupling integral, 80, 81, 87, 89, 94, 98
- Craig, R. A., 115, 238
- Crosscoupling, "cavity" and S-"loop" modes, 232
 "cavity" and "slot" modes, 224

- Crosscoupling, cloverleaf chain, 241, 444
 on surfaces between cavities, 142, 331
- Crosscoupling coefficient, centipede chain, 233
 cloverleaf chain, 241, 444
 resonant-loop coupled chain (S-loops), 232
 resonant-slot coupled chain, 224
- Crossenergy, of "cavity" and "slot" modes, 226
- Crosspower, between "cavity" and "slot" modes, 226
 between circuit and beam modes, 249
 between waves in beam tube, 295
- Current, beam, 351
 surface, electric, 76, 319; magnetic, 66, 69
- Currie, M. R., 377
- Cutler, C. C., 181
- Cutoff frequency, "loop" mode, 228
 open-circuit mode, 124
 short-circuit mode, 124
 "slot" mode, 104
 tape helix, 46
- Damping in ferrite, 389
- Decay constant, cutoff waveguide mode, 33
 due to wall loss, 319
 stopband of cavity chain, 100
- Delta function, Dirac, 30, 160
- Demagnetizing factor in ferrite, 388
- Density matrix, 396
- Density operator, 397
- Determinantal equation, *see* Dispersion equation
- Dipole moments, elliptical aperture, 78, 79, 86
- Dirac, P. A. M., 396
- Discontinuity, aperture in waveguide, 30-32, 80-83
 obstacle in waveguide, 30
 reactive cavity, 307
- Dispersion curve(s), 7, 14, 15
 branches, 14
 cavity chains in general, 103, 140
 centerhole-coupled chain, 13, 34, 193
- Dispersion curve(s), centipede, 237
 cloverleaf, 240
 contrawound helix, 189
 lumped-loaded transmission line, 161
 parametric amplifier chain, 382
 resonant-loop coupled, 228, 231-233
 resonant-slot coupled, 212, 216, 227;
 successive slots rotated, 216
 ridged waveguide, 56, 60, 193
 ring-plane helix, 182
 shaping, 425, 426
 sheath helix, 43
 table of properties, 103
 tape helix, 4, 45, 172, 177, 178
 tape ladder lines, 53
- Dispersion diagram, 45, 189
- Dispersion equation, centipede chain, 233
 chain and beam, sinusoidal cavity field, 299, 453; two mode-pairs, 261; uniform cavity field, 276, 451
 cloverleaf chain, 241, 442
 complex propagation constant in beam tube, 250, 254, 432
 first stopband of cavity chain, 100
 general cavity chain and beam, 255, 259, 266
 general "cold" chain, 267
 lumped-loaded transmission line, 34
 modified for wall loss effect, 321
 one passband of cavity chain, 34, 99, 101, 142, 145, 147, 198, 199
 parametric phase shift, 371
 Pierce, 251, 255, 259, 260, 266, 432
 resonant-loop coupled chain, S-loops, 233, 427; U-loops, 227
 resonant-slot coupled chain, 211, 214, 219, 224, 425; and beam, 263, 264; successive slots rotated, 219
 ridged waveguide, 193
 ring-plane helix, (+ +) wave, 184; (+ -) wave, 186
 sheath helix, 43
 single mode-pair, cavity chain, 34, 99, 101, 107
 trough waveguide, 56
 two mode-pairs, beam tube, 263; "cold" chain, 142, 143, 145

- Dispersion equation, unifilar tape helix,
 - predominant $n = 0$ mode, 48, 179, 180; predominant $n \neq 0$ mode, 174
- Divergence identity, 128, 145, 156
- Dyadic, Hermitian, 369
 - permeability, 369
 - permittivity, 369
 - susceptibility, 389
- "Easitron" structure, 51
- Eggimann, W. H., 75, 77
- Eigenfield, 417
- Eigenfunction, of Helmholtz (wave)
 - equation, 155, 158
 - of Schrödinger equation, 397
- Eigenvalue, 155
- Electric field, effective, 260, 277, 299
- Electromagnetic operation, ferromagnetic parametric amplifier, 387, 389
- Elliptical coupling hole, 77
- Energy, magnetic dipole, 398
 - stored, 108, 413
 - theorem, 17
 - velocity, 18
- Energy levels, molecular, 395, 398
- Englefield, G. C., 124
- Equivalence principle for cavity chains, 265-267
- Equivalent circuit elements, aperture, 32, 81
 - cavity chain, 113, 115
- Error, mean square, 153
- Euler, L., 15
- Everhart, T. E., 50, 181, 189, 424
- Excitation, antisymmetric, 30, 31
 - beam in cavity, 276, 326
 - symmetric, 30-32
 - waveguide aperture, 32, 81
- Expansions, adjoint field, 246, 253
 - contrawound helix, 189, 411
 - Fourier, 154
 - irrotational modes, 135, 208
 - mode-harmonics, 14
 - nonuniqueness, 132
 - open-circuit modes, 123, 125, 130
 - parametric amplifier fields, 369
 - ridged waveguide, 58
- Expansions, ring helix, 183, 185
 - sheath helix, 42, 44
 - short-circuit modes, 123, 125, 129, 208
 - solenoidal modes, 124, 137, 208
 - standing wave form, 9, 23, 35, 123
 - tape helix, 172, 173
 - tape ladder lines, 54
 - TE field, 14, 28
 - TM field, 14, 28
 - traveling wave form, 9, 23, 28, 41, 58
- Expectation value of operator, 397
- Faraday's law, 316, 329
- Feher, C., 394
- Felsen, L. B., 80
- Ferrite, 387
- Feshbach, H., 24, 107, 159, 417, 419
- Field expansions, *see* Expansions
- Filter, lumped-element line, 16, 19
- Floquet theorem, 9
- Forbidden zone, helix, 44, 45
- Fourier series, 6, 23, 153
- Frequency, cutoff, 19, 24, 27, 28
 - molecular transition, 395
 - operating, 271, 275
 - plasma, 109
 - precessional in ferrite, 387
 - pump, 362, 368, 395
 - reduced plasma, 109; relativistic correction to, 448
 - resonant, 24, 224, 229
- Frequency-squared sum rule, 145
- Fundamental propagation constant, 4, 6, 9, 11, 45
- Gain, electronic, 108
 - impedance wall effect, 320
 - in beam amplifier, maximization, 108; optimum, 288, 297
 - insertion (parametric chain), 384, 385
 - single-sideband in parametric amplifier, 367
 - transducer, 337, 349
 - unilateral, 381
- Gain-bandwidth product, of beam amplifier, 336, 343
 - of parametric amplifier, 375, 384

- Gain of beam amplifier models, definition, 337
 effect of intermediate cavity loading, 345
 gain and input Z , synchronously tuned, 338–340; stagger-tuned, 346
 table of buncher powers and bandwidths, 349, 350
- Gain parameter, C_B (uniform E_z), 109, 110, 277
 C_{B8} (sinusoidal E_z), 109, 110
 C_P (effective E_z), 261, 276, 440
 C_P (Pierce), 276, 438
 corrections for all the modes, 263, 266
 measurement, 439
 relation between C_P and C_B , 278
- Gain-passband width conservation relation, parametric amplifier, 375
- Gate functions, 420
- Generator, current, 302
- Ginzton, E. T., 437
- Goblick, T. J., 161
- Golde, H., 344
- Gould, R. W., 377
- Grabowski, K. P., 381
- Green's function, 67, 73
- Green's function technique, 191
- Green's vector theorems, 84, 187
- Group theory, 122, 411
- Group velocity, 5, 7, 16, 17, 108
- Growing wave, effect of impedance walls, 320
- Growing wave excitation, beyond detuned cavity, 308, 309
 in first cavity of amplifier, 302, 304, 305
- Growth factor, beam amplifier wave, 247, 259, 281–290, 295–301
 effect of impedance walls, 318, 320
 parametric amplifier wave, 374
- Gyromagnetic factor or ratio, 388, 401
- Hamiltonian or energy operator, 397
 of cavity chain and molecules, 398
 of field energy, 398
 of field-molecular coupling, 398
 of molecular-molecular coupling, 398
 of molecule, 400–402
- Hansen, W. W., 190
- Harmonic, parametric pump, 369
 spatial, 6, 14, 47, 58, 193
- Haus, H. A., 245, 364
- Heisenberg description, 396
- Heisenberg equation for operator, 397, 399, 401
- Helix, bifilar tape, 49, 50
 contrawound, 188, 189, 408
 multifilar, 49
 quadrifilar, 49
 ring-bar, 344
 ring-plane, 181, 182
 sheath, 40–44
 unifilar tape, 10, 44, 167
- Helmholtz (wave) equation, 106, 127, 155
- Hutter, R. G. E., 47, 61
- Impedance (inverse of admittance),
 beam, 246, 248, 448
 cavity output, 311, 316
 first cavity input, 334, 341
 ideal, 414, 415
 interaction (correction to measured), 438; (measurement), 437; (parametric chain), 374; (single mode-pair), 415
 interaction or Pierce, 50, 265
 match to backward wave, 315
 match to growing wave, 312, 322 (Prob. 10)
 normalized cavity output, 316
 Pierce or interaction, 50, 265
 ring-plane helix, 181, 185, 186
 transverse, 56
 unifilar tape helix, 178
 waveguide mode, TE, 28, 82; TM, 28, 82
- Inductance, slot, 210
- Input cavity excitation, beam amplifier, 302, 329
 buncher cavity, 355
 parametric amplifier, 383
- Interaction parameter, 413
 centipede chain, 237
 cloverleaf chain, 241, 242, 444
 ideal, 413, 415

- Interaction parameter, measurement, by
 - perturbation, 234, 438; correction to, 438, 440
 - resonant-loop coupled chains (S-loops), 234, 235
 - resonant-slot coupled chains, 214, 234
- Inversion of π -cutoff point, 233
- Inverter operation, parametric, 363
- Irrotational electric field, 275
- Irrotational mode, 127, 132, 134, 137, 215, 265
- Jaynes, E. T., 437
- Johnson, H. R., 50
- Kahn, W. K., 80
- Kinetic voltage, 446
- Kino, G. S., 208, 437, 438
- Kirschbaum, H. S., 57
- Kompfner dip, 439
- Ladder line, tape, 51
- Lagerstrom, R. P., 438
- Lagrange, 19
- Linear accelerator, Stanford, 86
- Load, capacitive on transmission line, 22, 23
 - external cavity, 311, 312, 316, 329, 345
 - inductive, 312-315
 - intermediate cavity of amplifier models, 345
 - matched, in beam amplifier, 311-318; in parametric amplifier, 383
 - maximum power in beam amplifier, 313
 - normalized external cavity, 316, 331
- Loop excitation of maser cavity, 399
- Louisell, W. H., 256, 361, 367, 439
- Magnetic field in ferrite, 388
- Magnetic moment operator, 395
 - equation of motion, 401
- Magnetization vector, 364, 387, 388
- Maier, L. C., 435
- Manley, J. M., 364
- Manley-Rowe relations, 364, 371
- Marcuvitz, N., 27, 80, 81
- Marcuvitz-Schwinger technique, 27
- Maser, cavity chain, 394
- Mass of electron, longitudinal, 447
- Matched load, "cold" circuit, 312
 - "hot" circuit, 312
 - input cavity, 311, 315
 - output cavity, 311
- Maxwell's equations, cgs units, 67, 390
 - mks units, 2, 14, 24, 26, 75, 85, 129, 166, 246, 370, 417
- Meander line, 37
- Meixner, J., 70
- Membrane, circular, 159
- Microwave amplifier, 1, 270, 323
- Midplanes, cavity, 122
- Mixer operation, parametric, 363
 - mks system of units, 71
- Mode, amplitude, 24, 26, 28, 35, 84, 110
 - "cavity," 96, 105, 138, 207, 221, 222, 229, 233
 - classification, 139
 - cloverleaf chain, 239
 - "coupling," 138
 - cutoff, 81
 - d.c., 127, 128, 132
 - definition, 3
 - dominant TM, 84
 - even symmetry, 123
 - expansions, 3, 5, 6, 11, 12, 35, 112
 - fast beam, 245, 248
 - faster than light, 12
 - irrotational, 127, 132, 134, 137, 215, 265
 - "loop," 229, 230, 233
 - magnetostatic, 388, 392
 - odd symmetry, 123
 - open-circuit, 22, 24, 26, 35, 64, 87, 96, 110, 258
 - parametric amplifier, 366, 369
 - pattern, *see* Pattern
 - resonant cavity, 6, 25, 65, 139
 - resonant frequency, 258
 - resonant on transmission line, 21
 - short-circuit, 22, 24, 26, 35, 96, 258
 - "slot," 104, 105, 138, 207, 222, 443
 - slow beam, 245, 248
 - solenoidal, 125
 - space charge, 274
 - spin, 388

- Mode, "straight through," 24, 72
 - symmetry, 123
 - TE_{10} , 31, 84
 - waveguide, TE, 14, 27, 31, 80, 88, 132; TM, 14, 27
- Mode-harmonic, 14, 36, 124, 189, 193
- Momentum, angular, 401
- Morse, P. M., 24, 107, 159, 417, 419
- Neumann function, 12, 192
- Noise figure, parametric amplifier, 367
- Normalization of patterns, 96
- Normalized parameters for cavity chain, 330
- Obstacle, waveguide, 29, 30, 82
- Oliner, A. A., 56
- Operator, electric and magnetic fields, 397
 - field-wall loss coupling, 398
 - Hermitian, 397
 - magnetic moment, 395
 - see also Hamiltonian, Energy operator
- Operator equation of motion, fields, 399, 400
 - magnetization vector, 401
 - molecular Hamiltonian, 402
- Orthogonality relations, beam modes, 250
 - between irrotational and solenoidal modes, 129
 - eigenfunctions of scalar wave equation, 158
 - irrotational modes, 128
 - magnetostatic modes, 393
 - solenoidal modes, 126, 127
- Orthonormality relations, irrotational modes, 128
 - solenoidal modes, 127
 - waveguide modes, 30
- Oscillation, "cutoff," 336, 340
- Output cavity, optimized, 347
- Parametric amplifier, ferroelectric (experimental), 384; (theoretical), 368
 - ferromagnetic (electromagnetic operation), 387, 389; (semistatic operation), 387, 391
- Parametric TEM line, 362
- Passband, 15, 35
 - "cavity," 102, 207, 224, 227, 240
 - "loop," 227
 - "slot," 105, 207, 215, 224, 240
- Passband parameter M , 98, 101, 105
- Passband width, 108, 278, 334
- Pattern, beam mode, 248
 - "cavity" (open- or short-circuit) mode, 25, 36, 84, 95, 96, 110, 224, 248, 278
 - irrotational mode, 127
 - "loop" mode, 230
 - "resonant" mode, 26
 - "slot" mode, 104, 222, 239
 - standing wave, 22, 23
 - waveguide mode, TE, 28; TM, 27
- Pearce, A. F., 228, 233
- Periodicity theorem, 9, 11, 97
- Periodic length, 6, 9
- Permeability, dyadic, 125, 369
 - scalar, 125
- Permittivity, dyadic, 125, 369
 - pumped component, 369
 - scalar, 125
- Perturbation formula for $\omega(\beta)$ of helix, 422, 423
- Perturbations, propagation constant, by dielectric rod, 437; by highly conducting rod, 438
 - resonant frequency, by dielectric rod, 202; by dielectric sphere, 436; by metallic sphere, 436; by needle, 435
- Slater formula, 204
- Perveance, beam, 279, 291
- Phase shift, "cold" circuit, 4, 7, 11, 13, 26, 31, 136
 - "hot" circuit, 11, 258
- incremental in beam tube, 259, 280–286, 295, 296, 298
- parametric chain, 370, 373, 374, 376, 377, 380
- Phase velocity, 10, 12
- Pierce, J. R., 12, 41, 43, 175, 250, 276, 289, 290, 415
- Pitch angle of helix, 9, 41
- Polarization, electric, 364
 - elliptical at ferrite post, 385
 - magnetic, 364

- Potential, for tape ladder lines, U , 53;
 V , 53
 scalar, electric, 127; magnetic, 67;
 magnetostatic mode, 392; TE
 mode, 28; TM mode, 27
 vector, electric, 67; magnetic, 75
- Power, buncher, 348
- Power flow, axial, in beam or para-
 metric amplifier, 18, 286, 292; in
 "cold" circuit, 18, 107, 175, 249,
 260, 431
 beam modes, 248, 249, 287, 334
 cloverleaf chain, 241
 continuity across coupling surface,
 294
 equivalence of single mode-pair ex-
 pressions, 292
 external load, 316, 334
 first cavity, 334
 noise, 367
 quantum expression for load power,
 404
 resonant-slot coupled chain, 226, 264
 single mode-pair approximation, 107
 transverse, 59
- Power flow-stored energy-group veloc-
 ity relation, 18, 107, 225, 429
- Power gain, cavity-chain beam tube,
 288, 290, 337
 helix tube, 289, 290
 parametric amplifier, 374
- Precession of magnetization vector,
 387, 389
- Propagation constant, cavity chain vs.
 helix tube, 289
 complex, in beam amplifier, 245; in
 parametric amplifier, 364-366
 free space, 12
 fundamental, 4, 6, 9, 11, 45
 incremental, 251
 mode or mode-harmonic, 4, 6, 12
 plasma, 447
 radial, 253
 reduced plasma, 245, 448; relativistic
 correction to, 448
 tape helix, 46
- Pupin, M., 19
- Quadrupole moment, coupling hole, 78
- Quality factor Q , loaded cavity, 336
 unloaded cavity, 319
- R_{sh}/Q , 437
- Ramo, S., 12, 210
- Rayleigh-Ritz technique, 165
- Reactive growing or decaying wave, 287
- Reduction factor, beam-circuit coupling
 coefficient C_B or C_P , 109, 277
 plasma frequency, 110, 276
- Relativistic correction, beam-circuit
 coupling coefficient C_B or C_P , 110
 beam equations, 226, 447
 space charge electric field, 447
- Resistance, shunt, 344
- Resistive wall amplifier effect, 320
- Resonant frequency, effective of cavity,
 332
 normalized, 331, 338
- Resonant mode, *see* Mode, resonant
- Ridged waveguide, 57, 190
- Ring-bar helix beam amplifier, 344
- Rotman, W., 56
- Rowe, H. E., 364
- Schelkunoff, S. A., 85, 131
- Schelkunoff's procedure, 85
- Schiff, L. I., 401
- Schmidt process, 126
- Schrödinger description, 396
- Schrödinger equation for operator, 397
- Schwinger, J., 27
- Scovil, H. E. D., 394
- Seidel, H., 394
- Semistatic operation, ferromagnetic
 parametric amplifier, 387, 391
- Senitzky, I. R., 397, 400
- Sensiper, S., 15, 41, 47, 385
- Shaping, dispersion curve, 425
- Shorting planes, electric, 16, 25, 124
 magnetic, 17, 26, 124
- Shunt resistance, 344, 437; *see also*
 R_{sh}/Q
- Siegman, A. E., 50, 177
- Skirtron tube model, 347, 353, 354
- Slater, J. C., 1, 34, 190, 204, 398, 435

- Slow wave structure, 2, 102
 "easitron," 51
 ladder strip, 36(Prob. 1)
 loaded coaxial line, 117(Prob. 2)
 lumped-loaded transmission line, 20, 22
 meander line, 37(Prob. 9)
 ridged waveguide, 51, 56, 57, 190
 tape ladder lines, 51, 52, 54, 203(Prob. 8)
 trough waveguide, 56
 see also Cavity chain, Helix
 Smoothing parameter, 419
 Space charge electric field, 299
 Space charge energy, relativistic correction, 448
 Space charge modes, fast and slow, 248
 Space charge parameter $4(QC)_B$, 277
 $4(QC)_P$ of Pierce, 251
 Spin-lattice decay, 396
 Stagger-tuned chain, equations, 332, 333
 Stark, L., 46
 State, parametric, 363, 366
 Stimulated emission of radiation, 394
 Stopband, 15, 19
 Stored energy, 18, 108, 226
 Stratton, J. A., 70, 436
 Suhl, H., 389, 394
 Superposition, 15
 Susceptance, aperture, 82, 89
 see also Admittance
 Susceptibility dyadic, ferrite, 389
 Symmetry, longitudinal, 123
 mirror, 95, 100, 103, 108, 122, 125
 Symmetry operations, contrawound helix, 408-410
 Symmetry planes, 16
 Synchronism, beam-circuit, 248, 311, 313
 Synchronism condition, parametric line, 376
 Synchronism parameter b_B , 278
 b_P of Pierce, 250, 259
 Tape ladder lines, 52, 54, 55
 Taper, cavity chain, 341
 Tapered cavity beam amplifier, 342
 Tape width parameter, 173
 Teichmann, T., 135
 Terminals, waveguide, 31
 TE waveguide mode, 14, 27, 31, 80, 88, 132
 Theorems, field expansion, 148
 frequency-squared sum rule, 148
 irrotational mode effect on dispersion, 137
 narrow passband, 102
 Tien, P. K., 43, 175, 362
 Time constant, T_1 (molecular-lattice), 396
 T_2 (molecular-molecular), 396, 403
 Time dependence, 3
 TM waveguide mode, 14, 27
 Transmission line equations, ordinary
 TEM line, 160
 parametric line, 363
 resonant slot, 209
 waveguide modes, 28, 32
 Transmission line, lumped-loaded, 20
 Trial fields, adjoint beam, 248, 253
 adjoint circuit, 248, 253
 beam, 248, 256
 cavity chain, 196, 258
 "cold" circuit, 248, 253, 256
 continuity conditions in helix, 421
 contrawound helix, 189
 general helix, 429
 one passband of cavity chain, 198
 resonant-loop coupled chain, 231
 resonant-slot coupled chain, 221
 ridged waveguide, 192, 194
 ring-plane helix, 183-186
 serrated ridge waveguide, 57
 slot voltage, 219
 smoothing for continuity, 420
 tape helix, 172
 tape ladder lines, 203(Prob. 8)
 Trial function, for circular membrane, 160
 for scalar function, 153
 for scalar wave equation, 157
 Tsu, R., 57
 Tube model, coupled-cavity, 334, 348
 decoupled cavity, 334, 346, 348
 stagger-tuned, 346
 synchronously tuned, 334, 345

- Uncertainty principle, 399
- Uniqueness of field expansion, 131
- Upper-bound expression for $\omega(\beta)$ of
 - helix, 202, 417
 - failure of upper-bound nature, 422
- Variational analysis, critique, 201
- Variational expressions, cavities with
 - several discontinuity surfaces per period, 201
 - complex phase shift of beam-cavity chain system, 257
 - complex propagation constant of beam-circuit system, 246
 - $j\omega$, of cavity chain, 196; of periodically loaded transmission line, 160
 - k , of sheath helix, 171
 - k^2 , of cavity chain, 197, 200; of resonant-loop coupled chain, 201, 220; of resonant-slot coupled chain, 201, 218, 220
 - ω , of ridged waveguide, 193, 194; of ring-plane helix, 184; of tape and wire helices, 169
 - ω^2 , of periodically loaded transmission line, 164
 - $\omega(\beta)$, lower bound to, 418; upper bound to, 417
- Variational technique, examples, 153
- Vector, magnetization, 364, 387
 - polarization, 364
- Vector theorems, divergence identity, 128, 145, 156
 - Green's, 84
- Velocity, energy, 18
 - envelope, 18
 - group, 5, 7, 16, 17, 108
 - phase, 5, 7, 16, 43, 47
- Voltage, kinetic, 351
 - synchronous beam, 274, 277
- Voltage gain-bandwidth product, parametric, 375, 384
- Walker, G. B., 124
- Walker, R. L., 392
- Watkins, D. A., 177
- Wave, adjoint, 248
 - backward, 307, 315
 - "cold" circuit, 3, 12
 - constituents, 3, 12, 23, 28, 35
 - cutoff, 11, 100
 - decaying, 307
 - definition, 3
 - fast beam, 307
 - forward, 307
 - growing, 297, 307
 - "hot" in beam tube, 281-286, 295, 296, 298, 300, 301
 - multifilar helix, 49
 - parametric, 366, 373
 - pump, 362
 - ring helix (++) behavior, 186, 187
 - slow, 2
 - table of parametric amplifier wave properties, 373, 378-380
- Wave equation, scalar, 106, 127, 155
 - vector, 196
- Wavefunction, quantum, 397
- Waveguide, fin-loaded, 30
 - ridged, 57, 190
 - tape ladder line, 51
 - trough, 56
 - uniform, 27
- Wavelength, electronic, 290
 - free space, 34
- Weglein, R. D., 381
- Wessel-Berg, T., 343
- Whinnery, J. R., 12, 210
- Wigner, E. P., 135

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